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A note on a multiplicative hybrid problem

by

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1. Introduction and result. In what follows, $e(x) = e^{2\pi i x}$, $[x]$ is the integer part of x, $\psi(x) = x - [x] - 1/2$ and N is a natural number large enough.

In 1987, Iwaniec and Sárközy [5] dealt with the following problem: Let S_1 and S_2 be subsets of $]N, 2N] \cap \mathbb{Z}$. If $|S_1| \gg N$ and $|S_2| \gg N$, then they proved that there exist integers $n_1 \in S_1$, $n_2 \in S_2$ and b such that

$$n_1 n_2 = b^2 + O((b \log b)^{1/2}).$$

The following generalization was considered by Zhai ([9, 10]): Let $k \geq 4$ be an integer and $S_1, \ldots, S_k$ be subsets of $]N, 2N] \cap \mathbb{Z}$. If $|S_i| \gg N$ for $i = 1, \ldots, k$, then there exist integers $n_1 \in S_1, \ldots, n_k \in S_k$ and b such that

$$n_1 \cdots n_k = b^k + O(b^{k-3/2}). \tag{1}$$

That result can easily be related to the following multi-dimensional lattice point problem. Let $0 < \delta \leq 1/4$ be any small real number and define

$$\mathcal{R}_k = \mathcal{R}_k(N, \delta) := \Big|\Big\{(n_1, \ldots, n_k, b) \in \prod_{i=1}^{k} S_i \times \mathbb{Z} : |(n_1 \cdots n_k)^{1/k} - b| \leq \delta\Big\}\Big|$$

and suppose there exist $\beta_k \geq 0$ and $0 \leq \theta_k < k$ such that

$$\mathcal{R}_k = 2\delta |S_1| \cdots |S_k| + O(N^{\theta_k} (\log N)^{\beta_k}). \tag{2}$$

Then using $|S_i| \geq a_i N$ (with $a_i > 0$) and setting $A_k := \min_{1 \leq i \leq k} a_i$, we have

$$\mathcal{R}_k \geq 2\delta |S_1| \cdots |S_k| - c_k N^{\theta_k} (\log N)^{\beta_k} \geq 2\delta A_k^k N^k - c_k N^{\theta_k} (\log N)^{\beta_k}$$

with $c_k > 0$ depending only on k. Now taking $\delta = c_0 N^{\theta_k - k} (\log N)^{\beta_k}$ with $c_0 > 2^{-1} c_k A_k^{-k}$ gives $\mathcal{R}_k > 0$, which implies that, if N is sufficiently large, then there exist integers $n_1 \in S_1, \ldots, n_k \in S_k$ and b such that

$$n_1 \cdots n_k = b^k + O(b^{\theta_k - 1} (\log b)^{\beta_k}). \tag{3}$$

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If δ is sufficiently small, it is easy to see that $\mathcal{R}_k$ counts the number of integer points close to the hypersurface $x_{k+1} = (x_1 \cdots x_k)^{1/k}$ with $x_i \in S_i$ ($i = 1, \ldots, k$). In the one-dimensional case, upper bounds of such numbers can be obtained by using results dealing with divided differences (see [2, 4]). In the general case, estimate (2) can be attained by using exponential sums methods. In his work [9, 10], Zhai used a double large sieve inequality for bilinear forms first established by Bombieri and Iwaniec (see [1, 3, 7]). In this paper, we treat the resulting sums coming from the error term of (2) by making use of multi-dimensional exponent pairs introduced by Srinivasan (see [8, 6]). This leads to the following improvement of (1):

THEOREM 1.1. *Let $k \geq 2$ be an integer, N be a large natural number, and $S_1, \ldots, S_k$ be subsets of $]N, 2N] \cap \mathbb{Z}$. If $|S_1| \gg N, \ldots, |S_k| \gg N$, then there exist integers $n_1 \in S_1, \ldots, n_k \in S_k$ and b such that*

$$n_1 \cdots n_k = b^k + O(b^{k-5/3+r(k)})$$

where $r(k) = 2(9k+7)/(3(9k^2 - 3k + 10))$.

Although this result is valid for $k \geq 2$, it only improves on (1) as soon as $k \geq 5$.

2. Proof of Theorem 1.1. Clearly we have

$$\mathcal{R}_k = 2\delta|S_1| \cdots |S_k| + \sum_{(n_1,\ldots,n_k)\in S_1\times\cdots\times S_k} \{\psi((n_1 \cdots n_k)^{1/k} - \delta) - \psi((n_1 \cdots n_k)^{1/k} + \delta)\}.$$

The following result will be useful:

LEMMA 2.1. *Let $d, N \geq 1$ be integers, $\mathcal{D}_d \subset (]N, 2N] \cap \mathbb{Z})^d$, $X \geq 1$, and let $\alpha_1, \ldots, \alpha_d$ be nonzero real numbers satisfying*

$$u \sum_{i=1}^{d} \alpha_i + \sum_{i=1}^{d} \alpha_i \varepsilon_i \neq 1 + u + v$$

for any pair (u, v) of nonnegative integers and any $(\varepsilon_1, \ldots, \varepsilon_d) \in \{0, 1\}^d$. Let $\Delta \in \mathbb{R}$, $s_d = \alpha_1 + \cdots + \alpha_d$ and (l_0, l_1) be an exponent pair of dimension d. Suppose that

$$N^{l_1 - l_0(s_d - 1)} \geq X^{l_0}. \tag{4}$$

Then

$$\sum_{(n_1,\ldots,n_d)\in\mathcal{D}_d} \psi(X n_1^{\alpha_1} \cdots n_d^{\alpha_d} \pm \Delta) \ll (X^{l_0} N^{l_0(s_d+d-1)+1-l_1})^{d/(1+dl_0)}.$$

Proof. The starting point is the well-known inequality

$$-\frac{1}{2H}+\sum_{h\in\mathbb{Z}^*}c_h e(-hx)\le\psi(x)\le\frac{1}{2H}-\sum_{h\in\mathbb{Z}^*}c_h e(hx)$$

where $x\in\mathbb{R}$, H is any positive integer at our disposal and

$$c_h:=\frac{H}{2\pi ih}\int_0^{1/H}e(-ht)\,dt$$

so that

$$|c_h|\le\frac{1}{2\pi}\min\left(\frac{1}{|h|},\frac{H}{h^2}\right).$$

Now summing on $\mathcal{D}_d$ gives

$$\sum_{(n_1,\dots,n_d)\in\mathcal{D}_d}\psi(Xn_1^{\alpha_1}\cdots n_d^{\alpha_d}\pm\Delta)$$
$$\ll\frac{N^d}{H}+\sum_{h=1}^{\infty}\min\left(\frac{1}{h},\frac{H}{h^2}\right)\Big|\sum_{(n_1,\dots,n_d)\in\mathcal{D}_d}e(hXn_1^{\alpha_1}\cdots n_d^{\alpha_d})\Big|$$

and using the exponent pair (l_0,l_1) gives

$$\Big|\sum_{(n_1,\dots,n_d)\in\mathcal{D}_d}e(hXn_1^{\alpha_1}\cdots n_d^{\alpha_d})\Big|\ll\prod_{j=1}^{d}(XhN^{s_d-1})^{l_0}N^{1-l_1}$$
$$\ll(Xh)^{dl_0}N^{d\{l_0(s_d-1)+1-l_1\}}$$

so that

$$\sum_{(n_1,\dots,n_d)\in\mathcal{D}_d}\psi(Xn_1^{\alpha_1}\cdots n_d^{\alpha_d}\pm\Delta)\ll\frac{N^d}{H}+\sum_{h\le H}h^{-1}(Xh)^{dl_0}N^{d\{l_0(s_d-1)+1-l_1\}}$$
$$+H\sum_{h>H}h^{-2}(Xh)^{dl_0}N^{d\{l_0(s_d-1)+1-l_1\}}$$

and since $l_0\le(2d+2)^{-1}$ (see [8, Definition 2]) we have $-2+dl_0\le-3/2$ and hence

$$\sum_{(n_1,\dots,n_d)\in\mathcal{D}_d}\psi(Xn_1^{\alpha_1}\cdots n_d^{\alpha_d}\pm\Delta)\ll\frac{N^d}{H}+(XH)^{dl_0}N^{d\{l_0(s_d-1)+1-l_1\}}.$$

Taking $H=[(X^{-l_0}N^{l_1-l_0(s_d-1)})^{d/(1+dl_0)}]$ gives the desired result. ∎

To produce exponent pairs, one often uses A-B processes as described in [8] to transform a given exponent pair into a new one. For example, Theorem 4 of [8] (see also Theorem 1 of [6]) states that, if (λ_0,λ_1) is an exponent pair of dimension d, then so is

$$(l_0,l_1)=\left(\frac{\lambda_0}{2(1+d\lambda_0)},\frac{\lambda_0+\lambda_1}{2(1+d\lambda_0)}\right).\tag{5}$$

For our purpose, it will be convenient to regard these processes as linear transformations on projective space. To this end, set

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2d & 0 & 2 \end{pmatrix}.$$

Then

$$A \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \lambda_0 + \lambda_1 \\ 2(1 + d\lambda_0) \end{pmatrix}$$

from which we easily derive (5). In a similar way, if we set

$$B = \begin{pmatrix} 0 & -1 & \frac{1}{2} \\ -1 & -1 & 1 \\ 0 & -2d & d+2 \end{pmatrix}$$

then Theorem 6 of [8] (or Theorem 2 of [6]) implies that the pair (l_0, l_1) derived from the transformation

$$B \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ 1 \end{pmatrix}$$

is an exponent pair of dimension d provided $\lambda_1 - \lambda_0 \le 1/(3d)$. Now define

$$\Gamma = BA = \begin{pmatrix} d-1 & -1 & 1 \\ 2(d-1) & -1 & 2 \\ 2d(d+1) & -2d & 2(d+2) \end{pmatrix}.$$

We have the following result:

Lemma 2.2. *Let (λ_0, λ_1) be an exponent pair of dimension d such that*

$$d(3\lambda_1 - 2\lambda_0) \le 2. \tag{6}$$

Then the pair (l_0, l_1) derived from the transformation

$$\Gamma \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ 1 \end{pmatrix}$$

is an exponent pair of dimension d satisfying (6) *with (λ_0, λ_1) replaced by (l_0, l_1).*

Proof. By (5) the pair

$$(\mu_0, \mu_1) = \left(\frac{\lambda_0}{2(1+d\lambda_0)}, \frac{\lambda_0+\lambda_1}{2(1+d\lambda_0)}\right)$$

is an exponent pair of dimension d and condition (6) ensures that $\mu_1 - \mu_0 \le 1/(3d)$, which proves the first part of the lemma by using

$$B\begin{pmatrix}\mu_0\\ \mu_1\\ 1\end{pmatrix} = \Gamma\begin{pmatrix}\lambda_0\\ \lambda_1\\ 1\end{pmatrix}.$$

Furthermore,

$$d(3l_1 - 2l_0) = 2 - \frac{d(8\lambda_0 - 3\lambda_1) + 8}{2\{\lambda_0 d(d+1) - \lambda_1 d + d + 2\}}$$

and using (6) we have

$$\begin{aligned} d(8\lambda_0 - 3\lambda_1) + 8 &\ge -2 + 8 = 6,\\ \lambda_0 d(d+1) - \lambda_1 d + d + 2 &\ge (d+1)(-1 + 3d\lambda_1/2) - \lambda_1 d + d + 2\\ &= \tfrac{1}{2}(3\lambda_1 d^2 + \lambda_1 d + 2) > 0 \end{aligned}$$

so that $d(3l_1 - 2l_0) \le 2$ as asserted. ■

An easy induction gives the following corollary:

COROLLARY 2.3. *For every positive integer h, the pair (l_0, l_1) derived from the transformation*

$$\Gamma^h\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$

is an exponent pair of dimension d. In particular, for the first values of h, the following pairs are exponent pairs of dimension d:

h	(l_0, l_1)
1	$\left(\frac{1}{2d+4}, \frac{1}{d+2}\right)$
2	$\left(\frac{3d+1}{2(3d^2+7d+8)}, \frac{3d+2}{3d^2+7d+8}\right)$

REMARK. The first exponent pair above has already been given by Srinivasan (see [8, Theorem 9]).

Corollary 2.4. *Let* $k \geq 2$ *be an integer and* $\Delta \in \mathbb{R}$. *If* $N \geq 2^{1/k+3/(3k-2)}$ *and* $|S_i| \gg N$ *for* $i = 1, \ldots, k$ *then*

$$\sum_{(n_1,\ldots,n_k)\in S_1\times\cdots\times S_k} \psi((n_1\cdots n_k)^{1/k} \pm \Delta) \ll N^{k-2/3+r(k)}$$

where $r(k)$ *is defined in Theorem* 1.1.

Proof. Write the sum on the left-hand side as

$$\sum_{n_k\in S_k} \sum_{(n_1,\ldots,n_{k-1})\in S_1\times\cdots\times S_{k-1}} \psi(X(n_1\cdots n_{k-1})^{1/k} \pm \Delta)$$

where $X = n_k^{1/k}$ and apply Lemma 2.1 with $d = k-1$, $\mathcal{D}_{k-1} = S_1\times\cdots\times S_{k-1}$ and $\alpha_i = 1/k$ $(i = 1, \ldots, k-1)$ so that $s_{k-1} = 1 - 1/k$. The number

$$u\sum_{i=1}^{k-1}\alpha_i + \sum_{i=1}^{k-1}\alpha_i\varepsilon_i - (1+u+v)$$

is equal to

$$\frac{1}{k}\Big(\sum_{i=1}^{k-1}\varepsilon_i - u\Big) - 1 - v$$

and is clearly nonzero for every pair (u,v) of nonnegative integers and every $\varepsilon_i \in \{0,1\}$. Furthermore, since $n_k \leq 2N$, we see that hypothesis (4) is satisfied as soon as $N^{l_1} \geq 2^{l_0/k}$, so that Lemma 2.1 implies that

$$\sum_{(n_1,\ldots,n_k)\in S_1\times\cdots\times S_k} \psi((n_1\cdots n_k)^{1/k} \pm \Delta)$$

$$\ll \sum_{n_k\in S_k} n_k^{\frac{(k-1)l_0}{k\{1+(k-1)l_0\}}} N^{\frac{(k-1)\{l_0(k-1-1/k)+1-l_1\}}{1+(k-1)l_0}} \ll N^{1+\frac{(k-1)\{(k-1)l_0+1-l_1\}}{1+(k-1)l_0}}$$

and the desired result follows by using the $(k-1)$-dimensional exponent pair

$$(l_0,l_1) = \left(\frac{3k-2}{2(3k^2+k+4)}, \frac{3k-1}{3k^2+k+4}\right)$$

of Corollary 2.3. ∎

Now Theorem 1.1 follows at once from Corollary 2.4 and (3).

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