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Simultaneous Pellian equations with a single or no solution

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1. Introduction. A *system of simultaneous Pellian equations* is a system of Diophantine equations of the form

$$ax^2 - by^2 = \delta_1, \quad cy^2 - dz^2 = \delta_2, \tag{1.1}$$

where $a, b, c, d, \delta_1, \delta_2$ are nonzero integers, and $\gcd(ab, \delta_1) = \gcd(cd, \delta_2) = 1$. In 1969, Baker and Davenport [1] used the theory of linear forms in logarithms of algebraic numbers to solve equations (1.1) in the particular instance $(a, b, c, d, \delta_1, \delta_2) = (1, 3, 8, 1, -2, 7)$. Since then, many systems of simultaneous Pell equations have been studied.

Many authors have obtained upper bounds for the number of solutions of (1.1) (see for example [2], [21], [22], [3], [7]). In 1996, Ono [17] remarked that the existence of only trivial solutions of the system of simultaneous Pellian equations

$$x^2 - ay^2 = z^2 - by^2 = 1 \tag{1.2}$$

is a consequence of the related elliptic curve

$$y^2 = x(x + a)(x + b)$$

having Mordell–Weil rank zero over $\mathbb{Q}$. Two years later, Bennett [2] proved that the system of simultaneous Pell equations (1.2) has at most three positive integer solutions, where a, b are two distinct positive integers. In 2002, Yuan [21] strengthened this result by proving that these equations have at most two solutions in positive integers (x, y, z) if $\max\{a, b\} > 1.4 \cdot 10^{57}$. This result was sharpened by Bennett–Cipu–Mignotte–Okazaki [3] by removing the above condition.

Progress has been made in the study of some particular cases giving at most one positive solution (see [10], [20], [6], [23], [12], [4], [19], [11] and [14]). Moreover, very recently, Li, Xia, and Yuan [13] studied a special case

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of system of simultaneous Pellian equations

$$\begin{cases} (m+\delta)x^2 - my^2 = \delta, \\ y^2 - bz^2 = 1, \end{cases} \tag{1.3}$$

where $\delta = 1$ or 4, and $2 \nmid m$ if $\delta = 4$. They proved that equations (1.3) then have at most one solution in positive integers (x, y, z).

In this paper, we consider an extension of the above problem. In fact, we study the system of simultaneous Pellian equations

$$(m+\delta)x^2 - my^2 = \delta, \quad y^2 - bz^2 = 1, \quad \delta \in \{\pm 1, \pm 2, \pm 4\}, \tag{1.4}$$

where $\min\{m, m+\delta\} \geq 1$, and $2 \nmid m$ if $\delta \neq \pm 1$. Our main result is the following.

THEOREM 1.1. *Equations* (1.4) *have at most one solution in positive integers* (x, y, z).

From Theorem 1.1, we get the following result.

THEOREM 1.2. *For any positive integer* m *and* $\delta \in \{\pm 1, \pm 2, \pm 4\}$, *the system of simultaneous Pellian equations*

$$x^2 - mdz^2 = y^2 - (m+\delta)dz^2 = 1 \tag{1.5}$$

has at most one positive integer solution (x, y, z).

In fact, if we multiply the second equation of (1.4) by m and add the resulting equation to the first equation of (1.4), we obtain

$$(m+\delta)x^2 - mbz^2 = m + \delta.$$

Taking $b = d(m+\delta)$ and simplifying by $m+\delta$, we obtain (1.5). Moreover, when d has the form $2^f p^g$ where p is an odd prime, we can deduce that equations (1.5) have at most one positive solution (x, y, z). Theorem 1.2 generalizes a theorem of Walsh [19] on equations (1.2). He proved this result for the special case $m = 1$ and $\delta = 1$.

The organization of this paper is as follows. In Section 2, we recall or prove some useful results following the work independently done by Yuan and Walsh. The proof of Theorem 1.1 is given in Section 3 by applying a result due to Walsh [20]. Finally, in Section 4, we study a particular case. In fact, we assume $b = b'|4m/\delta + 4|$ with $b' \in \{1, 2, p\}$ where p is an odd prime and we determine all solutions to equations (1.4).

2. Some lemmas. Let $n = \min\{m, m+\delta\}$, i.e.

$$n = \begin{cases} m & \text{if } \delta = 1, 2, 4, \\ m+\delta & \text{if } \delta = -1, -2, -4, \end{cases} \tag{2.1}$$

where $2 \nmid m$ if $\delta \neq \pm 1$. Then equations (1.4) can be rewritten as

$$(n+c)x^2 - ny^2 = c, \quad c \in \{1, 2, 4\}, \tag{2.2}$$

$$y^2 - bz^2 = 1 \ (\text{if } \delta = c) \quad \text{or} \quad x^2 - bz^2 = 1 \ (\text{if } \delta = -c). \tag{2.3}$$

In fact, when $\delta = -1, -2, -4$, one can interchange x and y to obtain equations (2.2) and (2.3). Therefore if (x, y, z) is a solution of (2.2) and (2.3) when $\delta = 1, 2, 4$, then (y, x, z) is a solution of (2.2) and (2.3) when $\delta = -1, -2, -4$, and vice versa.

We consider the following result of Yuan [22].

LEMMA 2.1. *Let $x_1\sqrt{a}+y_1\sqrt{b}$ be the smallest solution of $ax^2 - by^2 = \delta$, with $\delta \in \{1, 2, 4\}$. Then every positive integer solution (x, y) of this equation can be given by*

$$\frac{x\sqrt{a}+y\sqrt{b}}{\sqrt{\delta}} = \left(\frac{x_1\sqrt{a}+y_1\sqrt{b}}{\sqrt{\delta}}\right)^n, \qquad n > 0, \tag{2.4}$$

with $2\nmid n$ if $\min\{a, b\} > 1$ or if $(a, \delta) \neq (1, 1), (1, 4)$.

We put $N = 4n/c$. Then equation (2.2) becomes

$$(N + 4)x^2 - Ny^2 = 4. \tag{2.5}$$

Let us consider

$$\alpha = \frac{\sqrt{N+4}+\sqrt{N}}{2}, \qquad \overline{\alpha} = \frac{\sqrt{N+4}-\sqrt{N}}{2}.$$

By Lemma 2.1, every positive integer solution (x, y) of (2.2) or (2.5) can be represented as

$$\frac{x\sqrt{N+4}+y\sqrt{N}}{2} = \alpha^n, \qquad n > 0,\ 2\nmid n. \tag{2.6}$$

Moreover, let $\beta = \alpha^2$. Then β is the smallest solution of the equation

$$\tau^2x^2 - N(N + 4)y^2 = 4, \tag{2.7}$$

where $\tau = 1$ or 2 when N is odd or even respectively.

Now let $\gamma = v_1 + u_1\sqrt{b}$ be the smallest solution of the equation $v^2 - bu^2 = 1$. For integers $j, k, l \geq 1$ with $2\nmid j$, we define the sequences $\{T_j\}$, $\{W_j\}$, $\{V_k\}$, $\{U_k\}$, $\{v_l\}$ and $\{u_l\}$ by

$$\alpha^j = \frac{T_j\sqrt{N+4}+W_j\sqrt{N}}{2}, \tag{2.8}$$

$$\beta^k = \frac{V_k + U_k\sqrt{N(N+4)}}{2}, \tag{2.9}$$

$$\gamma^l = v_l + u_l\sqrt{b}. \tag{2.10}$$

Notice that $(V, U) = (x, y)$ if $2\nmid N$, and $(V, U) = (2x, y)$ if $2 \mid N$.

The following lemma lists some properties of Lehmer sequences. They are true not only for $(\{V_k\}, \{U_k\})$, but also for $(\{T_j\}, \{W_j\})$ and $(\{v_l\}, \{u_l\})$.

LEMMA 2.2. *Let $d = \gcd(m, n)$ for some integers m and n.*

(1) *If $U_m \neq 1$, then $U_m \mid U_n$ if and only if $m \mid n$.*
(2) *If $m > 1$, then $V_m \mid V_n$ if and only if n/m is an odd integer.*

(3) $\gcd(U_m, U_n) = U_d$.
(4) $\gcd(V_m, V_n) = V_d$ *if* m/d *and* n/d *are odd, and* 1 *otherwise.*
(5) $\gcd(U_m, V_n) = V_d$ *if* m/d *is even, and* 1 *otherwise.*

Proof. See Lemma 2.1 of [23], [20], or Lemma 2.2 of [13]. ∎

One can see that if we extend the above sequences to negative indices, the definition is still effective. In fact, we have

$$T_{-n} = T_n,\ W_{-n} = -W_n,\ V_{-n} = V_n,\ U_{-n} = -U_n,\ v_{-n} = v_n,\ u_{-n} = -u_n.$$

LEMMA 2.3. *Let* $k_0, k_1, k_2, p \in \mathbb{Z}$ *with* $k_1, k_2, p > 0$. *If* k_i $(i = 0, 1, 2)$ *are all odd and* $k_2 = 2pk_1 + k_0$, *then*

(i) $T_{2pk_1+k_0} \equiv (-1)^p T_{k_0} \pmod{T_{k_1}}$;
(ii) $W_{2pk_1+k_0} \equiv W_{k_0} \pmod{W_{k_1}}$.

Proof. (i) If $2 \nmid p$, then

$$\begin{aligned} T_{2pk_1+k_0} + T_{k_0} &= \frac{\alpha^{2pk_1+k_0} + \overline{\alpha}^{2pk_1+k_0} + \alpha^{k_0} + \overline{\alpha}^{k_0}}{\sqrt{N+4}} \\ &= \frac{(\alpha^{pk_1+k_0} + \overline{\alpha}^{pk_1+k_0})(\alpha^{pk_1} + \overline{\alpha}^{pk_1})}{\sqrt{N+4}} = V_{(pk_1+k_0)/2} T_{pk_1}. \end{aligned}$$

Therefore $T_{2pk_1+k_0} \equiv -T_{k_0} \pmod{T_{k_1}}$.

If $2 \mid p$, we have

$$\begin{aligned} T_{2pk_1+k_0} - T_{k_0} &= \frac{\alpha^{2pk_1+k_0} + \overline{\alpha}^{2pk_1+k_0} - \alpha^{k_0} - \overline{\alpha}^{k_0}}{\sqrt{N+4}} \\ &= \frac{(\alpha^{pk_1+k_0} - \overline{\alpha}^{pk_1+k_0})(\alpha^{pk_1} - \overline{\alpha}^{pk_1})}{\sqrt{N+4}} = N W_{pk_1+k_0} U_{pk_1/2}. \end{aligned}$$

From Lemma 2.2(1), we have $U_{k_1} \mid U_{pk_1/2}$ and by (2.8) and (2.9), we get $U_{k_1} = T_{k_1} W_{k_1}$. Thus $T_{2pk_1+k_0} \equiv T_{k_0} \pmod{T_{k_1}}$.

(ii) The proof is similar to that of (i). We have

$$\begin{aligned} W_{2pk_1+k_0} - W_{k_0} &= \frac{\alpha^{2pk_1+k_0} - \overline{\alpha}^{2pk_1+k_0} - \alpha^{k_0} + \overline{\alpha}^{k_0}}{\sqrt{N}} \\ &= \frac{(\alpha^{pk_1+k_0} + \overline{\alpha}^{pk_1+k_0})(\alpha^{pk_1} - \overline{\alpha}^{pk_1})}{\sqrt{N}}. \end{aligned}$$

If $2 \nmid p$, then $W_{2pk_1+k_0} - W_{k_0} = V_{(pk_1+k_0)/2} W_{pk_1}$. As $W_{k_1} \mid W_{pk_1}$, we have $W_{2pk_1+k_0} \equiv W_{k_0} \pmod{W_{k_1}}$. If $2 \mid p$, then

$$W_{2pk_1+k_0} - W_{k_0} = (N+4) T_{pk_1+k_0} U_{pk_1/2}.$$

Therefore, we get the same result. ∎

Now we assume that positive integer solutions of (2.2) and (2.3) exist. Let (x_1, y_1, z_1) be the positive solution with the smallest z_1, and (x_2, y_2, z_2)

be any other solution. Then there exist positive integers j_i, l_i $(i = 1, 2)$ with $2 \nmid j_i$ such that

$$x_i = T_{j_i}, \quad y_i = W_{j_i}, \quad y_i \text{ or } x_i = v_{l_i}, \quad z_i = u_{l_i}. \tag{2.11}$$

The following result is similar to Lemma 2.4 of [23] and Lemma 2.4 of [13]. In [22], Yuan proved that for positive integers k_0, k_1, k_2, p, we have $v_{2pk_1 \pm k_0} \equiv \pm v_{k_0} \pmod{v_{k_1}}$. We use it and Lemma 2.3 to get

Lemma 2.4. *In the notations of* (2.11), *we have*

$$y_1 \mid y_2, \quad j_1 \mid j_2, \quad \text{and} \quad l_1 \mid l_2.$$

Furthermore, j_2/j_1 *and* l_2/l_1 *are odd integers. This implies* $x_1 \mid x_2$ *and* $z_1 \mid z_2$.

Define

$$R_{2k+1}^{(\lambda)} = \begin{cases} T_{2k+1} & \text{if } \lambda = 1, \\ W_{2k+1} & \text{if } \lambda = -1. \end{cases} \tag{2.12}$$

Then from the definition of α and (2.8) we obtain

$$R_{2k+1}^{(\lambda)} = \frac{\alpha^{2k+1} + \lambda \overline{\alpha}^{2k+1}}{\sqrt{N + 2 + 2\lambda}}. \tag{2.13}$$

Lemma 2.5. *We have* $(R_{2k+1}^{(\lambda)})^2 - 1 = (N + 2 - 2\lambda) U_k U_{k+1}$.

Proof. Since $\alpha + \lambda\overline{\alpha} = \sqrt{N + 2 + 2\lambda}$, we get

$$\begin{aligned}
(R_{2k+1}^{(\lambda)})^2 - 1 &= \left(\frac{\alpha^{2k+1} + \lambda\overline{\alpha}^{2k+1}}{\sqrt{N+2+2\lambda}}\right)^2 - 1 = \frac{\alpha^{4k+2} + \overline{\alpha}^{4k+2} + 2\lambda}{N+2+2\lambda} - 1 \\
&= \frac{\alpha^{4k+2} + \overline{\alpha}^{4k+2} - (N+2)}{N+2+2\lambda} = \frac{\beta^{2k+1} + \overline{\beta}^{2k+1} - (\beta + \overline{\beta})}{N+2+2\lambda} \\
&= \frac{(\beta^{k+1} - \overline{\beta}^{k+1})(\beta^k - \overline{\beta}^k)}{N+2+2\lambda} \\
&= \frac{N(N+4)}{N+2+2\lambda} \cdot \frac{\beta^k - \overline{\beta}^k}{\sqrt{N(N+4)}} \cdot \frac{\beta^{k+1} - \overline{\beta}^{k+1}}{\sqrt{N(N+4)}} \\
&= (N + 2 - 2\lambda) U_k U_{k+1}. \ \blacksquare
\end{aligned}$$

Definition 2.6. Let $\{U_k\}$ be defined by (2.9). If there is a prime factor p of U_k that does not divide U_j for all $1 \le j \le k - 1$, then we say that p is a *primitive prime factor* of U_k.

Notice that there are two (slightly) different definitions of primitive prime factor. According to the definition in [5], p should not divide $N(N + 4)$ and U_j for all $1 \le j \le k - 1$. This was used in [23] and [13]. But the above definition is enough for our proof.

Lemma 2.7. *For $k > 1$, U_k has a primitive prime factor p except for $\beta = (1+\sqrt{5})/2$ and $k = 6$. Moreover, $p \mid U_{k'}$ if and only if $k \mid k'$.*

Proof. See Lemma 2.4 of [20]. ■

The following result is an adaptation of Lemma 2.5 of [20]. One can get it from some results on $AX^2 - By^4 = 1, 4$ due to Ljunggren [15], Cohn [8], [9] and the first author [18].

Lemma 2.8. *Let $\tau = 1$ if $2 \nmid N$, and $\tau = 2$ if $2 \mid N$. Then for any positive integer A, there is at most one positive solution (x, y) to the equation*

$$\tau^2 x^2 - N(N+4)y^2 = 4$$

with $y = Au^2$ for some integer u, except in the following cases:

(1) $N = 1$, $A = 1$, *in which case* $y \in \{1, 12^2\}$.
(2) $N = 336$, $A = 1$, *in which case* $y \in \{1, 6214^2\}$.
(3) $N = d^2 - 2$, $A = 1$, *in which case* $y \in \{1, d^2\}$.

Proof. Take $M = N + 2$, $X = \tau x$, and $Y = y$ in Lemma 2.5 of [20]. Moreover, if N is even, one can take $N = 2M - 2$, and if N is odd, $N = M - 2$. ■

Next, we recall the following result due to Ljunggren [16].

Lemma 2.9. *The Diophantine equation*

$$x^4 - py^2 = 1,$$

where p denotes any odd prime, has no solutions in positive integers x and y if $p \neq 5$ and $p \neq 29$. When $p = 5$ or $p = 29$ there is only one solution, i.e. $(x, y) = (3, 4)$ and $(x, y) = (99, 1820)$ respectively.

3. Proof of Theorem 1.1. In this section, we will prove the main theorem of this paper. We assume that (x_1, y_1, z_1) is the positive solution with the smallest positive z_1, and (x_2, y_2, z_2) is any other positive solution of equations (2.2) and (2.3). Then there exist positive integers j_i, l_i $(i = 1, 2)$ with $2 \nmid j_i$ such that

$$x_i = T_{j_i}, \quad y_i = W_{j_i}, \quad y_i \text{ or } x_i = v_{l_i}, \quad z_i = u_{l_i}. \tag{3.1}$$

We notice that $j_1 > 1$, otherwise $T_1 = W_1 = 1$. This implies $l_1 = 1$, $v_{l_1} = 1$ and $z = 0$. Let $j_i = 2k_i + 1$ $(i = 1, 2)$ with $0 < k_1 < k_2$. From (2.3) and (3.1), we have $T_{2k_i+1}^2 - 1 = bz_i^2$ or $W_{2k_i+1}^2 - 1 = bz_i^2$. Using (2.12) and Lemma 2.5, we get

$$bz_1^2 = \left(R_{2k_1+1}^{(\lambda)}\right)^2 - 1 = (N + 2 - 2\lambda)U_{k_1}U_{k_1+1}, \tag{3.2}$$

$$bz_2^2 = \left(R_{2k_2+1}^{(\lambda)}\right)^2 - 1 = (N + 2 - 2\lambda)U_{k_2}U_{k_2+1}. \tag{3.3}$$

From Lemma 2.4, we have $z_1 \mid z_2$, so (3.2) and (3.3) give

$$\frac{U_{k_2}U_{k_2+1}}{U_{k_1}U_{k_1+1}} = \left(\frac{z_2}{z_1}\right)^2. \tag{3.4}$$

Before discussing the above equation, let us express U_k $(1 \le k \le 6)$ using the recurrence relation $U_{k+2} = (N+2)U_{k+1} - U_k$ for $k \ge 1$:

$$\begin{aligned}
U_1 &= 1,\\
U_2 &= N+2,\\
U_3 &= N^2+4N+3,\\
U_4 &= N^3+6N^2+10N+4,\\
U_5 &= N^4+8N^3+21N^2+20N+5,\\
U_6 &= N^5+10N^4+36N^3+56N^2+35N+6.
\end{aligned}$$

First, we assume that $N=1$, $k_1 = 5$ or 6. If $k_1 = 5$, then $U_{k_1+1} = 144 = 2^4 \cdot 3^2$. By Lemma 2.7, U_{k_1} has a primitive prime factor p, so that $U_{k_1} \mid U_{k_2}$ or $U_{k_1} \mid U_{k_2+1}$. If $U_{k_1} \mid U_{k_2}$, since $\gcd(U_{k_2}, U_{k_2+1}) = 1$, equation (3.4) implies the existence of positive integers s and t such that $U_{k_2}/(144U_{k_1}) = s^2$, $U_{k_2+1} = t^2$ or $U_{k_2}/(16U_{k_1}) = s^2$, $U_{k_2+1}/9 = t^2$ or $U_{k_2}/(9U_{k_1}) = s^2$, $U_{k_2+1}/16 = t^2$ or $U_{k_2}/U_{k_1} = s^2$, $U_{k_2+1}/144 = t^2$. The above cases give us $U_{k_2+1} = \square$. Using Lemma 2.8 and $U_1 = 1$, one can see that $U_{k_2+1} = 144$ and $k_2 = 5$. This contradicts the fact that $k_1 < k_2$. If $U_{k_1} \mid U_{k_2+1}$, then $k_2 = 6$. This is impossible. In the same way, if $k_1 = 6$, we also get a contradiction.

Assume now $N > 1$ or $N = 1$, $k_1 \ne 5, 6$. If $k_1 > 1$, by Lemma 2.7, U_{k_1} and U_{k_1+1} have primitive prime factors p and q respectively. By Lemma 2.7 again, equation (3.4) implies that

$$(k_1 \mid k_2 \text{ or } k_1 \mid k_2+1) \quad \text{and} \quad (k_1+1 \mid k_2 \text{ or } k_2 \mid k_2+1). \tag{3.5}$$

If $k_1 = 1$, then U_2 has primitive prime factor q, and properties (3.5) also hold. Moreover, since $j_1 \mid j_2$, we have

$$2k_1+1 \mid 2k_2+1. \tag{3.6}$$

Note that $\gcd(U_{k_2}, U_{k_2+1}) = 1$ by Lemma 2.2(3). Then properties (3.4) and (3.5) give us the following four cases:

$$\text{(i) } U_{k_2+1}/(U_{k_1}U_{k_1+1}) = s^2, \quad U_{k_2} = t^2, \tag{3.7}$$

$$\text{(ii) } U_{k_2}/(U_{k_1}U_{k_1+1}) = s^2, \quad U_{k_2+1} = t^2, \tag{3.8}$$

$$\text{(iii) } U_{k_2}/U_{k_1+1} = s^2, \quad U_{k_2+1}/U_{k_1} = t^2, \tag{3.9}$$

$$\text{(iv) } U_{k_2}/U_{k_1} = s^2, \quad U_{k_2+1}/U_{k_1+1} = t^2. \tag{3.10}$$

Case (i). Since $(V,U) = (N+2, 1)$ is a solution of $V^2 - N(N+4)U^2 = 4$, using equations (3.7) one can see that the equations

$$(3.11) \qquad \tau^2 x^2 - N(N+4)y^2 = 4, \quad y = u^2$$

have two solutions $u = 1$ and $u = t > 1$. By Lemma 2.8, we obtain $N = 1$, 336, or $d^2 - 2$.

• If $N = 1$, then we get $t = 12$. Therefore, equations (3.11) imply $x^2 - 5y^2 = 4$. Any solution (x, y) is given by

$$\frac{x + y\sqrt{5}}{2} = \left(\frac{3+\sqrt{5}}{2}\right)^k.$$

The solution with $y = 144$ implies $k_2 = 6$. On the other hand, the first equation of (3.7) gives us $k_1(k_1+1) \mid k_2 + 1 = 7$. This is impossible.

• If $N = 336$, then $t = 6214$. From $U_4 = N^3 + 6N^2 + 10N + 4 = 6214^2$, we obtain $k_2 = 4$. Since $k_1(k_1+1) \mid k_2 + 1 = 5$, we can find no positive integer k_1.

• If $N = d^2 - 2$, then $t = d$. From $U_2 = N + 2 = t^2$, we get $k_2 = 2$. But there is no positive integer k_1 satisfying $k_1(k_1+1) \mid k_2 + 1 = 3$.

Case (ii). This is similar to Case (i). By Lemma 2.8, the second equation of (3.8) implies $N = 1$, 336, or $d^2 - 2$.

• If $N = 1$, then $k_2 + 1 = 6$. We have already discussed this case and it is impossible.

• If $N = 336$, then $k_2 + 1 = 4$. But $k_1(k_1+1) \mid k_2 = 3$ is also impossible.

• If $N = d^2 - 2$, then $k_2 + 1 = 2$. But there is no integer k_1 such that $0 < k_1 < k_2$.

Case (iii). From (3.9), we have

$$(3.12) \qquad U_{k_1+1} = As_1^2, \quad U_{k_2} = As_2^2, \quad U_{k_1} = Bt_1^2, \quad U_{k_2+1} = Bt_2^2$$

for some positive integers A, B, s_1, s_2, t_1, t_2 such that $s = s_2/s_1$ and $t = t_2/t_1$. If $k_1+1 = k_2$, from (3.6) we get $2k_1+1 \mid 2k_1+3$, which is impossible. Therefore we consider $k_1 + 1 < k_2$. Thus $U_{k_1} < U_{k_1+1} < U_{k_2} < U_{k_2+1}$. But from Lemma 2.8, $A = B = 1$ and $N = 1$, 336 or $d^2 - 2$. Then U_{k_1}, U_{k_1+1}, U_{k_2}, and U_{k_2+1} are all perfect squares. This leads to a contradiction.

Case (iv). From (3.10), as in Case (iii), we have

$$(3.13) \qquad U_{k_1} = As_1^2, \quad U_{k_2} = As_2^2, \quad U_{k_1+1} = Bt_1^2, \quad U_{k_2+1} = Bt_2^2.$$

Since $k_1 < k_2$, from Lemma 2.8 we have $A = B = 1$ and $N = 1$, 336, or $d^2 - 2$. We get a contradiction as before. This completes the proof of Theorem 1.1.

4. A particular case.

Now we consider equations (1.4) with

$$(4.1) \qquad b = b'|4m/\delta + 4|, \quad b' \in \{1, 2, p\}.$$

Then we have the following result.

Proposition 4.1. *If equations* (1.4) *have a solution* (x, y, z) *with the condition* (4.1), *then* $U_k U_{k+1} = b'z^2$ *when either*

$$N = b'd^2 - 2, \quad b' \in \{1, 2, p\}, \quad k = 1, \quad z = d;$$

or

$$N = 7, \quad b' = 5, \quad k = 2, \quad z = 12;$$

or

$$N = 9799, \quad b' = 29, \quad k = 2, \quad z = 180180.$$

Proof. Suppose a positive integer solution (x, y, z) of (1.4) exists. Then there are positive integers j, l with $2 \nmid j$ such that

$$x = T_j, \quad y = W_j, \quad y \text{ or } x = v_l, \quad z = u_l. \tag{4.2}$$

If $j = 1$ then $z = 0$. Let $j = 2k+1$ for $k > 0$. From (2.3) and (4.2) we obtain $T_{2k+1}^2 - 1 = bz^2$ or $W_{2k+1}^2 - 1 = bz^2$. Using (2.12) and Lemma 2.5, we get $bz^2 = (R_{2k+1}^{(\lambda)})^2 - 1 = (N + 2 - 2\lambda)U_k U_{k+1}$. Thus

$$(N + 2 - 2\lambda)U_k U_{k+1} = b'|4m/\delta + 4|z^2, \quad b' \in \{1, 2, p\}. \tag{4.3}$$

We recall that $N = 4n/c$, $n = \min\{m, m + \delta\}$ and $c = |\delta|$.

If $\delta \in \{1, 2, 4\}$, then equations (1.4) give us the first equation in (2.3) and $n = m$, $c = \delta$. Thus we need to consider $W_{2k+1}^2 - 1 = bz^2$. By the definition of $R_{2k+1}^{(\lambda)}$ in (2.12), we have $\lambda = -1$. Therefore one can see that

$$N + 2 - 2\lambda = 4m/c + 4 = |4m/\delta + 4|.$$

If $\delta \in \{-1, -2, -4\}$, then equations (1.4) give us the second equation in (2.3) and $n = m - c$, $c = -\delta$. Thus we need to consider $T_{2k+1}^2 - 1 = bz^2$ and $\lambda = 1$. Therefore we also obtain

$$N + 2 - 2\lambda = 4(m - c)/c = 4m/c - 4 = |4m/\delta + 4|.$$

Then equation (4.3) implies

$$U_k U_{k+1} = b'z^2, \quad b' \in \{1, 2, p\}. \tag{4.4}$$

By Lemma 2.2(3), we have $\gcd(U_k, U_{k+1}) = 1$. So we obtain either

$$U_k = s^2, \qquad U_{k+1} = b't^2, \tag{4.5}$$

or

$$U_k = b't^2, \quad U_{k+1} = s^2, \tag{4.6}$$

where $z = st$, $s, t \in \mathbb{N}$.

If equations (4.5) hold, then from Lemma 2.8 one can see that $U_k = s^2$ implies $k = 1$, except for $N = 1$, 336, or $d^2 - 2$. First, we suppose $k = 1$. Then from the second equation of (4.5) we have $U_2 = N + 2 = b't^2$. Thus $N = b't^2 - 2$ with $b' \in \{1, 2, p\}$. Second, we suppose $k > 1$ and we discuss the following three cases.

• If $N = 1$, then U_k is a perfect square when $k = 6$. But $b't^2 = U_{k+1} = U_7 = (N+2)U_6 - U_5 = 377 = 13 \cdot 29$ is impossible.

• If $N = 336$, then $k = 4$. The fact that $b't^2 = U_{k+1} = U_5 = N^4 + 8N^3 + 21N^2 + 20N + 5 = 13051348805 = 5 \cdot 11 \cdot 19 \cdot 109 \cdot 149 \cdot 769$ also leads to a contradiction.

• If $N = d^2 - 2$, then $k = 2$ and $s = d$. Therefore, from $b't^2 = U_{k+1} = U_3 = N^2 + 4N + 3 = (N+2)^2 - 1$, we have

$$d^4 - b't^2 = 1, \quad b' \in \{1, 2, p\}. \tag{4.7}$$

It is easy to see that (4.7) has no positive integer solution when $b' = 1$. If $b' = 2$, then (4.7) implies $d = 1$, $t = 0$, which is impossible. If $b' = p$, then by Lemma 2.9, equation (4.7) has a positive integer solution if and only if either $b' = 5$, $(d,t) = (3,4)$, or $b' = 29$, $(d,t) = (99, 1820)$. Since $z = st$, we get $z = 12$ or 180180 respectively.

Now we suppose equations (4.6) hold. In a similar way, $U_{k+1} = s^2$ implies $k = 0$, except for $N = 1$, 336, or $d^2 - 2$. But $k = 0$ leads to a contradiction. Now we discuss the following three cases when $k > 0$.

• If $N = 1$, then $k + 1 = 6$. But $b't^2 = U_k = U_5 = N^4 + 8N^3 + 21N^2 + 20N + 5 = 55 = 5 \cdot 11$ gives a contradiction.

• If $N = 336$, then $k + 1 = 4$. Thus $b't^2 = U_k = U_3 = N^2 + 4N + 3 = 114243 = 3 \cdot 113 \cdot 337$ is impossible.

• If $N = d^2 - 2$, then $k + 1 = 2$. Then from $b't^2 = U_1 = 1$, we get $b' = 1$ and $t = 1$. This is also impossible. ∎

Finally, we use Proposition 4.1 to prove the following result which is a particular case of Theorem 1.1.

THEOREM 4.2. *If p is an odd prime and $b = b'|4m/\delta + 4|$, $b' \in \{1, 2, p\}$, then the system of simultaneous equations* (1.4) *has no positive integer solution (x, y, z), except in the following cases.*

(1) *If $(\delta, b') \neq (\pm 1, 1), (\pm 1, p)$, then there is a positive integer d such that*

$$m = \begin{cases} \delta(b'd^2 - 2)/4 & \text{if } \delta > 0, \\ -\delta(b'd^2 + 2)/4 & \text{if } \delta < 0, \end{cases}$$

and equations (1.4) *have one solution*

$$(x, y, z) = (|4m/\delta + 1|, |4m/\delta + 3|, d).$$

(2) *If $(m, \delta, b) = (7, 4, 55)$, then the solution is $(x, y, z) = (71, 89, 12)$; if $(m, \delta, b) = (11, -4, 35)$, then the solution is $(x, y, z) = (89, 71, 12)$.*

(3) *If $(m, \delta, b) = (9799, 4, 29 \cdot 9803)$, then the solution is*

$$(x, y, z) = (96049799, 96069401, 180180);$$

if $(m, \delta, b) = (9803, -4, 29 \cdot 9799)$, *then the solution is*

$$(x, y, z) = (96069401, 96049799, 180180).$$

Proof. Suppose that there exists a positive integer solution (x, y, z) of equations (1.4) with the condition (4.1). From Proposition 4.1, we have $U_k U_{k+1} = b' z^2$, $b' \in \{1, 2, p\}$ and $N = 4n/|\delta| = b'd^2 - 2$, 7 or 9799, where $n = \min\{m, m + \delta\}$.

First, let $N = b'd^2 - 2$. If $\delta > 0$, then $n = m$, thus $m = \delta(b'd^2 - 2)/4$. We have $k = 1$ and $z = d$ by Proposition 4.1. From $y^2 = bz^2 + 1$ we obtain

$$y^2 = b'|4m/\delta + 4|z^2 + 1 = b'(4m/\delta + 4)z^2 + 1$$
$$= b'(b'd^2 + 2)d^2 + 1 = (b'd^2 + 1)^2.$$

Thus we have $y = b'd^2 + 1 = 4m/\delta + 3$. Consequently, we get the solution $(x, y, z) = (4m/\delta + 1, 4m/\delta + 3, d)$.

If $\delta < 0$, then $n = m + \delta$, thus $m = -\delta(b'd^2 - 2)/4 - \delta = -\delta(b'd^2 + 2)/4$. In a similar way, knowing that $\delta + m \geq 1$ we have

$$y^2 = b'|4m/\delta + 4|z^2 + 1 = b'(4m/(-\delta) - 4)z^2 + 1$$
$$= b'(b'd^2 - 2)d^2 + 1 = (b'd^2 - 1)^2.$$

It follows that $y = b'd^2 - 1 = 4m/(-\delta) - 3$, and we get the solution $(x, y, z) = (4m/(-\delta) - 1, 4m/(-\delta) - 3, d)$. This proves the first exceptional case.

Finally, let $N = 7$ or 9799. Since $N = 4n/|\delta|$, we have $|\delta| = 4$. Noticing $k = 2$, by direct computations, it is easy to get the second and third exceptional cases. This completes the proof of Theorem 4.2. ■

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References

[1] A. Baker and H. Davenport, *The equations* $3x^2 - 2 = y^2$ *and* $8x^2 - 7 = z^2$, Quart. J. Math. Oxford Ser. (2) 20 (1969), 129–137.

[2] M. A. Bennett, *On the number of solutions of simultaneous Pell equations*, J. Reine Angew. Math. 498 (1998), 173–199.

[3] M. A. Bennett, M. Cipu, M. Mignotte and R. Okazaki, *On the number of solutions of simultaneous Pell equations II*, Acta Arith. 122 (2006), 407–417.

[4] M. A. Bennett and P. G. Walsh, *Simultaneous quadratic equations with few or no solutions*, Indag. Math. 11 (2000), 1–12.

[5] Yu. Bilu, G. Hanrot and P. M. Voutier (with an appendix by M. Mignotte), *Existence of primitive divisors of Lucas and Lehmer numbers*, J. Reine Angew. Math. 539 (2001), 75–122.

[6] M. Cipu, *Pairs of Pell equations having at most one common solution in positive integers*, An. Ştiinţ. Univ. "Ovidius" Constanţa 15 (2007), 55–66.

[7] M. Cipu and M. Mignotte, *On the number of solutions of simultaneous hyperbolic Diophantine equations*, J. Number Theory 125 (2007), 356–392.

[8] J. H. E. Cohn, *Eight Diophantine equations*, Proc. London Math. Soc. 16 (1966), 153–166.

[9] —, *Five Diophantine equations*, Math. Scand. 21 (1967), 61–70.

[10] X. L. Dong, W. C. Shiu, C. I. Chu and Z. F. Cao, *The simultaneous Pell equations* $y^2 - Dz^2 = 1$ *and* $x^2 - 2Dz^2 = 1$, Acta Arith. 126 (2007), 115–123.

[11] B. He, *On the number of solutions of simultaneous Pell equations* $x^2 - ay^2 = 1$ *and* $y^2 - bz^2 = 1$, Acta Math. Sinica 51 (2008), 721–726 (in Chinese).

[12] S.-I. Katayama and C. Levesque, *On simultaneous diophantine equations*, Acta Arith. 108 (2003), 369–377.

[13] Z. G. Li, J. Y. Xia and P. Z. Yuan, *On some special forms of simultaneous Pell equations*, ibid. 128 (2007), 55–67.

[14] Z. G. Li and P. Z. Yuan, *On the number of solutions of some special simultaneous Pell equations*, Acta Math. Sinica 50 (2007) 1349–1356 (in Chinese).

[15] W. Ljunggren, *Einige Eigenschaften der Einheiten reeller quadratischer und reinbiquadratischer Zahlkörper mit Anwendung auf die Lösung einer Klasse unbestimmter Gleichungen vierten Grades*, Oslo Vid.-Akad. Skrifter 1936, no. 12, 1–73.

[16] —, *Some remarks on the diophantine equations* $x^2 - dy^4 = 1$ *and* $x^4 - dy^2 = 1$, J. London Math. Soc. 41 (1966), 542–544.

[17] K. Ono, *Euler's concordant forms*, Acta Arith. 78 (1996), 101–123.

[18] A. Togbé, P. M. Voutier and P. G. Walsh, *Solving a family of Thue equations with an application to the equation* $x^2 - Dy^4 = 1$, ibid. 120 (2005), 39–58.

[19] P. G. Walsh, *On integer solutions to* $x^2 - dy^2 = 1$, $z^2 - 2dy^2 = 1$, ibid. 82 (1997), 69–76.

[20] —, *Sharp bounds for the number of solutions to simultaneous Pellian equations*, ibid. 126 (2007), 125–137.

[21] P. Z. Yuan, *On the number of solutions of simultaneous Pell equations*, ibid. 101 (2002), 215–221.

[22] —, *Simultaneous Pell equations*, ibid. 115 (2004), 119–131.

[23] —, *On the number of solutions of* $x^2 - 4m(m+1)y^2 = y^2 - bz^2 = 1$, Proc. Amer. Math. Soc. 132 (2004), 1561–1566.

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