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On metric theory of Diophantine approximation for complex numbers

by

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1. Introduction. In 1941, R. J. Duffin and A. C. Schaeffer [DS] made a conjecture on a Diophantine approximation problem. The conjecture states that the inequality

$$\left|\alpha - \frac{m}{n}\right| < \frac{\psi(n)}{n}, \qquad (m,n) = 1, \tag{1.1}$$

has infinitely many solutions in positive integers m and n for almost all real numbers α if and only if $\sum_{n=2}^{\infty} \phi(n)\psi(n)n^{-1} = \infty$. If this series converges, then we can easily see that (1.1) has only finitely many solutions in positive integers m and n for almost all α. So the only difficulty is proving that (1.1) has infinitely many solutions for almost all α whenever $\sum_{n=2}^{\infty} \phi(n)\psi(n)n^{-1} = \infty$. R. J. Duffin and A. C. Schaeffer also gave a sufficient condition on $\psi(n)$ for (1.1) to have infinitely many solutions a.e., which is called the Duffin–Schaeffer theorem. In 1950, J. W. S. Cassels [C] showed that the inequality $|\alpha - m/n| < \psi(n)/n$ without the condition $(m,n) = 1$ has infinitely many solutions for either almost all α or almost no α. Then in 1961, P. X. Gallagher [G] added the condition $(m,n) = 1$, and proved that (1.1) has infinitely many solutions for either almost all α or almost no α. In 1970, P. Erdős [E] showed that if $\psi(n) = 0$ or $\psi(n) = \varepsilon n^{-1}$ for all $n \in \mathbb{N}$ and some $\varepsilon > 0$, then (1.1) has infinitely many solutions in positive integers m and n for almost all α if $\sum_{n=2}^{\infty} \phi(n)\psi(n)n^{-1}$ diverges. In 1978, J. D. Vaaler [V] gave a more general result following P. Erdős' idea. More precisely, he proved that (1.1) has infinitely many solutions in positive integers m and n for almost all α if $\psi(n) = \mathcal{O}(n^{-1})$ and $\sum_{n=2}^{\infty} \phi(n)\psi(n)n^{-1}$ diverges.

Diophantine approximation of complex numbers was first considered in 1887 by A. Hurwitz [Hu], who discussed Diophantine approximation by con-

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tinued fractions over the quadratic fields $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$. Since then, a number of papers on this subject have appeared, such as [F], [P] and [N]. In 1982, D. Sullivan [S] gave a metric result on Diophantine approximation over an imaginary quadratic field under a condition similar to the condition of the Duffin–Schaeffer theorem. In 1991, H. Nakada and G. Wagner [NW] proved a Duffin–Schaeffer type theorem over an imaginary quadratic field, as well as a Gallagher type theorem.

In this paper, we discuss a further development of the metric theory of Diophantine approximation over an imaginary quadratic field. Our main result indicates that the difficulty of the complex number version of the Duffin–Schaeffer conjecture is similar to that of the one-dimensional real case. Indeed, we will show that a Vaaler type theorem holds in this case, and then we find the same difficulty as in the case of real numbers for proving the complex version of the Duffin–Schaeffer conjecture. We refer to [HPV] and [BHHV] for the recent developments on the original Duffin–Schaeffer conjecture.

For a given square-free negative integer d, we consider

$$\mathbb{Q}(\sqrt{d}) = \{p + q\sqrt{d} : p, q \in \mathbb{Q}\}$$

and its maximal order

$$\mathbb{Z}[\omega] = \{m + n\omega : m, n \in \mathbb{Z}\}$$

where

$$\omega = \begin{cases} (1+\sqrt{d})/2 & \text{if } d \equiv 1 \pmod 4, \\ \sqrt{d} & \text{if } d \equiv 2, 3 \pmod 4). \end{cases}$$

In order to avoid the problem of different prime factor decompositions of an integer in $\mathbb{Z}[\omega]$, we consider ideals to get the uniqueness of prime factor decomposition. For $a \in \mathbb{Z}[\omega]$, we denote by (a) the principal ideal generated by a. Then we can give a complex number version of the Duffin–Schaeffer conjecture as follows: the inequality

$$\left| z - \frac{a}{r} \right| < \frac{\Psi((r))}{|r|}, \qquad (r, a) = (1), \tag{1.2}$$

has infinitely many solutions in $r, a \in \mathbb{Z}[\omega]$ for almost all $z \in \mathbb{C}$ if and only if $\sum \Phi((r))\Psi^2((r))|r|^{-2} = \infty$. Here (r, a) denotes the ideal in $\mathbb{Z}[\omega]$ generated by r and a, and $(r, a) = (1)$ means that r and a are coprime in terms of ideals. We set

$$\mathbb{F} = \{z \in \mathbb{C} : z = x + y\omega,\ x, y \in \mathbb{R},\ 0 \le x, y < 1\}.$$

Without loss of generality, we discuss our problems for almost all $z \in \mathbb{F}$ instead of $z \in \mathbb{C}$. The function $\Psi((r))$ is a non-negative real-valued function defined on the set of principal ideals of $\mathbb{Z}[\omega]$. The function $\Phi((r))$ is a complex number version of Euler's function over $\mathbb{Z}[\omega]$, which counts the number of integers $a \in \mathbb{Z}[\omega]$ relatively prime to r and $a/r \in \mathbb{F}$, and it equals the number of residue classes modulo the principal ideal (r). Then we have $\Phi((r)) = |r|^2 \prod_{P|(r)}(1 - \mathrm{N}^{-1}(P))$, where P denotes prime ideals of $\mathbb{Z}[\omega]$ and $\mathrm{N}(\cdot)$ denotes the norm of ideals.

Our main theorem is the following.

THEOREM 1.1. *If $\Psi((r)) = \mathcal{O}(|r|^{-1})$, then* (1.2) *has infinitely many solutions in $r, a \in \mathbb{Z}[\omega]$ for almost all $z \in \mathbb{C}$ whenever $\sum \Phi((r))\Psi^2((r))|r|^{-2} = \infty$.*

We define $\mathcal{E}_{(r)}$ as the set of complex numbers z which satisfy (1.2) for a given $r \in \mathbb{Z}[\omega]$, i.e.

$$\mathcal{E}_{(r)} = \bigcup_{\substack{a\in\mathbb{Z}[\omega] \\ a/r\in\mathbb{F} \\ (a,r)=(1)}} \left\{ z : \left| z - \frac{a}{r} \right| < \frac{\Psi((r))}{|r|},\ z \in \mathbb{F} \right\}.$$

To prove Theorem 1.1, it is enough to show

$$\lambda\Big(\bigcap_{N=1}^{\infty} \bigcup_{|r|^2=N}^{\infty} \mathcal{E}_{(r)}\Big) = \lim_{N\to\infty} \lambda\Big(\bigcup_{|r|^2=N}^{\infty} \mathcal{E}_{(r)}\Big) = 1 \tag{1.3}$$

whenever $\Psi((r)) = \mathcal{O}(|r|^{-1})$ and $\sum \Phi((r))\Psi^2((r))|r|^{-2} = \infty$. Here λ denotes the normalized Lebesgue measure on $\mathbb{F}$.

We extend two theorems of Vaaler [V, Theorems 2 and 3] to imaginary quadratic fields:

THEOREM 1.2. *Suppose there exist an integer $k \geq 2$ and a real number $\eta > 0$ such that the following condition holds: every finite subset $\mathbf{Z}$ of $\{k, k+1, \ldots\}$ with $0 \leq \Lambda(\mathbf{Z}) \leq \eta$ satisfies*

$$\sum_{\substack{|r|^2\in\mathbf{Z}}} \sum_{\substack{|s|^2\in\mathbf{Z} \\ (r)\neq(s)}} \lambda(\mathcal{E}_{(r)} \cap \mathcal{E}_{(s)}) \leq \Lambda(\mathbf{Z}), \tag{1.4}$$

where $\Lambda(\mathbf{Z}) = \sum_{|r|^2\in\mathbf{Z}} \lambda(\mathcal{E}_{(r)})$. Then $\sum \Phi((r))\Psi^2((r))|r|^{-2} = \infty$ implies (1.3).

THEOREM 1.3. *If $\Psi((r)) = \mathcal{O}(|r|^{-1})$, then there exists $\eta > 0$ such that if $\mathbf{Z}$ is a finite subset of $\{2, 3, \ldots\}$ with $0 < \Lambda(\mathbf{Z}) \leq \eta$, then*

$$\sum_{\substack{|r|^2\in\mathbf{Z}}} \sum_{\substack{|s|^2\in\mathbf{Z} \\ (r)\neq(s)}} \lambda(\mathcal{E}_{(r)} \cap \mathcal{E}_{(s)}) \ll \Lambda^2(\mathbf{Z})\Big(\ln\ln \frac{1}{\Lambda(\mathbf{Z})}\Big)^2. \tag{1.5}$$

We note that (1.5) is stronger than (1.4) since there exists a large rational integer k such that $\Lambda(\mathbf{Z})(\ln\ln \Lambda(\mathbf{Z})^{-1})^2 < 1$ with $\mathbf{Z} = \{k, k+1, \ldots\}$.

In the next section, we will prove Theorem 1.3 and then prove Theorem 1.2, which will complete the proof of Theorem 1.1. We note that we do need the condition $\Psi((r)) = \mathcal{O}(|r|^{-1})$ in the proof of Theorem 1.3, but we do not need it in the proof of Theorem 1.2.

2. Proof of main results. Throughout this section we will use $\mathrm{N}(\cdot)$ for the norm of an ideal over $\mathbb{Z}[\omega]$, and use P (and P_j) for prime ideals. We also use $\Psi(\cdot)$ to denote the number of residue classes modulo ideals, the complex number version of Euler's function.

We denote by $g(R)$, for an ideal R of $\mathbb{Z}[\omega]$, the smallest positive integer v that satisfies

$$\sum_{\substack{P\mid R\\ \mathrm{N}(P)>v}} \frac{1}{\mathrm{N}(P)} < 1.$$

Before proving Theorem 1.3, we give some lemmas similar to Vaaler's estimates [V].

LEMMA 2.1. *If R is an ideal of $\mathbb{Z}[\omega]$ and $g(R) = v$, then*

$$\prod_{\substack{P\mid R\\ \mathrm{N}(P)\le v}} \left(1 - \frac{1}{\mathrm{N}(P)}\right) \ll \frac{\Phi(R)}{\mathrm{N}(R)} \qquad \text{as } v \to \infty.$$

Proof. From the formula for Euler's function over ideals, we have

$$\Phi(R) = \mathrm{N}(R) \prod_{P\mid R}\left(1 - \frac{1}{\mathrm{N}(P)}\right).$$

Then

$$\begin{aligned}
\prod_{\substack{P\mid R\\ \mathrm{N}(P)\le v}} \left(1 - \frac{1}{\mathrm{N}(P)}\right) &= \frac{\Phi(R)}{\mathrm{N}(R)} \prod_{\substack{P\mid R\\ \mathrm{N}(P)>v}} \left(1 - \frac{1}{\mathrm{N}(P)}\right)^{-1}\\
&= \frac{\Phi(R)}{\mathrm{N}(R)} \exp\Bigg\{ \sum_{\substack{P\mid R\\ \mathrm{N}(P)>v}} \ln\left(1 - \frac{1}{\mathrm{N}(P)}\right)^{-1}\Bigg\}\\
&\le \frac{\Phi(R)}{\mathrm{N}(R)} \exp\Bigg\{ \sum_{\substack{P\mid R\\ \mathrm{N}(P)>v}} \frac{1}{\mathrm{N}(P)} + \sum_{P}\sum_{j=2}^{\infty} \frac{1}{j\mathrm{N}^j(P)}\Bigg\}.
\end{aligned}$$

Moreover,

$$\sum_{P}\sum_{j=2}^{\infty} \frac{1}{j\mathrm{N}^j(P)} \le \sum_{P} \frac{1}{\mathrm{N}(P)(\mathrm{N}(P)-1)} < \sum_{S} \frac{1}{\mathrm{N}^2(S)}. \tag{2.1}$$

Here $\sum_S$ is a sum over all ideals of $\mathbb{Z}[\omega]$. In order to show the right side of (2.1) converges, we first estimate the number $T(N)$ of ideals whose norm is less than or equal to a given rational integer N. By [H], there exists a constant $k(d)$ such that

$$\lim_{N\to\infty}\frac{T(N)}{N}=k(d),$$

which shows that $u_N=T(N)/N$ is bounded. Let T_i denote the number of ideals whose norm is equal to $i\in\mathbb{N}$. Then $T(N)=\sum_{i=1}^N T_i$. From $T_N=Nu_N-(N-1)u_{N-1}$, we have

$$\begin{aligned}\sum_{\mathrm{N}(S)=1}^{N}\frac{1}{\mathrm{N}^2(S)}&=\sum_{i=1}^{N}\frac{T_i}{i^2}\\ &<\frac{u_N}{N}+\frac{2}{N^2}u_{N-1}+\frac{2}{(N-1)^2}u_{N-2}+\cdots+\frac{2}{2^2}u_1\\ &\ll\sum_{i=1}^{N}\frac{1}{i^2}\quad\text{ as } N\to\infty.\end{aligned}$$

So the right side of (2.1) converges, which implies

$$\prod_{\substack{P\mid R\\ \mathrm{N}(P)\le v}}\left(1-\frac{1}{\mathrm{N}(P)}\right)\ll\frac{\Phi(R)}{\mathrm{N}(R)}\quad\text{ as } v\to\infty.\ \blacksquare$$

We now give a corollary of Lemma 2.1 which we will use later.

Corollary 2.2. *If R is an ideal of $\mathbb{Z}[\omega]$ and $g(R)=v$, then*

$$1\ll\frac{\Phi(R)}{\mathrm{N}(R)}\ln(1+v)\quad\textit{ as } v\to\infty.$$

Proof. Here we need M. Rosen's [R] result on Mertens' theorem for an algebraic number field $\mathbb{K}$:

$$\prod_{\mathrm{N}(P)\le x}\left(1-\frac{1}{\mathrm{N}(P)}\right)^{-1}=e^{\gamma}\alpha_{\mathbb{K}}\ln x+\mathcal{O}_{\mathbb{K}}(1)$$

where $\gamma>0$ and $\alpha_{\mathbb{K}}$ are constants. From Lemma 2.1 we have

$$\begin{aligned}1&\ll\frac{\Phi(R)}{\mathrm{N}(R)}\prod_{\substack{P\mid R\\ \mathrm{N}(P)\le v}}\left(1-\frac{1}{\mathrm{N}(P)}\right)^{-1}\le\frac{\Phi(R)}{\mathrm{N}(R)}\prod_{\mathrm{N}(P)\le v}\left(1-\frac{1}{\mathrm{N}(P)}\right)^{-1}\\ &\ll\frac{\Phi(R)}{\mathrm{N}(R)}\ln(1+v)\quad\text{ as } v\to\infty.\ \blacksquare\end{aligned}$$

For $\xi,x,v>0$, we define a collection $\mathcal{N}(\xi,x,v)$ of ideals of $\mathbb{Z}[\omega]$ by

$$\mathcal{N}(\xi,x,v)=\Big\{R:\sum_{\substack{P\mid R\\ \mathrm{N}(P)\ge v}}\frac{1}{\mathrm{N}(P)}\ge\xi,\ \mathrm{N}(R)\le x\Big\}.$$

We denote by $\#\mathcal{N}(\xi, x, v)$ the number of ideals in $\mathcal{N}(\xi, x, v)$. Then we can extend Vaaler's estimate [V] to the complex number case as follows:

LEMMA 2.3. *For any $\varepsilon, \xi, x > 0$, we have*

$$\#\mathcal{N}(\xi, v, x) \ll \frac{x}{e^{v^{\beta(1-\varepsilon)}}} \qquad \text{as } v \to \infty \text{ with } \beta = e^{\xi}. \tag{2.2}$$

Proof. Suppose $0 < \varepsilon < 1 - 1/e^{\xi} = 1 - 1/\beta$. It is enough to show the conclusion for such ε since the right side of (2.2) becomes larger if ε gets larger. Let $[v, w]$ be an interval with $w = v^{\beta(1-2\varepsilon/3)}$. Let $\{P_1, \ldots, P_M\}$ be the set of all prime ideals whose norms are in $[v, w]$ with $\mathrm{N}(P_1) \le \cdots \le \mathrm{N}(P_M)$. Let π be the prime-counting function in the sense of ideals of $\mathbb{Z}[\omega]$, i.e. $\pi(w)$ is the number of prime ideals whose norm is less than or equal to w. Then we see $M \ge \pi(w) - \pi(v)$. We have the equality

$$\frac{v^{\beta(1-2\varepsilon/3)}}{\frac{w}{\ln w} - \frac{v}{\ln v}} = \frac{\beta(1-\varepsilon/3)\ln v}{v^{\beta\varepsilon/3} - \beta(1-\varepsilon/3)v^{1-\beta(1-2\varepsilon/3)}}.$$

Since $\varepsilon < 1 - 1/\beta$, we have $1 - \beta(1-2\varepsilon/3) < 0$. Hence there exists an integer $v_0(\varepsilon, \xi) > 0$ such that $\frac{w}{\ln w} - \frac{v}{\ln v} \ge v^{\beta(1-2\varepsilon/3)}$ for any $v \ge v_0$ and we have

$$M \ge \pi(w) - \pi(v) \gg \frac{w}{\ln w} - \frac{v}{\ln v} \ge v^{\beta(1-2\varepsilon/3)} \qquad \text{as } v \to \infty$$

by the prime ideal theorem.

Next, we divide all the ideals in $\mathcal{N}(\xi, x, v)$ into two classes.

CLASS 1: There are no less than M different prime ideal factors of R and their norms are all in the interval $[v, e^w]$. By using Mertens' theorem on algebraic number fields [R],

$$\sum_{\mathrm{N}(P) \le x} \frac{1}{\mathrm{N}(P)} = \ln\ln x + B_{\mathbb{K}} + \mathcal{O}_{\mathbb{K}}\left(\frac{1}{\ln x}\right)$$

where $B_{\mathbb{K}}$ is a constant depending only on the algebraic number field $\mathbb{K}$. We denote by N_1 the number of ideals of class 1. Then

$$N_1 \ll x \frac{\left(\sum_{v \le \mathrm{N}(P) \le e^w} \frac{1}{\mathrm{N}(P)}\right)^M}{M!} \ll x \frac{(\ln w)^M}{M!} \qquad \text{as } v \to \infty,$$

where $w \le M^2$. From Stirling's formula we have

$$\begin{aligned} x\frac{(\ln w)^M}{M!} &\ll \frac{2^M (e^{\ln\ln M})^M}{M!} \ll x \frac{2^M e^{M + M\ln\ln M}}{M^M \sqrt{2\pi M}} \\ &\ll \frac{x}{e^{v^{\beta(1-2\varepsilon/3)}}} \qquad \text{as } v \to \infty. \end{aligned} \tag{2.3}$$

CLASS 2: There are less than M different prime ideal factors of R and their norms are all in $[v, e^w]$. By using Mertens' theorem on algebraic number fields [R], we have

$$\sum_{j=1}^{M} \frac{1}{\mathrm{N}(P_j)} \ll \ln\ln w - \ln\ln v = \xi + \ln\left(1 - \frac{\varepsilon}{3}\right) < \xi - \frac{\varepsilon}{3} \quad \text{as } v \to \infty.$$

From

$$\sum_{\substack{P\mid R\\ \mathrm{N}(P)\ge v\ge g(R)}} \frac{1}{\mathrm{N}(P)} = \sum_{\substack{P\mid R\\ v\le \mathrm{N}(P)\le w}} \frac{1}{\mathrm{N}(P)} + \sum_{\substack{P\mid R\\ w<\mathrm{N}(P)\le e^w}} \frac{1}{\mathrm{N}(P)} + \sum_{\substack{P\mid R\\ \mathrm{N}(P)>e^w}} \frac{1}{\mathrm{N}(P)} \ge \xi$$

and the definition of class 2, we see that

$$\sum_{\substack{P\mid R\\ v\le \mathrm{N}(P)\le w}} \frac{1}{\mathrm{N}(P)} + \sum_{\substack{P\mid R\\ w<\mathrm{N}(P)\le e^w}} \frac{1}{\mathrm{N}(P)} \le \sum_{v\le \mathrm{N}(P)\le w} \frac{1}{\mathrm{N}(P)} \ll \xi - \frac{\varepsilon}{3} \quad \text{as } v \to \infty.$$

So we have the estimate

$$\sum_{\substack{P\mid R\\ \mathrm{N}(P)>e^w}} \frac{1}{\mathrm{N}(P)} \gg \frac{\varepsilon}{3} \quad \text{as } v \to \infty.$$

The number of ideals R of class 2 is less than $\sum_{\mathrm{N}(R)\le x} 1$ and so

$$\begin{aligned}
\text{(2.4)} \qquad \sum_{\mathrm{N}(R)\le x} 1 &\ll \sum_{\mathrm{N}(R)\le x} \frac{3}{\varepsilon} \sum_{\substack{P\mid R\\ \mathrm{N}(P)>e^w}} \frac{1}{\mathrm{N}(P)} \\
&\ll \frac{1}{\varepsilon} \sum_{\mathrm{N}(P)>e^w} \frac{1}{\mathrm{N}(P)} \cdot \frac{x}{\mathrm{N}(P)} \\
&< \frac{x}{\varepsilon}\left(\frac{1}{(e^w)^2} + \frac{1}{e^w(e^w+1)} + \frac{1}{(e^w+1)(e^w+2)} + \cdots\right) \\
&\ll \frac{1}{\varepsilon} \cdot \frac{x}{e^{v^{\beta(1-2\varepsilon/3)}}} \quad \text{as } v \to \infty.
\end{aligned}$$

The estimates (2.3) and (2.4) imply (2.2). ■

For a fixed $r \in \mathbb{Z}[\omega]$ and $\xi, v > 0$ we define two collections $\mathcal{A}_r(\xi, v)$ and $\mathcal{B}_r(\xi, v)$ of ideals by

$$\mathcal{A}_r(\xi, v) = \left\{A : A \mid (r),\ \sum_{\substack{P\mid A\\ \mathrm{N}(P)\ge v\ge g((r))}} \frac{1}{\mathrm{N}(P)} \ge \xi\right\},$$

$$\mathcal{B}_r(\xi, v) = \left\{B : B \mid (r),\ \sum_{\substack{P\mid B\\ \mathrm{N}(P)\ge v\ge g((r))}} \frac{1}{\mathrm{N}(P)} < \xi\right\}.$$

Lemma 2.4. *For any* $\varepsilon, \xi > 0$ *and* $v \geq g((r))$,

$$\sum_{A\in\mathcal{A}_r(\xi,v)} \frac{1}{\mathrm{N}(A)} \ll \frac{\ln(1+g((r)))}{e^{v^{\beta(1-\varepsilon)}}} \quad \text{as } v\to\infty \text{ with } \beta = e^{\xi}.$$

Proof. Let $w = v^{\beta(1-\varepsilon/3)}$, where $0 < \varepsilon < 1 - e^{-\xi} = 1 - \beta^{-1}$. Suppose there are M different prime ideals $P_1, \ldots, P_M$ with $v \leq \mathrm{N}(P_1) \leq \cdots \leq \mathrm{N}(P_M) \leq w$. Let $\mathcal{J}$ be any collection of M different prime ideals whose norms are in $[v, \infty)$. Then from the proof of Lemma 2.3, we have

$$\sum_{P\in\mathcal{J}} \frac{1}{\mathrm{N}(P)} \leq \sum_{j=1}^{M} \frac{1}{\mathrm{N}(P_j)} \ll \xi - \frac{\varepsilon}{3} \quad \text{as } v\to\infty.$$

For any $A \in \mathcal{A}_r(\xi, v)$, we have

$$\sum_{\substack{P|A\\ \mathrm{N}(P)\geq v\geq g((r))}} \frac{1}{\mathrm{N}(P)} \geq \xi.$$

This implies that for all large v, there are at least M different prime ideal factors of A whose norms are all in $[v, \infty)$. Let $Q_1, \ldots, Q_J$ be all different prime ideal factors of (r).

Case 1: $J < M$. From the discussion above, $\mathcal{A}_r(\xi, v) = \emptyset$ for all large v, which means $\sum_{A\in\mathcal{A}_r(\xi,v)} \mathrm{N}^{-1}(A) = 0$.

Case 2: $J \geq M$. Since $v \geq g((r))$ and $\sum_{j=1}^{J} \mathrm{N}^{-1}(Q_j) < 1$, we have

$$\sum_{A\in\mathcal{A}_r(\xi,v)} \frac{1}{\mathrm{N}(A)} \leq \sum_{A|(r)} \frac{1}{\mathrm{N}(A)} \cdot \frac{\left(\sum_{j=1}^{J} \frac{1}{\mathrm{N}(Q_j)}\right)^M}{M!} < \left(\sum_{A|(r)} \frac{1}{\mathrm{N}(A)}\right) \frac{1}{M!}. \tag{2.5}$$

Suppose $(r) = Q_1^{\gamma_1} \cdots Q_J^{\gamma_J}$ where $Q_1, \ldots, Q_J$ are all different prime ideal factors of (r) and $\gamma_1, \ldots, \gamma_J$ are positive integers. By Corollary 2.2, we have

$$\begin{aligned}\sum_{A|(r)} \frac{1}{\mathrm{N}(A)} &= \left(1 + \frac{1}{\mathrm{N}(Q_1)} + \frac{1}{\mathrm{N}^2(Q_1)} + \cdots + \frac{1}{\mathrm{N}^{\gamma_1}(Q_1)}\right) \\ &\quad \cdot \left(1 + \frac{1}{\mathrm{N}(Q_2)} + \frac{1}{\mathrm{N}^2(Q_2)} + \cdots + \frac{1}{\mathrm{N}^{\gamma_2}(Q_2)}\right) \\ &\quad \cdots \left(1 + \frac{1}{\mathrm{N}(Q_J)} + \frac{1}{\mathrm{N}^2(Q_J)} + \cdots + \frac{1}{\mathrm{N}^{\gamma_J}(Q_J)}\right) \\ &\leq \prod_{\substack{Q|(r)\\ Q \text{ a prime ideal}}} \left(1 - \frac{1}{\mathrm{N}(Q)}\right)^{-1} \ll \ln(1+g((r))) \quad \text{as } v\to\infty.\end{aligned} \tag{2.6}$$

From (2.3), (2.5), and (2.6), we have

$$\sum_{A\in\mathcal{A}_r(\xi,v)} \frac{1}{\mathrm{N}(A)} \ll \frac{\ln(1+g((r)))}{e^{v^{\beta(1-\varepsilon)}}} \quad \text{as } v\to\infty \text{ with } \beta=e^{\xi}. \ \blacksquare$$

Lemma 2.5. *Suppose* $(s),(r)$ *are two principal ideals with* $s,r\in\mathbb{Z}[\omega]$ *and* $U=(s,r)$. *Then for* $\varepsilon,\xi,x>0$, $y\geq 2$, *and* $v\geq g((r))$, *we have*

$$\sum_{\substack{(s)^v\\ x\mathrm{N}(U)<|s|^2<xy\mathrm{N}(U)}} \frac{1}{|s|^2} \ll \frac{\ln(1+g((r)))\ln y}{e^{v^{\beta(1-\varepsilon)}}} \quad \textit{as } v\to\infty, \tag{2.7}$$

$$\sum_{\substack{(s)^v\\ x\mathrm{N}^{-1}(U)<|s|^2<xy\mathrm{N}^{-1}(U)}} \frac{1}{|s|^2} \ll \frac{\ln(1+g((r)))\ln y}{e^{v^{\beta(1-\varepsilon)}}} \quad \textit{as } v\to\infty, \tag{2.8}$$

with $\beta=e^{\xi}$. *Here* $\sum_{(s)^v}$ *means the sum over* (s) *satisfying* $g((s))=v$.

Proof. The right sides of (2.7) and (2.8) are both independent of U and x. Thus, by choosing x properly, we see that (2.7) and (2.8) are equivalent. So we only need to prove (2.7). Let $(s)=US'$ and $(r)=UR'$. Then

$$\sum_{\substack{(s)^v\\ x\mathrm{N}(U)<|s|^2<xy\mathrm{N}(U)}} \frac{1}{|s|^2} = \sum_{U|(r)} \sum_{\substack{(s)^v\\ (s,r)=U\\ x\mathrm{N}(U)<|s|^2<xy\mathrm{N}(U)}} \frac{1}{|s|^2} \tag{2.9}$$

$$= \sum_{U\in\mathcal{A}_r(1/2,v)} \sum_{\substack{(s)^v\\ (s,r)=U\\ x\mathrm{N}(U)<|s|^2<xy\mathrm{N}(U)}} \frac{1}{|s|^2} + \sum_{U\in\mathcal{B}_r(1/2,v)} \sum_{\substack{(s)^v\\ (s,r)=U\\ x\mathrm{N}(U)<|s|^2<xy\mathrm{N}(U)}} \frac{1}{|s|^2}.$$

Here we have

$$\sum_{U\in\mathcal{A}_r(1/2,v)} \sum_{\substack{(s)^v\\ (s,r)=U\\ x\mathrm{N}(U)<|s|^2<xy\mathrm{N}(U)}} \frac{1}{|s|^2} = \sum_{U\in\mathcal{A}_r(1/2,v)} \sum_{\substack{(US')^v\\ (s,r)=U\\ x<\mathrm{N}(S')<xy}} \frac{1}{\mathrm{N}(U)}\frac{1}{\mathrm{N}(S')}$$

$$\leq \Big(\sum_{U\in\mathcal{A}_r(1/2,v)} \frac{1}{\mathrm{N}(U)}\Big)\Big(\sum_{\substack{S'\\ x<\mathrm{N}(S')<xy}} \frac{1}{\mathrm{N}(S')}\Big).$$

By using the same method as in the proof of Lemma 2.1, we estimate

$$\sum_{\substack{S'\\ x<\mathrm{N}(S')<xy}} \frac{1}{\mathrm{N}(S')} \ll \ln y \quad \text{as } y\to\infty.$$

From Lemma 2.4 we have

$$\sum_{U\in\mathcal{A}_r(1/2,v)}\frac{1}{\mathrm{N}(U)}\ll\frac{\ln(1+g((r)))}{e^{v^{\beta(1-\varepsilon)}}}\qquad\text{as } v\to\infty,$$

and so

$$\sum_{U\in\mathcal{A}_r(1/2,v)}\ \sum_{\substack{(s)^v\\(s,r)=U\\ x\mathrm{N}(U)<|s|^2<xy\mathrm{N}(U)}}\frac{1}{|s|^2}\ll\frac{\ln(1+g((r)))\ln y}{e^{v^{\beta(1-\varepsilon)}}}\qquad\text{as } v\to\infty. \tag{2.10}$$

Thus we get the desired estimate of the first term on the right side of (2.9) with $U\in\mathcal{A}_r(1/2,v)$.

Now we consider the second term of the right side of (2.9) with U in $\mathcal{B}_r(1/2,v)$ and $g((s))=v$. In this case we have

$$\begin{aligned}1\le\sum_{\substack{P|(s)\\ \mathrm{N}(P)\ge v=g((s))}}\frac{1}{\mathrm{N}(P)}&\le\sum_{\substack{P|U\\ \mathrm{N}(P)\ge v}}\frac{1}{\mathrm{N}(P)}+\sum_{\substack{P|S'\\ \mathrm{N}(P)\ge v}}\frac{1}{\mathrm{N}(P)}\\&<\frac12+\sum_{\substack{P|S'\\ \mathrm{N}(P)\ge v}}\frac{1}{\mathrm{N}(P)},\end{aligned}$$

which shows $\sum_{P|S',\,\mathrm{N}(P)\ge v}\mathrm{N}^{-1}(P)>1/2$. From Lemma 2.3, we see that

$$\begin{aligned}\sum_{\substack{S'\\ x<\mathrm{N}(S')\le 2x}}\frac{1}{\mathrm{N}(S')}&<\Bigg(\sum_{\substack{S'\\ x<\mathrm{N}(S')\le 2x}}1\Bigg)\frac1x\le\frac{\#\mathcal{N}(1/2,v,2x)}{x}\\&\ll\frac{1}{e^{v^{\beta(1-\varepsilon)}}}\qquad\text{as } v\to\infty.\end{aligned}$$

Thus we have

$$\begin{aligned}\sum_{\substack{S'\\ x<\mathrm{N}(S')<xy}}\frac{1}{\mathrm{N}(S')}&<\frac1x\sum_{k=1}^{[y]}\frac1k\Bigg(\sum_{\substack{S'\\ kx<\mathrm{N}(S')\le(k+1)x}}1\Bigg)\\&\le\frac1x\left(1-\frac12\right)\#\mathcal{N}\left(\frac12,v,2x\right)+\frac1x\left(\frac12-\frac13\right)\#\mathcal{N}\left(\frac12,v,3x\right)\\&\quad+\cdots+\frac1x\frac{1}{[y]}\#\mathcal{N}\left(\frac12,v,([y]+1)x\right)\\&\ll\left(1+\frac12+\frac13+\cdots+\frac{1}{[y]}\right)\cdot\frac{1}{e^{v^{\beta(1-\varepsilon)}}}\\&\ll\ln y\cdot\frac{1}{e^{v^{\beta(1-\varepsilon)}}}\qquad\text{as } y\to\infty.\end{aligned}$$

This gives the estimate of the second term as follows:

$$
\begin{aligned}
(2.11)\qquad & \sum_{U\in\mathcal{B}_r(1/2,v)} \sum_{\substack{(s)^{(v)}\\(s,r)=U\\ x\mathrm{N}(U)<|s|^2<xy\mathrm{N}(U)}} \frac{1}{|s|^2}\\
&\le \Bigg(\sum_{U\in\mathcal{B}_r(1/2,v)} \frac{1}{\mathrm{N}(U)}\Bigg)\Bigg(\sum_{\substack{S'\\ x<\mathrm{N}(S')<xy}} \frac{1}{\mathrm{N}(S')}\Bigg) \ll \Bigg(\sum_{U\mid(r)} \frac{1}{\mathrm{N}(U)}\Bigg)\frac{\ln y}{e^{v^{\beta(1-\varepsilon)}}}\\
&\le \frac{|r|^2}{\Phi((r))}\cdot\frac{\ln y}{e^{v^{\beta(1-\varepsilon)}}} \ll \ln(1+g((r)))\frac{\ln y}{e^{v^{\beta(1-\varepsilon)}}} \qquad \text{as } v\to\infty.
\end{aligned}
$$

Now we can deduce the assertion of Lemma 2.5 from (2.10) and (2.11). ■

Proof of Theorem 1.3. Let $r,s\in\mathbb{Z}[\omega]$ with $(r)\ne(s)$. Set

$$
\delta=\min\bigg\{\frac{\Psi((r))}{|r|},\frac{\Psi((s))}{|s|}\bigg\},\qquad \Delta=\max\bigg\{\frac{\Psi((r))}{|r|},\frac{\Psi((s))}{|s|}\bigg\},
$$
$$
t=\max\{g((r)),g((s))\}.
$$

For $a,b\in\mathbb{Z}[\omega]$ and given r and s, let

$$
\mathcal{R}_a=\bigg\{z\in\mathbb{F}:\bigg|z-\frac{a}{r}\bigg|<\frac{\Psi((r))}{|r|}\bigg\},\quad \mathcal{S}_b=\bigg\{z\in\mathbb{F}:\bigg|z-\frac{b}{s}\bigg|<\frac{\Psi((s))}{|s|}\bigg\}.
$$

Then

$$
\mathcal{E}_{(r)}=\bigcup_{\substack{a\in\mathbb{Z}[\omega]\\ a/r\in\mathbb{F}\\ (a,r)=(1)}}\mathcal{R}_a,\qquad \mathcal{E}_{(s)}=\bigcup_{\substack{b\in\mathbb{Z}[\omega]\\ b/s\in\mathbb{F}\\ (b,s)=(1)}}\mathcal{S}_b.
$$

If $\Psi((r))\le 1/2$ and $\Psi((s))\le 1/2$, then for any $a_1\ne a_2$ we have $\mathcal{R}_{a_1}\cap\mathcal{R}_{a_2}=\emptyset$, and similarly for $\mathcal{S}_b$. Then

$$
\begin{aligned}
(2.12)\qquad \lambda(\mathcal{E}_{(r)}\cap\mathcal{E}_{(s)}) &= \sum_{\substack{a/r\in\mathbb{F}\\(a,r)=(1)}}\sum_{\substack{b/s\in\mathbb{F}\\(b,s)=(1)}}\lambda(\mathcal{R}_a\cap\mathcal{S}_b)\\
&\le \delta^2\sum_{\substack{a/r\in\mathbb{F}\\(a,r)=(1)}}\sum_{\substack{b/s\in\mathbb{F}\\(b,s)=(1)\\ |a/r-b/s|<\Delta}}1=\delta^2\sum_{\substack{a/r\in\mathbb{F}\\(a,r)=(1)}}\sum_{\substack{b/s\in\mathbb{F}\\(b,s)=(1)\\ |as-br|<|r|\,|s|\Delta}}1.
\end{aligned}
$$

We define

$$
H(k)=\big\{\{a,b\}:a,b\in\mathbb{Z}[\omega],\ as-br=k,(a,r)=(b,s)=(1),\ \text{with } a/r,b/s\in\mathbb{F}\big\}.
$$

We will estimate the cardinality $\#H(k)$. Let $U=(r,s)$ and S' and R' be the ideals determined by $(s)=US'$ and $(r)=UR'$. Since $((a),R')=(1)$ and $(S',R')=(1)$, we have $(a)S'\ne(b)R'$, which shows $\#H(0)=0$. Since

$U \mid (as)$ and $U \mid (br)$ imply $U \mid (k)$, we have $\#H(k) = 0$ if $U \nmid (k)$. So we only need to consider $k \in \mathbb{Z}[\omega]$ with $U \mid (k)$. In this case, the principal ideal (k) can be uniquely represented as $(k) = U \cdot U_{(k)} \cdot K_1$. Here $U_{(k)}$ is the ideal whose prime ideal factors are all also prime ideal factors of U, and $(K_1, U) = (1)$.

If $(K_1, UR'S') \neq (1)$, then we can find a prime ideal P such that $P \mid K_1$ and $P \mid UR'S'$. Since $(K_1, U) = (1)$, either $P \mid R'$ or $P \mid S'$. If $P \mid R'$, then $P \mid (br)$ and $P \nmid (s)$. Here $P \mid R'$ implies $P \nmid (a)$ and we have $P \nmid (as)$, which is impossible since $P \mid (k)$. We can use the same approach for the case of $P \mid S'$ and get the same conclusion. Hence if $(K_1, UR'S') \neq (1)$, then $\#H(k) = 0$.

If $(U_{(k)}, R'S') \neq (1)$, then we can find a prime ideal P with $P \mid U_{(k)}$ and $P \mid R'S'$. If $P \mid R'$, there exists a positive integer n such that $P^n \mid U$ and $P^{n+1} \nmid U$. Then $P^{n+1} \mid (r)$, which means $br \in P^{n+1}$. From $P^{n+1} \mid (k)$, we see that $P \mid (a)$, which is impossible since $((a), R') = (1)$ and $P \mid R'$. We can use the same method for the case $P \mid S'$ and get the same conclusion. So if $(U_{(k)}, R'S') \neq (1)$, then $\#H(k) = 0$.

Consequently, we only need to estimate $\#H(k)$ in the case $(K_1, UR'S') = (1)$, $(U_{(k)}, R'S') = (1)$ and $\mathrm{N}(U) \leq |k|^2$. Suppose $\{a_1, b_1\}$ and $\{a_2, b_2\}$ are two different pairs of integers in $H(k)$ for a given $k \in \mathbb{Z}[\omega]$. Then $(a_1 - a_2)(s) = (b_1 - b_2)(r)$. So we have

$$R' \mid (a_1 - a_2) \quad \text{and} \quad S' \mid (b_1 - b_2). \tag{2.13}$$

We consider the set of pairs (a, b) with $a/r, b/s \in \mathbb{F}$ such that any two of them satisfy (2.13). Then its cardinality is $|r|^2 \mathrm{N}^{-1}(R') = \mathrm{N}(U)$.

Next, we estimate the number of pairs of integers a, b in the above set with $(a, U) = (1)$ and $(b, U) = (1)$. For this purpose we consider the pairs of integers a, b with $(a, U) \neq (1)$ or $(b, U) \neq (1)$ and exclude them from the pairs of integers a, b in the above set with $|a| \leq |r|$ and $|b| \leq |s|$. Here we assume a_j, b_j and a_l, b_l are two different pairs of solutions of (2.13). Now we estimate the number of pairs of integers a, b with $(a, U) \neq (1)$ or $(b, U) \neq (1)$. Since U can be decomposed into $U = P_1^{\gamma_1} \cdots P_j^{\gamma_j}$, we consider two cases for P $(= P_j)$.

Case 1: $P \mid U$, $P \nmid U_{(k)}$, and $P \nmid R'S'$. We will show that $P \mid (a_j)$ implies $P \nmid (b_j)$, which means that a, b are in different residue classes modulo P. Indeed, since $R' \mid (a_j - a_l)$, $S' \mid (b_j - b_l)$ and $\gcd(\mathrm{N}(P), \mathrm{N}(R')) = \gcd(\mathrm{N}(P), \mathrm{N}(S')) = 1$, we have $P \mid (a_j - a_l)$ and $P \mid (b_j - b_l)$. These show $UP \nmid (k)$ and $UP \mid (a_j s)$, which means $UP \nmid (b_j r)$ and thus $P \nmid (b_j)$.

Case 2: $P \mid U$ and either $P \mid U_{(k)}$ or $P \mid R'S'$.

(i) $P \mid U_{(k)}$ and $P \nmid R'S'$. As in Case 1, we have $P \mid (a_j - a_l)$ and $P \mid (b_j - b_l)$. Since $UP \mid (k)$ and $UP \mid (a_j s)$, we have $UP \mid (b_j r)$, which implies $P \mid (b_j)$. So in this case $P \mid (a_j)$ implies $P \mid (b_j)$, which means that a, b are in the same residue class modulo P.

(ii) $P \mid R'S'$. Assume $P \mid R'$ and $P \nmid S'$. Note that then $P \nmid (a_j)$. Since $(P, S') = (1)$, all the integers b are in the same residue class modulo P. In this case, we only need to exclude the pairs a, b with $P \mid (b)$. Similarly, for $P \nmid R'$ and $P \mid S'$, we only need to exclude a, b with $P \mid (a)$.

From the above discussion, we have

$$
\begin{aligned}
\#H(k) &\le \mathrm{N}(U) \prod_{\substack{P \mid U \\ P \nmid U_{(k)} \\ P \nmid R'S'}} \left(1 - \frac{2}{\mathrm{N}(P)}\right) \prod_{\substack{P \mid U \\ P \mid U_{(k)} R'S'}} \left(1 - \frac{1}{\mathrm{N}(P)}\right) \\
&\le \mathrm{N}(U) \prod_{\substack{P \mid U \\ P \nmid U_{(k)} \\ P \nmid R'S'}} \left(1 - \frac{1}{\mathrm{N}(P)}\right) \prod_{\substack{P \mid U \\ P \nmid U_{(k)} \\ P \nmid R'S'}} \left(1 - \frac{1}{\mathrm{N}(P)}\right) \prod_{\substack{P \mid U \\ P \nmid U_{(k)} \\ P \mid R'S'}} \left(1 - \frac{1}{\mathrm{N}(P)}\right) \\
&\quad \times \prod_{\substack{P \mid U \\ P \mid U_{(k)} \\ P \nmid R'S'}} \left(1 - \frac{1}{\mathrm{N}(P)}\right) \prod_{\substack{P \mid U \\ P \mid U_{(k)} \\ P \mid R'S'}} \left(1 - \frac{1}{\mathrm{N}(P)}\right) \\
&= \Phi(U) \prod_{\substack{P \mid U \\ P \nmid R'S'}} \left(1 - \frac{1}{\mathrm{N}(P)}\right) \prod_{P \mid U_{(k)}} \left(1 - \frac{1}{\mathrm{N}(P)}\right)^{-1}.
\end{aligned}
\tag{2.14}
$$

Now we use some notation following Vaaler [V]:

$$
\begin{aligned}
\mathcal{J}_0 &= \{P : P \mid U,\ P \nmid R'S'\}, \\
\mathcal{J}_1 &= \{P : P \in \mathcal{J}_0,\ \mathrm{N}(P) \le t\}, \\
\mathcal{J}_2 &= \{P : P \in \mathcal{J}_0,\ \mathrm{N}(P) > t\}, \\
\mathcal{I}_m &= \{I : I = P_1^{\gamma_1} \cdots P_k^{\gamma_k},\ P_1, \ldots, P_k \in \mathcal{J}_m,\ \gamma_1, \ldots, \gamma_k \in \mathbb{Z}\} \\
&\qquad \text{with } m = 0, 1, 2.
\end{aligned}
$$

Since $U_{(k)} \in \mathcal{I}_0$, we divide $U_{(k)}$ into two parts $I_1 \in \mathcal{I}_1$ and $I_2 \in \mathcal{I}_2$, with $U_{(k)} = I_1 I_2$. Then, together with (2.14), we have the following estimate:

$$
\begin{aligned}
\#H(k) &\le \Phi(U) \prod_{P \in \mathcal{J}_1} \left(1 - \frac{1}{\mathrm{N}(P)}\right) \prod_{P \mid I_1} \left(1 - \frac{1}{\mathrm{N}(P)}\right)^{-1} \frac{\prod_{P \in \mathcal{J}_2} \left(1 - \frac{1}{\mathrm{N}(P)}\right)}{\prod_{P \mid I_2} \left(1 - \frac{1}{\mathrm{N}(P)}\right)} \\
&\le \Phi(U) \prod_{P \in \mathcal{J}_1} \left(1 - \frac{1}{\mathrm{N}(P)}\right) \prod_{P \mid I_1} \left(1 - \frac{1}{\mathrm{N}(P)}\right)^{-1}.
\end{aligned}
\tag{2.15}
$$

Let

$$
K = I_2 K_1, \qquad Q = \prod_{\substack{P \mid R'S'U \\ \mathrm{N}(P) \le t}} P.
$$

Since $(K_1,U)=(1)$ and $(U_{(k)},R'S')=(1)$, we have $(K_1,R'S'U)=(1)$ and $(I_2,R'S')=(1)$, which implies $(K,Q)=(1)$. Then by using (2.12) and (2.15), we get

$$\lambda(\mathcal{E}_{(r)}\cap\mathcal{E}_{(s)})\le\delta^2\sum_{\substack{k\in\mathbb{Z}[\omega]\\ 1\le|k|\le|r|\,|s|\Delta}}\#H(k) \tag{2.16}$$

$$\le\delta^2\sum_{I_1\in\mathcal{I}_1}\sum_{\substack{K\\ 1\le\mathrm{N}(K)\le\frac{|r|^2|s|^2\Delta^2}{\mathrm{N}(U)\mathrm{N}(I_1)}\\ (K,Q)=(1)}}\Phi(U)\prod_{P\in\mathcal{J}_1}\Big(1-\frac{1}{\mathrm{N}(P)}\Big)\cdot\prod_{P|I_1}\Big(1-\frac{1}{\mathrm{N}(P)}\Big)^{-1}$$

$$=\delta^2\Phi(U)\prod_{P\in\mathcal{J}_1}\Big(1-\frac{1}{\mathrm{N}(P)}\Big)\cdot\sum_{I_1\in\mathcal{I}_1}\Big(\prod_{P|I_1}\Big(1-\frac{1}{\mathrm{N}(P)}\Big)^{-1}\sum_{\substack{K\\ 1\le\mathrm{N}(K)\le\frac{|r|^2|s|^2\Delta^2}{\mathrm{N}(U)\mathrm{N}(I_1)}\\ (K,Q)=(1)}}1\Big).$$

By the Landau prime ideal theorem [L], $\pi(y)=\mathrm{Li}(y)+\mathcal{O}_{\mathbb{K}}(ye^{-c_{\mathbb{K}}\sqrt{\ln y}})$, we have $(\pi(y)(\ln 2+\ln y)+\ln\ln y)y^{-1}\ll 1$ as $y\to\infty$. Then there exists $b\ge 0$ such that for any $y\ge b$, we have $\pi(y)(\ln 2+\ln y)+\ln\ln y\le y\ln 3$. We will estimate

$$\sum_{\substack{|r|^2\in\mathbf{Z}\\ (r)\ne(s)}}\sum_{|s|^2\in\mathbf{Z}}\lambda(\mathcal{E}_{(r)}\cap\mathcal{E}_{(s)})$$

by considering two cases.

Case A: $t\ge b$ and $|t|^2|s|^2\Delta^2\ge 3^t\mathrm{N}(U)$. By the sieve method for imaginary quadratic fields, we see that

$$\sum_{\substack{K\\ 1\le\mathrm{N}(K)\le\frac{|r|^2|s|^2\Delta^2}{\mathrm{N}(U)\mathrm{N}(I_1)}\\ (K,Q)=(1)}}1=\sum_{D|Q}\mu(D)T\Big(\Big[\frac{|r|^2|s|^2\Delta^2}{\mathrm{N}(U)\mathrm{N}(I_1)\mathrm{N}(D)}\Big]\Big) \tag{2.17}$$

$$\ll\sum_{D|Q}\frac{\mu(D)}{\mathrm{N}(D)}\,\frac{|r|^2|s|^2\Delta^2}{\mathrm{N}(U)\mathrm{N}(I_1)}-\sum_{D|Q}\mu(D)\Big\{\frac{|r|^2|s|^2\Delta^2}{\mathrm{N}(U)\mathrm{N}(I_1)\mathrm{N}(D)}\Big\}$$

$$\le\frac{|r|^2|s|^2\Delta^2}{\mathrm{N}(U)\mathrm{N}(I_1)}\prod_{P|Q}\Big(1-\frac{1}{\mathrm{N}(P)}\Big)+2^{\pi(t)}\qquad\text{as } t\to\infty,$$

where μ is the ideal version of the Möbius function, that is,

$$\mu(D)=\begin{cases}(-1)^k & \text{if } D=P_1\cdots P_k,\\ 0 & \text{if there exists } P \text{ such that } P^2\mid D,\end{cases}$$

and $T(\cdot)$ is the function we have used in the proof of Lemma 2.1. Next we use Mertens' theorem for algebraic number fields [R] to get

$$\lim_{t\to\infty}(\ln t)\prod_{\mathrm{N}(P)\le t}\left(1-\frac{1}{\mathrm{N}(P)}\right)=e^{-\gamma_{\mathbb{K}}}.$$

From this formula we have

$$\begin{aligned}(2.18)\qquad 2^{\pi(t)} &\le \frac{|r|^2|s|^2\varDelta^2}{\mathrm{N}(U)}\cdot\frac{1}{t^{\pi(t)}\ln t}\\ &\ll \frac{|r|^2|s|^2\varDelta^2}{\mathrm{N}(U)}\prod_{P\mid Q}\left(1-\frac{1}{\mathrm{N}(P)}\right)\frac{1}{t^{\pi(t)}}\qquad \text{as } t\to\infty.\end{aligned}$$

If $\mathrm{N}(I_1)\le t^{\pi(t)}$, we insert (2.17) and (2.18) into (2.16) to obtain

$$\begin{aligned}(2.19)\qquad \lambda(\mathcal{E}_{(r)}\cap\mathcal{E}_{(s)}) &\ll \delta^2\frac{\varPhi(U)}{\mathrm{N}(U)}|r|^2|s|^2\varDelta^2\prod_{P\mid Q}\left(1-\frac{1}{\mathrm{N}(P)}\right)\\ &\quad\cdot\prod_{P\in\mathcal{J}_1}\left(1-\frac{1}{\mathrm{N}(P)}\right)\left(1+\frac{\mathrm{N}(P)}{(\mathrm{N}(P)-1)^2}\right)\\ &\le \varPhi((r))\frac{\varPsi^2((r))}{|r|^2}\varPhi((s))\frac{\varPsi^2((s))}{|s|^2}\prod_{\substack{P\mid U\\ P\nmid R'S'\\ \mathrm{N}(P)\le t}}\left(1+\frac{1}{\mathrm{N}(P)(\mathrm{N}(P)-1)}\right)\\ &\ll \lambda(\mathcal{E}_{(r)})\lambda(\mathcal{E}_{(s)})\qquad \text{as } t\to\infty.\end{aligned}$$

If $\mathrm{N}(I_1)>t^{\pi(t)}$, then there exist a prime ideal $P\in\mathcal{J}_1$ and $\gamma\in\mathbb{Z}$ such that $P^\gamma\mid I_1$, $\mathrm{N}(P)\le t$, and $\mathrm{N}^\gamma(P)>t$. This implies that there exists an ideal D such that $D^2\mid I_1$ and $\mathrm{N}^2(D)\ge t^{2/3}$. Then we have

$$\begin{aligned}(2.20)\qquad \lambda(\mathcal{E}_{(r)}\cap\mathcal{E}_{(s)}) &\ll \delta^2\varPhi(U)\sum_{\substack{I_1\in\mathcal{I}_1\\ \mathrm{N}(I_1)>t^{\pi(t)}}}\ \sum_{\substack{K\\ 1\le\mathrm{N}(K)\le\frac{|r|^2|s|^2\varDelta^2}{\mathrm{N}(U)\mathrm{N}(I_1)}}}1\\ &\le \delta^2\varPhi(U)\sum_{\substack{D\\ [t^{1/3}]\le\mathrm{N}(D)<\infty}}\ \sum_{\substack{J\\ D^2\mid J\\ 1\le\mathrm{N}(J)\le\frac{|r|^2|s|^2\varDelta^2}{\mathrm{N}(U)}}}1\\ &\ll \delta^2\varDelta^2\frac{\varPhi(U)}{\mathrm{N}(U)}|r|^2|s|^2\sum_{\substack{D\\ [t^{1/3}]\le\mathrm{N}(D)<\infty}}\frac{1}{\mathrm{N}^2(D)}\qquad \text{as } t\to\infty.\end{aligned}$$

We use a method similar to the proof of Lemma 2.1 to estimate

$$\sum_{[t^{1/3}]\le \mathrm{N}(D)<\infty} \frac{1}{\mathrm{N}^2(D)} \ll \sum_{n=[t^{1/3}]}^{\infty} \frac{1}{n^2} \ll \frac{1}{t^{1/3}} \quad \text{as } t\to\infty.$$

We insert this estimate into (2.20) and with Corollary 2.2 we get

$$\begin{aligned}(2.21)\qquad \lambda(\mathcal{E}_{(r)}\cap\mathcal{E}_{(s)}) &\ll \Psi^2((r))\Psi^2((s))\frac{1}{t^{1/3}}\\ &\ll \Phi((r))\frac{\Psi^2((r))}{|r|^2}(\ln t)\Phi((s))\frac{\Psi^2((s))}{|s|^2}(\ln t)\frac{1}{t^{1/3}}\\ &\ll \lambda(\mathcal{E}_{(r)})\lambda(\mathcal{E}_{(s)}) \quad \text{as } t\to\infty.\end{aligned}$$

Together with (2.19) and (2.21), we conclude, in case A, that

$$(2.22)\qquad \sum_{\substack{|r|^2\in\mathbf{Z}\\ (r)\ne(s)}}\sum_{|s|^2\in\mathbf{Z}} \lambda(\mathcal{E}_{(r)}\cap\mathcal{E}_{(s)}) \ll \sum_{\substack{|r|^2\in\mathbf{Z}\\ (r)\ne(s)}}\sum_{|s|^2\in\mathbf{Z}} \lambda(\mathcal{E}_{(r)})\lambda(\mathcal{E}_{(s)}).$$

Case B: $t<b$ or $|r|^2|s|^2\Delta^2 < 3^t\mathrm{N}(U)$. Let $\eta_0 = e^{-\max\{b,C,v_0\}}$ and suppose $0<\Lambda(\mathbf{Z})\le\eta_0$. We set $L=\ln(1/\Lambda(\mathbf{Z}))$ and obtain

$$\begin{aligned}(2.23)\qquad \lambda(\mathcal{E}_{(r)}\cap\mathcal{E}_{(s)}) &\ll \Psi^2((r))\Psi^2((s))\frac{\Phi(U)}{\mathrm{N}(U)}\prod_{P\in\mathcal{J}_1}\left(1-\frac{1}{\mathrm{N}(P)}\right)\sum_{I_1\in\mathcal{I}_1}\frac{1}{\Phi(I_1)}\\ &< \Psi^2((r))\Psi^2((s))\prod_{\substack{P\mid U\\ P\notin\mathcal{J}_1}}\left(1-\frac{1}{\mathrm{N}(P)}\right)\\ &\quad\cdot\prod_{P\in\mathcal{J}_1}\left(1-\frac{1}{\mathrm{N}(P)}+\frac{1}{\mathrm{N}(P)(\mathrm{N}(P)-1)}-\frac{1}{(\mathrm{N}(P))^2(\mathrm{N}(P)-1)}\right)\\ &\ll \frac{\Psi^2((r))}{|r|^2}\Phi((r))\ln(1+g((r)))\frac{\Psi^2((s))}{|s|^2}\Phi((s))\ln(1+g((s)))\\ &\ll \lambda(\mathcal{E}_{(r)})\lambda(\mathcal{E}_{(s)})\ln^2(1+t) \quad \text{as } t\to\infty.\end{aligned}$$

If $t<L$, which implies $L\ge b$, then from (2.23) we deduce that

$$\begin{aligned}(2.24)\qquad \sum_{\substack{|r|^2\in\mathbf{Z}\\ (r)\ne(s)}}\sum_{|s|^2\in\mathbf{Z}} \lambda(\mathcal{E}_{(r)}\cap\mathcal{E}_{(s)}) &\ll \sum_{\substack{|r|^2\in\mathbf{Z}\\ (r)\ne(s)\\ t<L}}\sum_{|s|^2\in\mathbf{Z}} \lambda(\mathcal{E}_{(r)})\lambda(\mathcal{E}_{(s)})\ln^2(1+t)\\ &< \Lambda^2(\mathbf{Z})\left(\ln\left(1+\ln\frac{1}{\Lambda(\mathbf{Z})}\right)\right)^2\\ &\ll \Lambda^2(\mathbf{Z})\left(\ln\ln\frac{1}{\Lambda(\mathbf{Z})}\right)^2 \quad \text{as } t\to\infty.\end{aligned}$$

If $t \geq L$ and $\mathrm{N}(U) < |r|^2|s|^2\Delta^2 < 3^t\mathrm{N}(U)$, then

$$\sum_{\substack{|r|^2\in\mathbf{Z}\\(r)\neq(s)}}\sum_{|s|^2\in\mathbf{Z}}\lambda(\mathcal{E}_{(r)}\cap\mathcal{E}_{(s)}) \tag{2.25}$$
$$\ll \sum_{m=L}^{\infty}\sum_{n=1}^{m}\Big(\sum_{\substack{(r)^m\\|r|^2\in\mathbf{Z}\\ \mathrm{N}(U)<|r|^2|s|^2\Delta^2<3^m\mathrm{N}(U)}}\sum_{\substack{(s)^n\\|s|^2\in\mathbf{Z}}}\Psi^2((r))\Psi^2((s))\Big)$$
$$= \sum_{m=L}^{\infty}\sum_{n=1}^{m}\Big(\sum_{\substack{(s)^n\\|s|^2\in\mathbf{Z}}}\Psi^2((s))\sum_{\substack{(r)^m\\|r|^2\in\mathbf{Z}\\ \mathrm{N}(U)<|r|^2|s|^2\Delta^2<3^m\mathrm{N}(U)}}\Psi^2((r))\Big)$$
$$\ll \sum_{m=L}^{\infty}\sum_{n=1}^{m}\Big(\sum_{\substack{(s)^n\\|s|^2\in\mathbf{Z}}}\lambda(\mathcal{E}_{(s)})\ln(1+n)\sum_{\substack{(r)^m\\ \mathrm{N}(U)<|r|^2|s|^2\Delta^2<3^m\mathrm{N}(U)}}\Psi^2((r))\Big)$$
$$\ll \sum_{m=L}^{\infty}\ln(1+m)\sum_{n=1}^{m}\Big(\sum_{\substack{(s)^n\\|s|^2\in\mathbf{Z}}}\lambda(\mathcal{E}_{(s)})\sum_{\substack{(r)^m\\ \mathrm{N}(U)<|r|^2|s|^2\Delta^2<3^m\mathrm{N}(U)}}\Psi^2((r))\Big).$$

If $\Psi((r))|r|^{-1} \leq \Psi((s))|s|^{-1}$, then

$$\Delta = \Psi((s))|s|^{-1} \quad \text{and} \quad |r|^2|s|^2\Delta^2 = |r|^2\Psi^2((s)).$$

By using Lemma 4 with $\xi = 1/2$ and $e^{1/2}(1-\varepsilon) = 3/2$, we get

$$\sum_{\substack{(r)^m\\ \mathrm{N}(U)<|r|^2\Psi^2((s))<3^m\mathrm{N}(U)}}\Psi^2((r)) \ll C\sum_{\substack{(r)^m\\ \mathrm{N}(U)<|r|^2\Psi^2((s))<3^m\mathrm{N}(U)}}\frac{1}{|r|^2} \tag{2.26}$$
$$\ll C(\ln(1+n))(\ln 3^m)e^{-m^{\beta(1-\varepsilon)}}$$
$$\ll Cm(\ln(1+m))e^{-m^{3/2}},$$

where $C > 0$ is a constant which satisfies $\Psi((r)) \leq C|r|^{-1}$ for all principal ideals (r).

If $\Psi((r))|r|^{-1} > \Psi((s))|s|^{-1}$, then we can use the same approach of Vaaler's to divide the set $\mathbf{Z}$ into some small pieces, that is, let

$$W_j = \{e \in \mathbb{Z}[\omega] : C/2^{j+1} < |e|^2\Psi^2((e)) \leq C/2^j\}$$

with $j = 0, 1, 2\ldots$. For $r \in W_j$ and $\mathrm{N}(U) < |s|^2\Psi^2((r)) < 3^m\mathrm{N}(U)$, we see that

$$C|s|^2 2^{-j-1}3^{-m}\mathrm{N}^{-1}(U) < |r|^2 < C|s|^2 2^{-j}\mathrm{N}^{-1}(U).$$

From Lemma 2.5, we have

(2.27)
$$\sum_{\substack{(r)^m\\ \mathrm{N}(U)<|r|^2\Psi^2((s))<3^m\mathrm{N}(U)}} \Psi^2((r)) \le C\sum_{j=0}^{\infty}\frac{1}{2^j}\sum_{\substack{(r)^m\\ \frac{C|s|^2}{2^{j+1}3^m}\frac{1}{\mathrm{N}(U)}<|r|^2<\frac{C|s|^2}{2^j}\frac{1}{\mathrm{N}(U)}}}\frac{1}{|r|^2}$$
$$\ll C\sum_{j=0}^{\infty}\frac{1}{2^j}\ln(1+g((s)))\ln(3^m)e^{-v^{3/2}}$$
$$\ll Cm\ln(1+m)e^{-m^{3/2}}.$$

By using (2.25)–(2.27), we find that

(2.28)
$$\sum_{\substack{|r|^2\in\mathbf{Z}\\ (r)\neq(s)}}\sum_{|s|^2\in\mathbf{Z}}\lambda(\mathcal{E}_{(r)}\cap\mathcal{E}_{(s)})$$
$$\ll\sum_{m=L}^{\infty}\ln(1+m)\sum_{n=1}^{m}\Big(\sum_{\substack{(s)^n\\ |s|^2\in\mathbf{Z}}}\lambda(\mathcal{E}_{(s)})Cm(\ln(1+m))e^{-m^{3/2}}\Big)$$
$$< C\sum_{m=L}^{\infty}m\ln^2(1+m)e^{-m^{3/2}}\Big(\sum_{n=1}^{\infty}\sum_{\substack{(s)^n\\ |s|^2\in\mathbf{Z}}}\lambda(\mathcal{E}_{(s)})\Big)$$
$$\ll\frac{1}{e^L}\Lambda(\mathbf{Z})=\Lambda^2(\mathbf{Z}).$$

Then (2.24) and (2.28) imply

(2.29)
$$\sum_{\substack{|r|^2\in\mathbf{Z}\\ (r)\neq(s)}}\sum_{|s|^2\in\mathbf{Z}}\lambda(\mathcal{E}_{(r)}\cap\mathcal{E}_{(s)})\ll\Lambda^2(\mathbf{Z})\Big(\ln\ln\frac{1}{\Lambda(\mathbf{Z})}\Big)^2$$

in Case B. From (2.22) and (2.29), we get the assertion of Theorem 1.3. ∎

Proof of Theorem 1.2. Since $\sum\Phi((r))\Psi^2((r))|r|^{-2}=\infty$, by using a Gallagher type result for imaginary quadratic fields [NW], we conclude that $\lim_{N\to\infty}\lambda(\bigcup_{|r|^2=N}^{\infty}\mathcal{E}_{(r)})=0$ or 1.

Suppose $\lim_{N\to\infty}\lambda(\bigcup_{|r|^2=N}^{\infty}\mathcal{E}_{(r)})=0$. This also implies

(2.30)
$$\lim_{|r|^2\to\infty}\lambda(\mathcal{E}_{(r)})=0.$$

We can choose a large rational integer m where $\lambda(\bigcup_{|r|^2=m}^{\infty}\mathcal{E}_{(r)})\le\frac{1}{4}\eta$. Let $j=\max\{k,m\}$. From $\sum\Phi((r))\Psi^2((r))|r|^{-2}=\sum_{|r|^2=1}^{\infty}\lambda(\mathcal{E}_{(r)})=\infty$ and (2.30), it follows that there exists a finite subset $\mathbf{Z}$ of $\{j,j+1,\ldots\}$ such that

$\frac{2}{3}\eta \leq \Lambda(\mathbf{Z}) \leq \eta$. Since $\bigcup_{|r|^2\in\mathbf{Z}} \mathcal{E}_{(r)} \subseteq \bigcup_{|r|^2=m} \mathcal{E}_{(r)}$, we have

$$\begin{aligned}\frac{1}{4}\eta \geq \lambda\Big(\bigcup_{|r|^2=m} \mathcal{E}_{(r)}\Big) &\geq \lambda\Big(\bigcup_{|r|^2\in\mathbf{Z}} \mathcal{E}_{(r)}\Big)\\ &\geq \sum_{|r|^2\in\mathbf{Z}} \lambda(\mathcal{E}_{(r)}) - \frac{1}{2}\sum_{|r|^2\in\mathbf{Z}}\sum_{\substack{|s|^2\in\mathbf{Z}\\(r)\neq(s)}} \lambda(\mathcal{E}_{(r)}\cap\mathcal{E}_{(s)})\\ &\geq \Lambda(\mathbf{Z}) - \frac{1}{2}\Lambda(\mathbf{Z}) \geq \frac{1}{3}\eta,\end{aligned}$$

which is impossible. This implies $\lim_{N\to\infty} \lambda(\bigcup_{|r|^2=N}^{\infty} \mathcal{E}_{(r)}) \neq 0$, proving the assertion of Theorem 1.1. ∎

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