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Sums of three cubes, II

by

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1. Introduction. A heuristic application of the Hardy–Littlewood (circle) method suggests that the set of integers represented as the sum of three cubes of natural numbers should have positive density. Although intense effort over the past 75 years has delivered a reasonable approximation to this expectation, an unconditional proof remains elusive. However, each phase of progress has been accompanied by technological advances of value elsewhere in applications of the circle method, and so even modest advances remain of interest. The most recent progress [26] hinges on an extension of Vaughan's method [21] utilising smooth numbers, in which fractional moments of exponential sums are estimated non-trivially. In this paper, we make further progress on sums of three cubes by exploiting a new mean value estimate to improve earlier estimates for fractional moments of cubic smooth Weyl sums. Although these improvements are modest in scale, such estimates have found many applications (see, for example, [1], [5], [6]), and it seems reasonable to expect that our new bounds will also be of considerable utility.

We begin with a new lower bound for the number, $N(X)$, of integers not exceeding X which are the sum of three cubes of natural numbers.

THEOREM 1.1. *One has $N(X) \gg X^{\beta}$, where $\beta = 0.91709477$.*

Lower bounds for $N(X)$ are at least implicit in work of Hardy and Littlewood [10] from 1925. By developing methods based on diminishing ranges and their p-adic variants, Davenport [8] established the lower bound $N(X) \gg X^{13/15-\varepsilon}$, subsequently obtaining $N(X) \gg X^{47/54-\varepsilon}$ (see [9]). Thirty-five years later, Vaughan [19], [20] enhanced these methods, first proving that $N(X) \gg X^{8/9-\varepsilon}$, and later that $N(X) \gg X^{19/21-\varepsilon}$. His seminal introduction [21] of methods utilising smooth numbers led to the

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lower bound $N(X) \gg X^{11/12-\varepsilon}$ (see also Ringrose [18] for an intermediate result). The author's derivation of effective estimates for fractional moments of smooth Weyl sums [24] first delivered a lower bound of the shape $N(X) \gg X^{1-\xi/3-\varepsilon}$, where $\xi = 0.24956813\ldots$ denotes the positive root of the polynomial $\xi^3+16\xi^2+28\xi-8$. Subsequently, the author obtained a similar estimate in which $\xi = (\sqrt{2833}-43)/41 = 0.24941301\ldots$ (see [26]). With this value of ξ, one has $1-\xi/3 = 0.91686232\ldots$, which should be compared with the exponent 0.91709477 of Theorem 1.1. Subject to the truth of an unproved Riemann Hypothesis concerning a certain Hasse–Weil L-function, meanwhile, one has the conditional estimate $N(X) \gg X^{1-\varepsilon}$ due to Hooley [13], [14] and Heath-Brown [11].

Theorem 1.1 follows from an estimate for the sixth moment of a certain smooth Weyl sum. Define the set of R-smooth numbers of size at most P by

$$\mathcal{A}(P,R) = \{n \in [1,P] \cap \mathbb{Z} : p \mid n \text{ and } p \text{ prime} \Rightarrow p \le R\}.$$

Then, with $e(z) = e^{2\pi i z}$, we introduce the smooth and classical Weyl sums

$$f(\alpha;P,R) = \sum_{x\in\mathcal{A}(P,R)} e(\alpha x^3) \quad\text{and}\quad F(\alpha;P) = \sum_{1\le x\le P} e(\alpha x^3). \tag{1.1}$$

In §7 we establish the mean value estimate contained in the following theorem.

Theorem 1.2. *Write $\delta_6 = 0.24871567$. Then there exists a positive number η with the property that, whenever $R \le P^\eta$, one has*

$$\int_0^1 |F(\alpha;P)^2 f(\alpha;P,R)^4|\,d\alpha \ll P^{3+\delta_6}. \tag{1.2}$$

For comparison, [26, Theorem 1.2] yields a similar estimate with $\delta_6 = 0.24941301\ldots$, whilst the earlier work of Vaughan [21] provides an analogous sixth moment estimate for $f(\alpha;P,R)$ with associated exponent $\delta_6 = 1/4+\varepsilon$ for any $\varepsilon > 0$. Note that in many applications (see [5]–[7]), it is crucial that (1.2) hold with $\delta_6 < 1/4$, hence the significance of Theorem 1.2.

The bound (1.2) leads to improvement in estimates associated with the unrepresentation theory of Waring's problem for cubes. Let $E_s(X)$ denote the number of integers not exceeding X which are *not* the sum of s cubes of natural numbers. Then the arguments of Brüdern [3] and Kawada and Wooley [16] lead to the estimates recorded in the following theorem.

Theorem 1.3. *Write $\tau = \frac{2}{7}\left(\frac{1}{4} - 0.24871567\right) = 1/2725.15\ldots$. Then*

$$E_4(X) \ll X^{37/42-\tau}, \qquad E_5(X) \ll X^{5/7-\tau}, \qquad E_6(X) \ll X^{3/7-2\tau}.$$

The aforementioned work of Brüdern [3] yields the bound $E_4(X) \ll X^{37/42+\varepsilon}$, whilst Kawada and Wooley [16, Theorem 1.4] obtain a conclusion similar to that of Theorem 1.3, though with τ slightly smaller than $1/5962$. We will not discuss the (routine) proof of Theorem 1.3 further here, noting merely that the conclusion of Theorem 1.2 is the key input into the methods of [3].

We establish Theorem 1.2 as a consequence of estimates for the mean values

$$U_s(P,R)=\int_0^1 |f(\alpha;P,R)|^s\,d\alpha, \tag{1.3}$$

with $4\le s\le 8$. The iterative method of [24] obtains a bound for $U_s(P,R)$ in terms of corresponding bounds for $U_{s-2}(P,R)$ and $U_t(P,R)$, wherein t is a parameter to be chosen with $\frac{4}{3}(s-2)\le t\le 2(s-2)$. A key player in determining the strength of these estimates is an exponential sum of the shape

$$\widetilde{F}_1(\alpha)=\sum_{\substack{u\in\mathcal{A}(P^\theta R,R)\\ u>P^\theta}}\ \sum_{\substack{z_1,z_2\in\mathcal{A}(P,R)\\ z_1\equiv z_2\ (\mathrm{mod}\,u^3)\\ z_1\ne z_2}} e(\alpha u^{-3}(z_1^3-z_2^3)),$$

in which θ is a parameter with $0\le\theta\le 1/3$. This exponential sum is made awkward to handle by the constraint that the summands z_1 and z_2 be smooth. In this paper we estimate the auxiliary integral

$$\int_0^1 \widetilde{F}_1(\alpha)|f(\alpha;P^{1-\theta},R)|^{s-2}\,d\alpha$$

in terms of the mediating mean value

$$\int_0^1 |\widetilde{F}_1(\alpha)^2 f(\alpha;P^{1-\theta},R)^2|\,d\alpha.$$

By orthogonality, the latter counts the number of solutions of an underlying Diophantine equation. By discarding the smoothness constraint implicit in the sum $\widetilde{F}_1(\alpha)$, much of the strength of the Hardy–Littlewood method may be preserved in the ensuing minor arc estimate. After preparing an auxiliary estimate in §2, we analyse this new mean value in §3, and indicate in §4 how it may be utilised in the method of [24]. Ideas relevant for the estimation of the mean value $U_s(P,R)$ when $s=6$, and when $s>6.5$, are presented in §5.

The Keil–Zhao device (see [17, p. 608] and the discussion leading to [27, equation (3.10)]) enables us in §6 to obtain stronger minor arc estimates for smooth Weyl sums than available hitherto. When $\mathfrak{m}\subseteq[0,1)$, $0<t\le 2$ and

$s \geq 6$, this idea delivers an estimate of the shape

$$\int_{\mathfrak{m}} |f(\alpha;P,R)|^{s+t}\,d\alpha \ll P^{t/2}\Big(\sup_{\alpha\in\mathfrak{m}} |F(\alpha;P)|\Big)^{t/2} \int_0^1 |f(\alpha;P,R)|^s\,d\alpha,$$

in place of

$$\int_{\mathfrak{m}} |f(\alpha;P,R)|^{s+t}\,d\alpha \ll \Big(\sup_{\alpha\in\mathfrak{m}} |f(\alpha;P,R)|\Big)^{t} \int_0^1 |f(\alpha;P,R)|^s\,d\alpha.$$

The ease with which classical Weyl sums can be estimated on sets of minor arcs ensures that this device is of utility when s lies between 6 and 8. In particular, in §7 we explain how to improve [4, Theorem 2], which establishes that when R is a small enough power of P, then $U_s(P,R) \ll P^{s-3}$ for $s \geq 7.691$.

Theorem 1.4. *Suppose that $\eta > 0$ and P is sufficiently large in terms of η, and further that $R \leq P^\eta$. Then, provided that $s \geq 7.5906$, one has*

$$\int_0^1 |f(\alpha;P,R)|^s\,d\alpha \ll P^{s-3}.$$

Our estimates for the mean values $U_s(P,R)$ depend on those for $U_t(P,R)$ for appropriate choices of t. In §7, we describe how computations associated with this complicated iteration were performed, and discuss the extent to which the computed exponents reflect the sharpest available from this circle of ideas. These conclusions are summarised in the following theorem.

Theorem 1.5. *Let (s,δ_s,Δ_s) be a triple listed in Table 1. Suppose that $\eta > 0$ and P is sufficiently large in terms of η, and further that $R \leq P^\eta$. Then*

$$\int_0^1 |f(\alpha;P,R)|^s\,d\alpha \ll P^{s/2+\delta_s} \quad \text{and} \quad \int_0^1 |f(\alpha;P,R)|^s\,d\alpha \ll P^{s-3+\Delta_s}.$$

Exponents may be derived for values of s between those in the table by linear interpolation using Hölder's inequality. Values of δ_s and Δ_s computed in §7 have been rounded up, as appropriate, in the final decimal place recorded.

In this paper, we adopt the convention that whenever ε, P or R appear in a statement, either implicitly or explicitly, then for each $\varepsilon > 0$, there exists a positive number $\eta = \eta(\varepsilon)$ such that the statement holds whenever $R \leq P^\eta$ and P is sufficiently large in terms of ε and η. Implicit constants in Vinogradov's notation $\ll$ and $\gg$ will depend at most on ε and η. Since our iterative methods involve only a finite number of statements (depending at most on ε), there is no danger of losing control of implicit constants. Finally, write $\|\theta\| = \min_{y\in\mathbb{Z}} |\theta - y|$.

Table 1. Associated and permissible exponents for $4 \le s \le 8$

s	δ_s	Δ_s	s	δ_s	Δ_s
4.0	0.00000000	1.00000000	6.0	0.24871567	0.24871567
4.1	0.00130000	0.95130000	6.1	0.27667792	0.22667792
4.2	0.00495852	0.90495852	6.2	0.30598066	0.20598066
4.3	0.01069296	0.86069296	6.3	0.33718632	0.18718632
4.4	0.01811263	0.81811263	6.4	0.36984515	0.16984515
4.5	0.02685074	0.77685074	6.5	0.40263501	0.15263501
4.6	0.03754195	0.73754195	6.6	0.43542486	0.13542486
4.7	0.04903470	0.69903470	6.7	0.46851012	0.11851012
4.8	0.06130069	0.66130069	6.8	0.50330866	0.10330866
4.9	0.07426685	0.62426685	6.9	0.53863866	0.08863866
5.0	0.08780854	0.58780854	7.0	0.57423853	0.07423853
5.1	0.10328796	0.55328796	7.1	0.61131437	0.06131437
5.2	0.11894874	0.51894874	7.2	0.64881437	0.04881437
5.3	0.13477800	0.48477800	7.3	0.68631437	0.03631437
5.4	0.15076406	0.45076406	7.4	0.72381437	0.02381437
5.5	0.16689626	0.41689626	7.5	0.76131437	0.01131437
5.6	0.18316493	0.38316493	7.6	0.80000000	0.00000000
5.7	0.19954296	0.34954296	7.7	0.85000000	0.00000000
5.8	0.21593386	0.31593386	7.8	0.90000000	0.00000000
5.9	0.23232477	0.28232477	7.9	0.95000000	0.00000000

2. An auxiliary mean value estimate. Before announcing our pivotal mean value estimate, we introduce some notation. Let ϕ be a real number with $0 \le \phi \le 1/3$, and write

$$M = P^{\phi}, \qquad H = PM^{-3}, \qquad Q = PM^{-1}. \tag{2.1}$$

Define the exponential sums

$$F_1(\alpha) = \sum_{1 \le z \le 2P} \sum_{1 \le h \le H} \sum_{M < m \le MR} e(2\alpha h(3z^2 + h^2 m^6)), \tag{2.2}$$

$$D(\alpha) = \sum_{1 \le h \le H} \Big| \sum_{1 \le z \le 2P} e(6\alpha h z^2) \Big|^2,$$

$$E(\alpha) = \sum_{1 \le h \le H} \Big| \sum_{M < m \le MR} e(2\alpha h^3 m^6) \Big|^2. \tag{2.3}$$

Also, when $\mathfrak{B} \subseteq [0,1)$, we introduce the mean value

$$\Upsilon(P,R;\phi;\mathfrak{B}) = \int_{\mathfrak{B}} |F_1(\alpha)^2 f(\alpha;2Q,R)^2|\, d\alpha, \tag{2.4}$$

and then write $\Upsilon(P,R;\phi) = \Upsilon(P,R;\phi;[0,1))$. We observe that an application of Cauchy's inequality to (2.2) yields the bound $|F_1(\alpha)|^2 \le D(\alpha)E(\alpha)$.

Consequently, when $t \ge 2$, we obtain the estimate

$$\Upsilon(P,R;\phi;\mathfrak{B}) \le \int_{\mathfrak{B}} (D(\alpha)E(\alpha))^{2/t}\,|F_1(\alpha)|^{2-4/t}|f(\alpha;2Q,R)|^2\,d\alpha. \tag{2.5}$$

Recall the definition (1.3) of the mean value $U_s(P,R)$. We say that an exponent μ_s is *permissible* whenever it has the property that, with the notational conventions introduced above, one has $U_s(P,R) \ll P^{\mu_s+\varepsilon}$. It follows that, for each positive number s, a permissible exponent μ_s exists with $s/2 \le \mu_s \le s$. We refer to the exponent δ_s as *associated* when $\mu_s = s/2+\delta_s$ is permissible, and Δ_s as *admissible* when $\mu_s = s-3+\Delta_s$ is permissible.

We require a Hardy–Littlewood dissection. Let $\mathfrak{m}$ denote the set of points $\alpha \in [0,1)$ with the property that, whenever there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a,q)=1$ and $|q\alpha - a| \le PQ^{-3}$, then one has $q > P$. Further, let $\mathfrak{M} = [0,1) \setminus \mathfrak{m}$.

LEMMA 2.1. *Suppose that $t \ge 4$ and $0 \le \phi \le 1/3$. Then whenever δ_t is an associated exponent, one has*

$$\Upsilon(P,R;\phi;\mathfrak{m}) \ll P^{1+\varepsilon}MH^{1+2/t}Q^{1+2\delta_t/t}.$$

Proof. We ultimately work outside the range $0 \le \phi \le 1/7$ in which the estimate

$$\sup_{\alpha\in\mathfrak{m}} |F_1(\alpha)| \ll P^{\varepsilon}(PM)^{1/2}H$$

follows from [21, Lemmata 3.1 and 3.4], and so we engineer a hybrid method combining elements of the Hardy–Littlewood method with a Diophantine interpretation of auxiliary equations. We begin by applying Hölder's inequality to (2.5), obtaining the bound

$$\Upsilon(P,R;\phi;\mathfrak{m}) \le \Big(\sup_{\alpha\in\mathfrak{m}} D(\alpha)\Big)^{2/t} I_1^{2/t} I_2^{1-4/t} U_t(2Q,R)^{2/t}, \tag{2.6}$$

where $U_t(2Q,R)$ is defined via (1.3), and

$$I_1 = \int_0^1 E(\alpha)|F_1(\alpha)|^2\,d\alpha, \qquad I_2 = \int_0^1 |F_1(\alpha)|^2\,d\alpha. \tag{2.7}$$

The estimates

$$I_2 \ll P^{1+\varepsilon}MH \quad\text{and}\quad U_t(2Q,R) \ll Q^{t/2+\delta_t+\varepsilon} \tag{2.8}$$

follow, respectively, from [21, Lemma 2.3] with $j=1$ and the definition of an associated exponent. Also, given $\alpha \in [0,1)$, we find from [21, Lemma 3.1] that whenever $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a,q)=1$ and $|\alpha - a/q| \le q^{-2}$, then

$$D(\alpha) \ll P^{\varepsilon}\bigg(\frac{P^2H}{q+Q^3|q\alpha-a|} + PH + q + Q^3|q\alpha - a|\bigg). \tag{2.9}$$

By Dirichlet's theorem on Diophantine approximation, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $0 \le a \le q \le P^{-1}Q^3$, $(a,q) = 1$ and $|q\alpha - a| \le PQ^{-3}$. When $\alpha \in \mathfrak{m}$, it follows that $q > P$, and hence we deduce via (2.1) that

$$\sup_{\alpha \in \mathfrak{m}} D(\alpha) \ll P^{\varepsilon}(PH + P^{-1}Q^3) \ll P^{1+\varepsilon}H. \tag{2.10}$$

Finally, by reference to (2.2), (2.3) and (2.7), it follows from orthogonality that I_1 counts the number of integral solutions of the equation

$$h_0^3(n_1^6 - n_2^6) = h_1(3z_1^2 + h_1^2 m_1^6) - h_2(3z_2^2 + h_2^2 m_2^6), \tag{2.11}$$

with

$$1 \le h_0, h_1, h_2 \le H, \quad M < n_1, n_2, m_1, m_2 \le MR, \quad 1 \le z_1, z_2 \le 2P.$$

Let N_1 denote the number of solutions of (2.11) counted by I_1 in which $n_1 = n_2$, let N_2 denote the corresponding number in which $h_1 z_1^2 \ne h_2 z_2^2$, and let N_3 denote the number with $n_1 \ne n_2$ and $h_1 z_1^2 = h_2 z_2^2$. Thus $I_1 \le N_1 + N_2 + N_3$.

By orthogonality, it follows from (2.2) and (2.11) with $n_1 = n_2$ that

$$N_1 \le HMR \int_0^1 |F_1(\alpha)|^2 \, d\alpha,$$

and hence we deduce from (2.7) and (2.8) that

$$N_1 \ll P^{1+\varepsilon} M^2 H^2. \tag{2.12}$$

When $\mathbf{h}, \mathbf{m}, \mathbf{n}, \mathbf{z}$ is a solution of (2.11) counted by N_2, the integer

$$L = h_0^3(n_1^6 - n_2^6) - h_1^3 m_1^6 + h_2^3 m_2^6$$

is non-zero. There are $O(H^3(MR)^4)$ possible choices for L, and we find from (2.11) that for each fixed choice one has $3(h_1 z_1^2 - h_2 z_2^2) = L$. With h_1 and h_2 already fixed, it follows from [23, Lemma 3.5] that the number of possible choices for z_1 and z_2 is $O((h_1 h_2 |L| P)^{\varepsilon})$. Thus we conclude that

$$N_2 \ll P^{\varepsilon} H^3 M^4. \tag{2.13}$$

Finally, consider a solution $\mathbf{h}, \mathbf{m}, \mathbf{n}, \mathbf{z}$ counted by N_3. Given h_2 and z_2, an elementary estimate for the divisor function shows that the number of possible choices for h_1 and z_1 satisfying $h_1 z_1^2 = h_2 z_2^2$ is $O((HP)^{\varepsilon})$. Fix any one amongst these $O((HP)^{1+\varepsilon})$ possible choices for h_1, h_2, z_1, z_2. One finds from (2.11) that $h_0, \mathbf{m}, \mathbf{n}$ satisfy the equation

$$(h_1 m_1^2)^3 - (h_2 m_2^2)^3 = h_0^3(n_1^6 - n_2^6).$$

Since $n_1 \ne n_2$, the right hand side here is non-zero, and likewise the left hand side. Thus, again applying a divisor function estimate, it follows that for any one amongst the $O((MR)^2)$ possible choices for m_1 and m_2, there are $O(P^{\varepsilon})$ possible choices for h_0, $n_1 - n_2$ and $n_1^5 + n_1^4 n_2 + \cdots + n_2^5$. We deduce that there are just $O(P^\varepsilon)$ possible choices for h_0, n_1 and n_2, and thus

$$N_3 \ll P^\varepsilon (HP)^{1+\varepsilon}(MR)^2 \ll P^{1+3\varepsilon}HM^2. \tag{2.14}$$

On combining (2.12)–(2.14), we conclude via (2.1) that

$$I_1 \le N_1 + N_2 + N_3 \ll P^\varepsilon(PM^2H^2 + H^3M^4) \ll P^{1+\varepsilon}M^2H^2.$$

Substituting this estimate together with (2.8) and (2.10) into (2.6), we arrive at the upper bound

$$\Upsilon(P,R;\phi;\mathfrak{m}) \ll P^\varepsilon (PH)^{2/t}(PM^2H^2)^{2/t}(PMH)^{1-4/t}Q^{1+2\delta_t/t},$$

and the conclusion of the lemma follows with a modicum of computation. ∎

We require a complementary major arc estimate.

Lemma 2.2. *Suppose that $t \ge 4$ and $0 \le \phi \le 1/3$. Then whenever δ_t is an associated exponent, one has*

$$\Upsilon(P,R;\phi;\mathfrak{M}) \ll P^{1+\varepsilon}MH^{1+2/t}Q^{1+2\delta_t/t}.$$

Proof. The major arcs $\mathfrak{M}$ are contained in the union of the intervals

$$\mathfrak{M}(q,a) = \{\alpha \in [0,1) : |q\alpha - a| \le PQ^{-3}\},$$

with $0 \le a \le q \le P$ and $(a,q) = 1$. Define $\Delta(\alpha)$ for $\alpha \in [0,1)$ by

$$\Delta(\alpha) = (q + Q^3|q\alpha - a|)^{-1}$$

when $\alpha \in \mathfrak{M}(q,a) \subseteq \mathfrak{M}$, and otherwise by setting $\Delta(\alpha) = 0$. Then it follows from (2.9) that when $\alpha \in \mathfrak{M}$, one has

$$D(\alpha) \ll P^{2+\varepsilon}H\Delta(\alpha) + P^{1+\varepsilon}H. \tag{2.15}$$

We apply Hölder's inequality to (2.5), just as in the treatment of the mean value $\Upsilon(P,R;\phi;\mathfrak{m})$ in the proof of Lemma 2.1. Thus, by comparing (2.10) and (2.15), we obtain

$$\Upsilon(P,R;\phi;\mathfrak{M}) \ll P^\varepsilon\big(PMH^{1+2/t}Q^{1+2\delta_t/t} + (P^2HT)^{2/t}\Upsilon(P,R;\phi;\mathfrak{M})^{1-2/t}\big),$$

where

$$T = \int_{\mathfrak{M}} \Delta(\alpha)E(\alpha)|f(\alpha;2Q,R)|^2\,d\alpha. \tag{2.16}$$

Thus we infer that

$$\Upsilon(P,R;\phi;\mathfrak{M}) \ll P^{1+\varepsilon}MH^{1+2/t}Q^{1+2\delta_t/t} + P^{2+\varepsilon}HT. \tag{2.17}$$

In preparation for the estimation of T, we consider the mean value

$$T_0 = \int_0^1 E(\alpha)|f(\alpha;2Q,R)|^2\,d\alpha.$$

By reference to (2.3), it follows from orthogonality that T_0 counts the number of integral solutions of the equation

$$2h^3(n_1^6 - n_2^6) = x_1^3 - x_2^3,$$

with $1 \le h \le H$, $M < n_1, n_2 \le MR$ and $x_1, x_2 \in \mathcal{A}(2Q, R)$. Here, the number of diagonal solutions with $x_1 = x_2$ and $n_1 = n_2$ is $O(HMRQ)$. There are $O(H(MR)^2)$ possible choices for h, n_1 and n_2 with $2h^3(n_1^6 - n_2^6) \neq 0$. For each fixed such choice, an elementary estimate for the divisor function shows that there are $O(Q^\varepsilon)$ possible choices for $x_1 - x_2$ and $x_1^2 + x_1x_2 + x_2^2$, hence also for x_1 and x_2. Then we conclude via (2.1) that

$$T_0 \ll P^\varepsilon(HMQ + HM^2) \ll P^{1+\varepsilon}H. \tag{2.18}$$

On recalling (2.3), one finds that

$$E(\alpha)|f(\alpha; 2Q, R)|^2 = \sum_{l\in\mathbb{Z}} \psi(l)e(l\alpha),$$

where $\psi(l)$ denotes the number of solutions of the equation

$$2h^3(n_1^6 - n_2^6) + x_1^3 - x_2^3 = l,$$

with $1 \le h \le H$, $M < n_1, n_2 \le MR$ and $x_1, x_2 \in \mathcal{A}(2Q, R)$. In view of (2.18), one has $\psi(0) = T_0 \ll P^{1+\varepsilon}H$. Moreover,

$$\sum_{l\in\mathbb{Z}} \psi(l) = E(0)f(0; 2Q, R)^2 \ll H(MR)^2Q^2.$$

Then by applying [2, Lemma 2] within (2.16), we deduce via (2.1) that

$$T \ll Q^{\varepsilon-3}(P(P^{1+\varepsilon}H) + H(MR)^2Q^2) \ll P^{2\varepsilon}.$$

On substituting this estimate into (2.17), we conclude that

$$\Upsilon(P, R; \phi; \mathfrak{M}) \ll P^{1+\varepsilon}MH^{1+2/t}Q^{1+2\delta_t/t} + P^{2+\varepsilon}H.$$

The proof of the lemma is completed by noting that the relations (2.1) ensure that the second term on the right hand side here is majorised by the first. ∎

We finish this section by combining the conclusions of Lemmata 2.1 and 2.2.

LEMMA 2.3. *Suppose that $t \ge 4$ and $0 \le \phi \le 1/3$. Then whenever δ_t is an associated exponent, one has*

$$\int_0^1 |F_1(\alpha)^2 f(\alpha; 2Q, R)^2|\, d\alpha \ll P^{1+\varepsilon}MH^{1+2/t}Q^{1+2\delta_t/t}.$$

Proof. On recalling (2.4), the desired conclusion follows from Lemmata 2.1 and 2.2 by means of the relation

$$\Upsilon(P, R; \phi) = \Upsilon(P, R; \phi; \mathfrak{M}) + \Upsilon(P, R; \phi; \mathfrak{m}). \ ∎$$

3. Further auxiliary mean value estimates. We now introduce notation more closely aligned with the author's work [24]–[26] on fractional moments of smooth Weyl sums. We define the modified set of smooth numbers $\mathcal{B}(L,\pi,R)$ for prime numbers π by

$$\mathcal{B}(L,\pi,R)=\{n\in\mathcal{A}(L\pi,R): n>L,\ \pi\,|\,n,\ p\,|\,n \text{ and } p \text{ prime} \Rightarrow \pi\le p\}.$$

Recall the notation (2.1). We define the exponential sums

$$\widetilde{F}_{d,e}(\alpha;\pi)=\sum_{u\in\mathcal{B}(M/d,\pi,R)}\ \sum_{\substack{x,y\in\mathcal{A}(P/(de),R)\\ (x,u)=(y,u)=1\\ x\equiv y\ (\mathrm{mod}\ u^3)\\ y<x}} e(\alpha u^{-3}(x^3-y^3)), \tag{3.1}$$

$$F_{d,e}(\alpha)=\sum_{1\le z\le 2P/(de)}\ \sum_{1\le h\le Hd^2/e}\ \sum_{M/d<u\le MR/d} e(2\alpha h(3z^2+h^2m^6)) \tag{3.2}$$

and

$$\widetilde{f}(\alpha;P,M,R)=\max_{m>M}\Big|\sum_{x\in\mathcal{A}(P/m,R)} e(\alpha x^3)\Big|. \tag{3.3}$$

Note here that $F_{d,e}(\alpha)=0$ whenever $e>Hd^2$. Finally, we set

$$\Upsilon_{d,e,\pi}(P,R;\phi)=\int_0^1 |\widetilde{F}_{d,e}(\alpha;\pi)^2\widetilde{f}(\alpha;P/(de),M/d,\pi)^2|\,d\alpha. \tag{3.4}$$

We begin by demystifying the mean value $\Upsilon_{d,e,\pi}(P,R;\phi)$.

Lemma 3.1. *When $\pi\le R$, one has*

$$\Upsilon_{d,e,\pi}(P,R;\phi)\ll P^{\varepsilon}\int_0^1 |F_{d,e}(\alpha)^2 f(\alpha;2Q/e,R)^2|\,d\alpha.$$

Proof. We first eliminate the maximal aspect underlying the exponential sum $\widetilde{f}(\alpha;P/(de),M/d,\pi)$ implicit in $\Upsilon_{d,e,\pi}(P,R;\phi)$. Define

$$\mathcal{D}_K(\theta)=\sum_{|m|\le K^3} e(m\theta)\quad\text{and}\quad \mathcal{D}_K^*(\theta)=\min\{2K^3+1,\|\theta\|^{-1}\},$$

and note that for $K\ge 1$ one has

$$\int_0^1 \mathcal{D}_K^*(\theta)\,d\theta\ll \log(2K). \tag{3.5}$$

On recalling (2.1), we find that whenever $m>M$, then one has

$$\sum_{x\in\mathcal{A}(P/m,R)} e(\alpha x^3)=\int_0^1 f(\alpha+\theta;Q,R)\mathcal{D}_{P/m}(\theta)\,d\theta.$$

Since $\mathcal{D}_{P/m}(\theta) \ll \mathcal{D}^*_{P/m}(\theta) \le \mathcal{D}^*_Q(\theta)$ for $m > M$, we thus infer from (3.3) that

$$\widetilde{f}(\alpha; P/(de), M/d, \pi) \ll \int_0^1 |f(\alpha+\theta; Q/e, \pi)| \mathcal{D}^*_Q(\theta)\, d\theta. \tag{3.6}$$

On substituting (3.6) into (3.4), we deduce that

$$\Upsilon_{d,e,\pi}(P, R; \phi) \ll \int_{[0,1)^3} |\widetilde{F}_{d,e}(\alpha; \pi)^2 f_{\theta_1}(\alpha) f_{\theta_2}(\alpha)| \mathcal{D}^*_Q(\theta_1)\mathcal{D}^*_Q(\theta_2)\, d\theta_1\, d\theta_2\, d\alpha,$$

where, temporarily, we abbreviate $f(\alpha+\theta; Q/e, \pi)$ to $f_\theta(\alpha)$. Write

$$\Xi_{d,e,\pi}(\theta) = \int_0^1 |\widetilde{F}_{d,e}(\alpha; \pi)^2 f(\alpha+\theta; Q/e, \pi)^2|\, d\alpha. \tag{3.7}$$

Then by applying the inequality $|z_1 z_2| \le |z_1|^2 + |z_2|^2$ and invoking symmetry, we infer via (3.5) that

$$\begin{aligned}\Upsilon_{d,e,\pi}(P, R; \phi) &\ll \int_0^1 \Xi_{d,e,\pi}(\theta_1)\mathcal{D}^*_Q(\theta_1)\, d\theta_1 \int_0^1 \mathcal{D}^*_Q(\theta_2)\, d\theta_2 \\ &\ll Q^\varepsilon \int_0^1 \Xi_{d,e,\pi}(\theta_1)\mathcal{D}^*_Q(\theta_1)\, d\theta_1.\end{aligned} \tag{3.8}$$

Consider next the integral solutions of the equation

$$u_1^{-3}(x_1^3 - y_1^3) - u_2^{-3}(x_2^3 - y_2^3) = w_1^3 - w_2^3 \tag{3.9}$$

with, for $i = 1$ and 2, the constraints

$$w_i \in \mathcal{A}(Q/e, \pi), \quad u_i \in \mathcal{B}(M/d, \pi, R), \quad x_i, y_i \in \mathcal{A}(P/(de), R),$$
$$(x_i, u_i) = (y_i, u_i) = 1, \quad x_i \equiv y_i \pmod{u_i^3}, \quad y_i < x_i.$$

Then by orthogonality, it follows from (3.1) and (3.7) that the mean value $\Xi_{d,e,\pi}(\theta)$ counts the number of such solutions, with each solution counted with weight $e(\theta(w_2^3 - w_1^3))$. The latter weight being unimodular, it follows that $|\Xi_{d,e,\pi}(\theta)|$ is bounded above by the corresponding number of unweighted solutions, and hence by the number of integral solutions of the equation (3.9) with, for $i = 1$ and 2, the constraints

$$w_i \in \mathcal{A}(Q/e, R), \quad M/d < u_i \le MR/d,$$
$$1 \le y_i < x_i \le P/(de), \quad x_i \equiv y_i \pmod{u_i^3}.$$

We now substitute $z_i = x_i + y_i$ and $h_i = (x_i - y_i)u_i^{-3}$ $(i = 1, 2)$ into (3.9). It follows that $1 \le h_i \le (P/(de))(M/d)^{-3}$ for $i = 1$ and 2. Moreover, we have $2x_i = z_i + h_i u_i^3$ and $2y_i = z_i - h_i u_i^3$ $(i = 1, 2)$. Then on noting that

$$u^{-3}((z + hu^3)^3 - (z - hu^3)^3) = 2h(3z^2 + h^2u^6),$$

and recalling (2.1), we see that $|\Xi_{d,e,\pi}(\theta)|$ is bounded above by the number of integral solutions of the equation

$$2h_1(3z_1^2+h_1^2u_1^6)-2h_2(3z_2^2+h_2^2u_2^6)=w_1^3-w_2^3$$

with, for $i=1$ and 2,

$$\begin{gathered} w_i\in\mathcal{A}(2Q/e,R),\qquad M/d<u_i\le MR/d,\\ 1\le z_i\le 2P/(de),\qquad 1\le h_i\le Hd^2/e. \end{gathered}$$

Then on recalling (3.2), it follows by orthogonality that

$$|\Xi_{d,e,\pi}(\theta)|\le\int_0^1|F_{d,e}(\alpha)^2f(\alpha;2Q/e,R)^2|\,d\alpha.$$

On substituting this estimate into (3.8), we conclude that

$$\Upsilon_{d,e,\pi}(P,R;\phi)\ll Q^\varepsilon\Big(\int_0^1 D_Q^*(\theta)\,d\theta\Big)\int_0^1|F_{d,e}(\alpha)^2f(\alpha;2Q/e,R)^2|\,d\alpha.$$

The conclusion of the lemma now follows on applying the bound (3.5). ∎

Lemma 3.2. *Suppose that*

$$\pi\le R,\qquad 1\le d\le M,\qquad 1\le e\le\min\{Q,Hd^2\},\qquad 0\le\phi\le 1/3.$$

Then, whenever $t\ge 4$ and δ_t is an associated exponent, one has

$$\Upsilon_{d,e,\pi}(P,R;\phi)\ll d^{4/t}e^{-3-2/t}P^{1+\varepsilon}MH^{1+2/t}Q^{1+2\delta_t/t}.$$

Proof. A comparison of (2.2) and (3.2) reveals that, as a consequence of Lemma 2.3 in combination with (2.1), whenever $t\ge 4$ and $M^3\le P$, one has

$$\int_0^1|F_{1,1}(\alpha)^2f(\alpha;2Q,R)^2|\,d\alpha\ll P^{1+\varepsilon}MH^{1+2/t}Q^{1+2\delta_t/t}.\tag{3.10}$$

We apply this conclusion with $P/(de)$ in place of P and M/d in place of M. In view of the relations (2.1), we also have Hd^2/e in place of H and Q/e in place of Q. The hypotheses of the lemma concerning e and ϕ then ensure that

$$(M/d)^3(P/(de))^{-1}=e/(Hd^2)\le 1,$$

whence $(M/d)^3\le P/(de)$, confirming the validity of the estimate (3.10) with these substitutions. Hence we obtain the bound

$$\begin{aligned}\int_0^1|F_{d,e}(\alpha)^2f(\alpha;2Q/e,R)^2|\,d\alpha&\ll\Big(\frac{P}{de}\Big)^{1+\varepsilon}\Big(\frac{M}{d}\Big)\Big(\frac{Hd^2}{e}\Big)^{1+2/t}\Big(\frac{Q}{e}\Big)^{1+2\delta_t/t}\\ &\ll d^{4/t}e^{-3-2/t-2\delta_t/t}P^{1+\varepsilon}MH^{1+2/t}Q^{1+2\delta_t/t}.\end{aligned}$$

Since Lemma 3.1 establishes the relation

$$\Upsilon_{d,e,\pi}(P,R;\phi) \ll P^{\varepsilon}\int_0^1 |F_{d,e}(\alpha)^2 f(\alpha;2Q/e,R)^2|\,d\alpha,$$

the conclusion of the lemma follows on noting that $\delta_t \ge 0$. ∎

We also have need of estimates for the mean values

$$\Lambda^{(m)}_{d,e,\pi}(P,R;\phi) = \int_0^1 |\widetilde{F}_{d,e}(\alpha;\pi)|^{2m}\,d\alpha \quad (m=1,2). \tag{3.11}$$

LEMMA 3.3. *When $1 \le d \le M$, $1 \le e \le \min\{Q, Hd^2\}$ and $\pi \le R$, one has*

$$\Lambda^{(1)}_{d,e,\pi}(P,R;\phi) \ll P^{1+\varepsilon}HMe^{-2} \quad \text{and} \quad \Lambda^{(2)}_{d,e,\pi}(P,R;\phi) \ll P^{2+\varepsilon}H^3M^4e^{-5}.$$

Proof. These estimates are given by [24, equations (3.25) and (3.26)]. ∎

Finally, we recall an estimate for the mean value

$$\widetilde{U}_s(P,M,R) = \int_0^1 \widetilde{f}(\alpha;P,M,R)^s\,d\alpha. \tag{3.12}$$

LEMMA 3.4. *Suppose that $s > 1$ and δ_s is an associated exponent. Then whenever $P > M$ and $R > 2$, one has $\widetilde{U}_s(P,M,R) \ll_s (P/M)^{s/2+\delta_s+\varepsilon}$.*

Proof. This is immediate from [24, Lemma 3.2]. ∎

4. New associated exponents, I: $4 \le s \le 6.5$. We now convert the mean value estimates of §2 into new associated exponents by means of the ideas of [24, §§2–4]. Write

$$\Omega_{d,e,\pi}(P,R;\phi) = \int_0^1 |\widetilde{F}_{d,e}(\alpha;\pi)\widetilde{f}(\alpha;P/(de),M/d,\pi)^{s-2}|\,d\alpha, \tag{4.1}$$

and then set

$$\mathcal{U}_s(P,R) = \sum_{1\le d\le D}\sum_{\pi\le R}\sum_{1\le e\le Q} d^{2-s/2}e^{s/2-1}\Omega_{d,e,\pi}(P,R;\phi). \tag{4.2}$$

The relevant results from [24] are summarised in the following lemma.

LEMMA 4.1. *Suppose that $s > 4$ and $0 < \phi \le 1/3$. Then whenever μ_{s-2} and μ_s are permissible exponents, and $1 \le D \le P^{1/3}$, one has*

$$U_s(P,R) \ll P^{\mu_s+\varepsilon}D^{s/2-\mu_s} + MP^{1+\mu_{s-2}+\varepsilon} + P^{\left(\frac{s-3}{s-2}\right)\mu_s+\varepsilon}V_s(P,R),$$

where

$$V_s(P,R) = \left(PM^{s-2}Q^{\mu_{s-2}} + M^{s-3}\mathcal{U}_s(P,R)\right)^{1/(s-2)}.$$

Proof. The desired result follows at once on substituting the conclusion of [24, Lemma 3.3] into that of [24, Lemma 2.3]. ∎

We are now equipped to announce our new associated exponents.

LEMMA 4.2. *Suppose that $s \geq 4$ and $0 \leq \gamma \leq 1/4$, and let t satisfy*

$$\frac{2s-6+8\gamma}{1+2\gamma} \leq t \leq \frac{2s-4}{1+2\gamma}. \tag{4.3}$$

Suppose that δ_{s-2} and δ_t are associated exponents, and define

$$\theta_0 = \frac{2s-4-t+2(s-2)\delta_t - 2t\delta_{s-2}}{6s-12+t-4\gamma t+2(s-2)\delta_t - 2t\delta_{s-2}}. \tag{4.4}$$

Then the exponent $\delta_s = \delta_{s-2}(1-\theta) + \frac{1}{2}(s-2)\theta$ is associated, where we write $\theta = \max\{0, \min\{\theta_0, 1/3\}\}$.

Proof. We begin by estimating the mean value $\Omega_{d,e,\pi}(P,R;\phi)$. Suppose that

$$d \leq M, \quad e \leq \min\{Q, Hd^2\}, \quad \pi \leq R, \quad 0 \leq \phi \leq 1/3.$$

Then on recalling (3.4), (3.11) and (3.12), an application of Hölder's inequality to (4.1) yields the bound

$$\Omega_{d,e,\pi}(P,R;\phi) \leq \Upsilon_{d,e,\pi}(P,R;\phi)^{\gamma_1} \widetilde{U}_t(P/(de), M/d, \pi)^{\gamma_2} \times \Lambda^{(1)}_{d,e,\pi}(P,R;\phi)^{\gamma_3} \Lambda^{(2)}_{d,e,\pi}(P,R;\phi)^{\gamma}, \tag{4.5}$$

where

$$\gamma_1 = \tfrac{1}{4}(2s-4-t-2t\gamma), \quad \gamma_2 = (s-2-2\gamma_1)/t, \quad \gamma_3 = \tfrac{1}{2} - \gamma_1 - 2\gamma.$$

A few words are in order to confirm that the above is indeed a valid application of Hölder's inequality. Observe first that the hypotheses $s > 4$ and $0 \leq \gamma \leq 1/4$, together with those concerning the value t, ensure that

$$2s-6+8\gamma \leq t(1+2\gamma) \leq 2s-4,$$

so that

$$0 \leq \gamma_1 \leq \tfrac{1}{4}\big((2s-4)-(2s-6+8\gamma)\big) = \tfrac{1}{2}(1-4\gamma) \leq 1.$$

Hence we deduce that

$$0 = \tfrac{1}{2} - \tfrac{1}{2}(1-4\gamma) - 2\gamma \leq \gamma_3 \leq \tfrac{1}{2} - 2\gamma < 1.$$

Also, since $s \geq 4$ and $\gamma_1 \leq \frac{1}{2}(1-4\gamma)$, one finds that

$$\gamma_2 \geq (s-3+4\gamma)/t > 0.$$

Moreover, since $t \geq (2s-6+8\gamma)/(1+2\gamma)$, we have

$$(1+2\gamma)(s-2-2\gamma_1-t) \leq 4-s-2\gamma_1-\gamma(12+4\gamma_1-2s).$$

When $4 \leq s \leq 6$, we therefore deduce that

$$t(1+2\gamma)(\gamma_2-1) \leq 4-s-2\gamma_1 \leq 0,$$

and when $s > 6$ we see instead that

$$t(1+2\gamma)(\gamma_2 - 1) \le 4 - s - 2\gamma_1 + \tfrac{1}{4}(2s-12) \le 1 - \tfrac{1}{2}s \le 0.$$

Thus, in all circumstances, one has $0 \le \gamma_2 \le 1$. Finally, the relations

$$\gamma + \gamma_1 + \gamma_2 + \gamma_3 = 1, \quad 4\gamma + 2\gamma_1 + 2\gamma_3 = 1, \quad 2\gamma_1 + t\gamma_2 = s - 2 \tag{4.6}$$

follow by direct computation.

By applying Lemmata 3.2–3.4, we deduce from (4.5) that

$$\begin{aligned}\Omega_{d,e,\pi}(P,R;\phi) \ll P^{\varepsilon}(d^{4/t}e^{-3-2/t}PMH^{1+2/t}Q^{1+2\delta_t/t})^{\gamma_1} \\ \times (PMHe^{-2})^{\gamma_3}(P^2M^4H^3e^{-5})^{\gamma}((Q/e)^{t/2+\delta_t})^{\gamma_2}.\end{aligned}$$

Thus, by making use of the relations (4.6) and

$$t \ge 2, \quad \gamma_1 \le \tfrac{1}{2}, \quad 3\gamma_1 + \tfrac{1}{2}t\gamma_2 + 2\gamma_3 + 5\gamma \ge \tfrac{1}{2}s, \quad 2\gamma_1 + t\gamma = s - 2 - \tfrac{1}{2}t,$$

we deduce that

$$\Omega_{d,e,\pi}(P,R;\phi) \ll de^{-s/2}P^{1/2+\varepsilon}M^{1/2+2\gamma}H^{(s-2)/t}Q^{s/2-1+(s-2)\delta_t/t}. \tag{4.7}$$

When $e > Hd^2$, one has $F_{d,e}(\alpha) = 0$, and hence $\Omega_{d,e,\pi}(P,R;\phi) = 0$. Thus, on substituting (4.7) into (4.2), we discern that

$$\mathcal{U}_s(P,R) \ll P^{1/2+\varepsilon}M^{1/2+2\gamma}H^{(s-2)/t}Q^{s/2-1+(s-2)\delta_t/t}\Sigma_0,$$

where

$$\Sigma_0 = \sum_{1\le d\le D}\sum_{\pi\le R}\sum_{1\le e\le \min\{Q,Hd^2\}} d^{3-s/2}e^{-1}.$$

We therefore conclude that

$$\mathcal{U}_s(P,R) \ll D^2P^{1/2+2\varepsilon}M^{1/2+2\gamma}H^{(s-2)/t}Q^{s/2-1+(s-2)\delta_t/t}.$$

In the notation of Lemma 4.1, therefore, we have

$$V_s(P,R)^{s-2} \ll P^{\varepsilon}M^{s-3}(\Psi_1 + D^2\Psi_2),$$

where

$$\Psi_1 = PMQ^{\mu_{s-2}} \quad \text{and} \quad \Psi_2 = P^{1/2}M^{1/2+2\gamma}H^{(s-2)/t}Q^{s/2-1+(s-2)\delta_t/t}.$$

On recalling (2.1) and the definition of an associated exponent, the equation $\Psi_1 = \Psi_2$ implicitly determines a linear equation for ϕ, namely

$$\begin{aligned}&1 + \phi + \left(\frac{1}{2}(s-2) + \delta_{s-2}\right)(1-\phi) \\ &= \frac{1}{2} + \left(\frac{1}{2} + 2\gamma\right)\phi + \left(\frac{s-2}{t}\right)(1-3\phi) + \left(\frac{1}{2}(s-2) + \left(\frac{s-2}{t}\right)\delta_t\right)(1-\phi).\end{aligned}$$

A modicum of computation reveals that this equation has solution $\phi = \theta_0$, where θ_0 is given by (4.4). Let $D = P^{\omega}$, where ω is any sufficiently small, but fixed, positive number. Then we may follow the discussion of [24, §4] to confirm via Lemma 4.1 that whenever $\mu_{s-2} = \frac{1}{2}(s-2)+\delta_{s-2}$ and $\mu_t = \frac{1}{2}t+\delta_t$ are permissible exponents, then so too is

$$\mu_s = \mu_{s-2}(1-\theta) + 1 + (s-2)\theta.$$

It follows that the exponent $\delta_s = \delta_{s-2}(1-\theta) + \frac{1}{2}(s-2)\theta$ is associated. ∎

We highlight three special cases of Lemma 4.2 for future use.

Corollary 4.3. *Suppose that $4 < s \le 5$. Then whenever $\delta_{2s-4} \le 2$ is an associated exponent, so too is $\delta_s = \frac{1}{2}(s-2)\theta$, where*

$$\theta = \frac{\delta_{2s-4}}{4+\delta_{2s-4}}.$$

Proof. We take $\gamma = 0$ and $t = 2s-4$, so that γ and t satisfy (4.3). It follows from Hua's lemma [22, Lemma 2.5] that

$$\int_0^1 |f(\alpha;Q,R)|^4\,d\alpha \ll Q^{2+\varepsilon},$$

and hence one may take $\delta_u = 0$ for $0 < u \le 4$. With these choices of s, γ and t, one finds that $\delta_{s-2} = 0$, and hence (4.4) gives

$$\theta_0 = \frac{2(s-2)\delta_t}{8s-16+2(s-2)\delta_t} = \frac{\delta_{2s-4}}{4+\delta_{2s-4}}.$$

But $0 \le \delta_{2s-4} \le 2$, and hence $0 \le \theta_0 \le 1/3$. The conclusion of the corollary is now immediate from Lemma 4.2. ∎

Corollary 4.4. *Suppose that $5 \le s \le 6$. Then whenever $\delta_6 \le 3/2$ is an associated exponent, so too is $\delta_s = \frac{1}{2}(s-2)\theta$, where*

$$\theta = \frac{s-5+(s-2)\delta_6}{3s-3+(s-2)\delta_6}.$$

Proof. We take $\gamma = 0$ and $t = 6$, so that s, γ and t satisfy (4.3). We again have $\delta_u = 0$ for $0 < u \le 4$, and hence $\delta_{s-2} = 0$. Thus (4.4) gives

$$\theta_0 = \frac{2s-10+2(s-2)\delta_6}{6s-6+2(s-2)\delta_6} = \frac{s-5+(s-2)\delta_6}{3s-3+(s-2)\delta_6}.$$

But by hypothesis, one has $0 \le \delta_6 \le 3/2$ and $5 \le s \le 6$, and hence

$$0 \le \theta_0 \le \frac{1+4\delta_6}{15+4\delta_6} \le \frac{1}{3}.$$

The conclusion of the corollary therefore follows from Lemma 4.2. ∎

Corollary 4.5. *Suppose that $6 \le s \le 13/2$. Then whenever $\delta_{s-2} \le \delta_6 \le 1/2$ is an associated exponent, so too is $\delta_s = \delta_{s-2}(1-\theta) + \frac{1}{2}(s-2)\theta$, where*

$$\theta = \frac{s-5+(s-2)\delta_6-6\delta_{s-2}}{33-3s+(s-2)\delta_6-6\delta_{s-2}}.$$

Proof. We take $\gamma = \frac{1}{2}(s-6)$, so that when $6 \le s \le 13/2$, one has $0 \le \gamma \le 1/4$, and in addition

$$\frac{2s-6+8\gamma}{1+2\gamma} = 6 \quad \text{and} \quad \frac{2s-4}{1+2\gamma} = 2 + \frac{6}{s-5} \ge 6.$$

We are therefore entitled to apply Lemma 4.2 with $t = 6$, in which case

$$\begin{aligned}\theta_0 &= \frac{2s-10+2(s-2)\delta_6 - 12\delta_{s-2}}{6s-6-12(s-6)+2(s-2)\delta_6 - 12\delta_{s-2}} \\ &= \frac{s-5+(s-2)\delta_6 - 6\delta_{s-2}}{33-3s+(s-2)\delta_6-6\delta_{s-2}}.\end{aligned}$$

By hypothesis, we have $6 \le s \le 13/2$ and $\delta_{s-2} \le \delta_6 \le 1/2$, and hence

$$\theta_0 \ge \frac{s-5-2\delta_6}{2s+3+(s-2)\delta_6-6\delta_{s-2}} \ge \frac{s-6}{2s+3+(s-2)\delta_6-6\delta_{s-2}} \ge 0,$$

and

$$\theta_0 \le \frac{s-5+\frac{9}{2}\delta_6}{13-2\delta_6} \le \frac{\frac{3}{2}+\frac{9}{4}}{12} < \frac{1}{3}.$$

The conclusion of the corollary therefore follows from Lemma 4.2. ∎

5. New associated exponents, II: $s = 6$ and $6.5 < s \le 8$. We turn next to methods yielding associated exponents when $s = 6$, and when $s > 6.5$, beginning with one generalising that of [26, Lemma 2.2].

Lemma 5.1. *Let t be a real number with $4 < t \le 8$. Then whenever $\delta_6 \le 2/3$ and $\delta_t \le \frac{1}{6}(t-4)$ are associated exponents, then so too is*

$$\delta_6' = 2\max\left\{\frac{8-t+8\delta_t}{24+t+8\delta_t}, \frac{\delta_6}{4+\delta_6}\right\}. \tag{5.1}$$

Moreover,

$$\int_0^1 |F(\alpha;P)^2 f(\alpha;P,R)^4|\, d\alpha \ll P^{3+\delta_6'+\varepsilon}. \tag{5.2}$$

Proof. On considering the Diophantine equation underlying (1.3), one sees that

$$U_6(P,R) \ll \int_0^1 |F(\alpha;P)^2 f(\alpha;P,R)^4|\, d\alpha.$$

Consequently, the confirmation of the estimate (5.2) suffices to establish that the exponent δ_6' defined in (5.1) is associated. We set

$$\phi = \max\left\{\frac{8-t+8\delta_t}{24+t+8\delta_t}, \frac{\delta_6}{4+\delta_6}\right\}.$$

Our hypotheses concerning t, δ_t and δ_6 ensure that

$$\frac{8-t+8\delta_t}{24+t+8\delta_t} \le \frac{8-t+\frac{4}{3}(t-4)}{24+t+\frac{4}{3}(t-4)} = \frac{8+t}{56+7t} = \frac{1}{7}, \qquad \frac{\delta_6}{4+\delta_6} \le \frac{2/3}{4+2/3} = \frac{1}{7},$$

so that $0 \le \phi \le 1/7$. Recall the definitions (2.1) and (2.2), and define $\mathfrak{m}$ and $\mathfrak{M}$ as in the preamble to Lemma 2.1. Also, when $\mathfrak{B} \subseteq [0,1)$, define

$$I(\mathfrak{B}) = \int_{\mathfrak{B}} |F_1(\alpha)f(\alpha;2Q,R)^4|\,d\alpha. \tag{5.3}$$

Then [24, inequality (5.3)] yields the estimate

$$\int_0^1 |F(\alpha;P)^2 f(\alpha;P,R)^4|\,d\alpha \ll P^\varepsilon M^3\big(PMQ^2 + I([0,1))\big). \tag{5.4}$$

We begin with a discussion of the minor arc contribution $I(\mathfrak{m})$. By applying Hölder's inequality to (5.3), one obtains

$$I(\mathfrak{m}) \ll U_t(2Q,R)^{4/t}\Big(\int_{\mathfrak{m}} |F_1(\alpha)|^{t/(t-4)}\,d\alpha\Big)^{1-4/t}. \tag{5.5}$$

Since we may assume that δ_t is an associated exponent, we have

$$U_t(2Q,R) \ll Q^{t/2+\delta_t+\varepsilon}.$$

Also, on recalling that $0 \le \phi \le 1/7$, it follows from [24, inequality (5.4)] together with the argument of the proof of [21, Lemma 3.7] that

$$\begin{aligned}\int_{\mathfrak{m}} |F_1(\alpha)|^{t/(t-4)}\,d\alpha &\le \Big(\sup_{\alpha\in\mathfrak{m}} |F_1(\alpha)|\Big)^{\frac{8-t}{t-4}} \int_0^1 |F_1(\alpha)|^2\,d\alpha \\ &\ll P^\varepsilon\left((PM)^{1/2}H\right)^{\frac{8-t}{t-4}} PMH.\end{aligned}$$

Thus we deduce from (5.5) that

$$I(\mathfrak{m}) \ll P^\varepsilon (PM)^{1/2} H^{4/t} Q^{2+4\delta_t/t}. \tag{5.6}$$

In order to estimate $I(\mathfrak{M})$, we have merely to follow the argument leading to [26, equation (2.10)]. Thus, again making use of the fact that $0 \le \phi \le 1/7$, the estimate preceding [26, equation (2.10)] gives

$$\begin{aligned}I(\mathfrak{M}) &\ll P^{1+\varepsilon}HM(PQ^{-2})^{2/3}(Q^5)^{1/3} \\ &\quad + P^{1+\varepsilon}HM^{1/2}(PQ^{-2})^{1/2}(Q^{3+\delta_6})^{1/2} \\ &\ll P^{1+\varepsilon}HMQ\big((PQ^{-1})^{2/3} + (P(QM)^{-1})^{1/2}Q^{\delta_6/2}\big).\end{aligned} \tag{5.7}$$

By combining (5.6) and (5.7), we obtain an estimate for $I([0,1))$. By substituting this into (5.4) and recalling (2.1), we deduce that

$$\int_0^1 |F(\alpha;P)^2 f(\alpha;P,R)^4|\,d\alpha \ll P^{3+\varepsilon}M^2(1+\Phi_1+\Phi_2+\Phi_3),$$

where

$$\Phi_1 = (PM)^{-1/2}H^{4/t}Q^{4\delta_t/t}, \quad \Phi_2 = M^{-4/3}, \quad \Phi_3 = M^{-2}Q^{\delta_6/2}.$$

In view of (2.1), one finds that the respective conditions

$$\phi \geq \frac{8-t+8\delta_t}{24+t+8\delta_t} \quad \text{and} \quad \phi \geq \frac{\delta_6}{4+\delta_6}$$

ensure that $\Phi_1 \leq 1$ and $\Phi_3 \leq 1$. Thus, our choice of ϕ ensures that

$$\int_0^1 |F(\alpha;P)^2 f(\alpha;P,R)^4|\,d\alpha \ll P^{3+\varepsilon}M^2 = P^{3+2\phi+\varepsilon},$$

confirming the estimate (5.2) and completing the proof of the lemma. ∎

We also recall an estimate for associated exponents δ_s of use for $s > 13/2$.

Lemma 5.2. *Suppose $s > 4$. Then whenever $\delta_{s-2} \leq 1/4$ and $\delta_{4(s-2)/3} \leq 1$ are associated exponents, so too is $\delta_s = \delta_{s-2}(1-\theta) + \frac{1}{2}(s-2)\theta$, where*

$$\theta = \frac{1+3\delta_{4(s-2)/3} - 4\delta_{s-2}}{9+3\delta_{4(s-2)/3} - 4\delta_{s-2}}.$$

Proof. This is immediate from [1, Corollary to Lemma 2]. ∎

Finally, we recall a simple consequence of convexity.

Lemma 5.3 ([4, Lemma 4.3]). *Suppose $s > 2$ and $t < s$. Then, whenever δ_{s-t} and δ_{s+t} are associated exponents, so too is $\delta_s = \frac{1}{2}(\delta_{s+t} + \delta_{s-t})$.*

6. The Keil–Zhao device. Lilu Zhao [27, equation (3.10)] has observed that, in wide generality, one may obtain an estimate of Weyl type for an exponential sum over an arbitrary set, provided this sum inhabits an appropriate mean value. The same idea is applied also in independent work of Keil [17, p. 608]. This observation is useful in obtaining permissible exponents μ_s when $s > 6$. Before announcing our conclusions, we introduce some notation useful in their proofs. Write

$$g(\alpha;P,R) = \sum_{\substack{x\in\mathcal{A}(P,R)\\ x>P/2}} e(\alpha x^3) \quad \text{and} \quad G(\alpha) = \sum_{P/2<x\leq P} e(\alpha x^3). \tag{6.1}$$

Lemma 6.1. *Suppose that $s \geq 6$ and the exponent Δ_s is admissible. Suppose also that $\frac{1}{16}(8-s) \leq \Delta_s \leq 1/4$ and $u > s+8\Delta_s$. Then there exist positive* *numbers η and c, depending at most on u, with the following property. Whenever P is sufficiently large in terms of η, and* $\exp(c(\log\log P)^2) \le R \le P^\eta$, *then*

$$\int_0^1 |f(\alpha; P, R)|^u \, d\alpha \ll P^{u-3}. \tag{6.2}$$

In particular, the exponent $\mu_w = w - 3$ is permissible for $w \ge u$.

Proof. We seek to show that whenever $v \ge s + 8\Delta_s$, then

$$\int_0^1 |f(\alpha; P, R)|^v \, d\alpha \ll P^{v-3+\varepsilon}. \tag{6.3}$$

When $u > v$, the bound (6.2) follows from this estimate via [4, Lemma 4.5]. Next, by applying a dyadic dissection, we deduce from (1.1) and (6.1) that

$$f(\alpha; P, R) = \sum_{\substack{j=0 \\ 2^j \le \sqrt{P}}}^{\infty} g(\alpha; 2^{-j}P, R) + O(\sqrt{P}),$$

whence an application of Hölder's inequality reveals that

$$\begin{aligned}\int_0^1 |f(\alpha; P, R)|^v \, d\alpha &\ll (\log P)^{v-1} \sum_{\substack{j=0 \\ 2^j \le \sqrt{P}}}^{\infty} \int_0^1 |g(\alpha; 2^{-j}P, R)|^v \, d\alpha + P^{v/2} \\ &\ll P^\varepsilon \max_{\sqrt{P} \le X \le P} \int_0^1 |g(\alpha; X, R)|^v \, d\alpha + P^{v/2}.\end{aligned}$$

Consequently, provided we are able to show that

$$\int_0^1 |g(\alpha; P, R)|^v \, d\alpha \ll P^{v-3+\varepsilon}, \tag{6.4}$$

the bound (6.3) follows. Henceforth, we abbreviate $g(\alpha; P, R)$ to $g(\alpha)$.

We establish (6.4) via the Hardy–Littlewood method. When $1 \le X \le P$, define the major arcs $\mathfrak{M}(X)$ to be the union of the intervals

$$\mathfrak{M}(q, a; X) = \{\alpha \in [0, 1) : |q\alpha - a| \le XP^{-3}\},$$

with $0 \le a \le q \le X$ and $(a, q) = 1$. Also, set $\mathfrak{m}(X) = [0, 1) \setminus \mathfrak{M}(X)$. Finally, write $\mathfrak{P} = \mathfrak{M}(P^{4/5})$, $\mathfrak{Q} = \mathfrak{M}(P^{3/8})$, $\mathfrak{p} = \mathfrak{m}(P^{4/5})$ and $\mathfrak{q} = \mathfrak{m}(P^{3/8})$.

We begin by observing that, as a consequence of [4, Corollary 3.2], one has

$$\int_{\mathfrak{Q}} |f(\alpha; P, R)|^6 \, d\alpha + \int_{\mathfrak{Q}} |f(\alpha; P/2, R)|^6 \, d\alpha \ll P^{3+\varepsilon},$$

so that

$$\int_{\mathfrak{Q}} |g(\alpha)|^6 \, d\alpha \ll P^{3+\varepsilon}.$$

Since $|g(\alpha)| = O(P)$, we find that whenever $v \geq 6$, one has

$$\int_{\mathfrak{Q}} |g(\alpha)|^v \, d\alpha \ll P^{v-3+\varepsilon}. \tag{6.5}$$

Suppose next that $\alpha \in \mathfrak{q}$. By Dirichlet's theorem on Diophantine approximation, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a,q) = 1$, $q \leq P^{11/5}$ and $|q\alpha - a| \leq P^{-11/5}$. An application of [4, Lemma 2.2] in concert with [4, equation (2.1)] delivers the estimate

$$g(\alpha) \ll \frac{q^{\varepsilon-1/6} P(\log P)^{5/2+\varepsilon}}{(1 + P^3|\alpha - a/q|)^{1/3}} + P^{9/10+\varepsilon}.$$

When $\alpha \in \mathfrak{p}$, it follows that $q > P^{4/5}$, and thus $g(\alpha) \ll P^{9/10+\varepsilon}$. Meanwhile, when $\alpha \in \mathfrak{P} \cap \mathfrak{q}$, we have either $q > P^{3/8}$ or $|q\alpha - a| > P^{-21/8}$, and hence $|g(\alpha)| \ll P^{15/16+\varepsilon}$. Consequently, since $\mathfrak{q} = \mathfrak{p} \cup (\mathfrak{P} \cap \mathfrak{q})$, we conclude that

$$\sup_{\alpha \in \mathfrak{q}} |g(\alpha)| \ll P^{15/16+\varepsilon}. \tag{6.6}$$

We now turn to the main task at hand. Suppose that $s \geq 6$ and that Δ_s is an admissible exponent. We consider the mean value

$$T_0 = \int_{\mathfrak{q}} |g(\alpha)|^{s+2} \, d\alpha. \tag{6.7}$$

By reference to (6.1), an application of Cauchy's inequality shows that

$$T_0 = \sum_{\substack{x \in \mathcal{A}(P,R) \\ x > P/2}} \sum_{\substack{y \in \mathcal{A}(P,R) \\ y > P/2}} \int_{\mathfrak{q}} |g(\alpha)|^s e(\alpha(x^3 - y^3)) \, d\alpha \leq P T_1^{1/2}, \tag{6.8}$$

where

$$T_1 = \sum_{\substack{P/2 < x,y \leq P \\ x,y \in \mathcal{A}(P,R)}} \left| \int_{\mathfrak{q}} |g(\alpha)|^s e(\alpha(x^3 - y^3)) \, d\alpha \right|^2.$$

We bound T_1 above by removing the condition $x, y \in \mathcal{A}(P,R)$, obtaining

$$T_1 \leq \sum_{P/2 < x,y \leq P} \int_{\mathfrak{q}} \int_{\mathfrak{q}} |g(\alpha)g(\beta)|^s e((\alpha - \beta)(x^3 - y^3)) \, d\alpha \, d\beta.$$

Thus, again recalling (6.1), we deduce by means of (6.8) that

$$T_0^2 \leq P^2 \int_{\mathfrak{q}} \int_{\mathfrak{q}} |g(\alpha)g(\beta)|^s |G(\alpha - \beta)|^2 \, d\alpha \, d\beta. \tag{6.9}$$

We analyse the mean value on the right hand side of (6.9) by applying the Hardy–Littlewood method. Let $\mathfrak{N} = \mathfrak{M}(P^{3/4})$ and $\mathfrak{n} = \mathfrak{m}(P^{3/4})$. Denote by $\kappa(q)$ the multiplicative function defined on prime powers by taking

$$\kappa(p^{3l})=p^{-l},\quad \kappa(p^{3l+1})=3p^{-l-1/2},\quad \kappa(p^{3l+2})=p^{-l-1}\qquad (l\ge 0).$$

Also, define the function $\Upsilon(\gamma)$ for $\gamma\in\mathfrak{N}$ by taking

$$\Upsilon(\gamma)=\kappa(q)^2(1+P^3|\gamma-a/q|)^{-1} \tag{6.10}$$

when $\gamma\in\mathfrak{M}(q,a;P^{3/4})\subseteq\mathfrak{N}$, and $\Upsilon(\gamma)=0$ when $\gamma\in\mathfrak{n}$. Then it follows from [15, Lemma 2.1] that $G(\gamma)^2\ll P^2\Upsilon(\gamma)+P^{3/2+\varepsilon}$. Substituting this estimate into (6.9), we deduce that

$$T_0^2\ll P^{7/2+\varepsilon}\Big(\int_0^1|g(\alpha)|^s\,d\alpha\Big)^2+P^4T_2, \tag{6.11}$$

where

$$T_2=\int_{\mathfrak{q}}\int_{\mathfrak{q}}\Upsilon(\alpha-\beta)|g(\alpha)g(\beta)|^s\,d\alpha\,d\beta.$$

By applying the trivial inequality $|z_1\cdots z_n|\le|z_1|^n+\dots+|z_n|^n$, we find that

$$|g(\alpha)g(\beta)|^s\ll|g(\alpha)g(\beta)^{s-1}|^2+|g(\beta)g(\alpha)^{s-1}|^2.$$

Hence, by symmetry, we obtain the estimate

$$T_2\ll\Big(\sup_{\beta\in\mathfrak{q}}|g(\beta)|\Big)^{s-4}\int_{\mathfrak{q}}\int_{\mathfrak{q}}\Upsilon(\alpha-\beta)|g(\beta)^{s+2}g(\alpha)^2|\,d\alpha\,d\beta.$$

By invoking (6.6), we thus deduce that

$$T_2\ll(P^{15/16+\varepsilon})^{s-4}\int_{\mathfrak{q}}|g(\beta)|^{s+2}\int_0^1\Upsilon(\alpha-\beta)|g(\alpha)|^2\,d\alpha\,d\beta. \tag{6.12}$$

On recalling the definitions (6.1) and (6.10), we discern that

$$\int_0^1\Upsilon(\alpha-\beta)|g(\alpha)|^2\,d\alpha=\int_{\mathfrak{N}}\Upsilon(\gamma)|g(\gamma+\beta)|^2\,d\gamma\le\sum_{1\le q\le P^{3/4}}\kappa(q)^2\Lambda(q),$$

where

$$\Lambda(q)=\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{-P^{-9/4}}^{P^{-9/4}}(1+P^3|\theta|)^{-1}\Big|\sum_{\substack{x\in\mathcal{A}(P,R)\\x>P/2}}e(x^3(\beta+\theta+a/q))\Big|^2\,d\theta.$$

Let $c_q(n)$ be Ramanujan's sum, which we define by

$$c_q(n)=\sum_{\substack{a=1\\(a,q)=1}}^{q}e(an/q).$$

Then it follows that

$$\sum_{\substack{a=1\\(a,q)=1}}^{q}\Big|\sum_{\substack{x\in\mathcal{A}(P,R)\\x>P/2}}e(x^3(\beta+\theta+a/q))\Big|^2=\sum_{\substack{P/2<x,y\le P\\x,y\in\mathcal{A}(P,R)}}c_q(x^3-y^3)e((\beta+\theta)(x^3-y^3)).$$

Thus, the well-known estimate $|c_q(n)|\le(q,n)$ yields the bound

$$\Lambda(q)\le\sum_{1\le x,y\le P}(q,x^3-y^3)\int_{-P^{-9/4}}^{P^{-9/4}}(1+P^3|\theta|)^{-1}\,d\theta,$$

and consequently

$$\int_0^1\Upsilon(\alpha-\beta)|g(\alpha)|^2\,d\alpha\ll P^{-3}\log(2P)\sum_{1\le q\le P^{3/4}}\kappa(q)^2\sum_{1\le x,y\le P}(q,x^3-y^3).$$

From here, the treatment following [4, equation (3.2)] delivers the upper bound

$$\int_0^1\Upsilon(\alpha-\beta)|g(\alpha)|^2\,d\alpha\ll P^{\varepsilon-1}. \tag{6.13}$$

Next, substituting (6.13) into (6.12), we infer that

$$T_2\ll P^{\varepsilon-1}(P^{15/16})^{s-4}\int_{\mathfrak{q}}|g(\beta)|^{s+2}\,d\beta.$$

In view of (6.7) and (6.11), the hypothesis that Δ_s is admissible yields

$$T_0^2\ll P^{7/2+\varepsilon}(P^{s-3+\Delta_s})^2+P^{3+\varepsilon}(P^{15/16})^{s-4}T_0,$$

whence

$$T_0\ll P^{s-1+\varepsilon}(P^{\Delta_s-1/4}+P^{-(s-4)/16}).$$

On recalling (6.7), application of Hölder's inequality and the trivial estimate $|g(\alpha)|\le P$ delivers the upper bound

$$\begin{aligned}\int_{\mathfrak{q}}|g(\alpha)|^v\,d\alpha&\le P^{v-(s+8\Delta_s)}T_0^{4\Delta_s}\Big(\int_0^1|g(\alpha)|^s\,d\alpha\Big)^{1-4\Delta_s}\\&\ll P^{v-s-8\Delta_s+\varepsilon}(P^{s+\Delta_s-5/4}+P^{s-(s+12)/16})^{4\Delta_s}(P^{s-3+\Delta_s})^{1-4\Delta_s}.\end{aligned}$$

Thus we deduce that whenever $\Delta_s\ge\frac{1}{16}(8-s)$, then

$$\int_{\mathfrak{q}}|g(\alpha)|^v\,d\alpha\ll P^{v-3+\varepsilon}(1+P^{-\Delta_s+(8-s)/16})^{4\Delta_s}\ll P^{v-3+\varepsilon}.$$

But the latter condition on s is ensured by the hypotheses of the lemma, and thus we conclude via (6.5) that

$$\int_0^1 |g(\alpha)|^v\,d\alpha = \int_{\mathfrak{Q}} |g(\alpha)|^v\,d\alpha + \int_{\mathfrak{q}} |g(\alpha)|^v\,d\alpha \ll P^{v-3+\varepsilon}.$$

This confirms the estimate (6.4), and the conclusion of the lemma follows. ■

7. Computations. We now address the problem of how to implement the computation of associated exponents δ_s for $4 \le s \le 8$. Let h be a small positive number that we view as a step size, and put $J = \lceil 16/h \rceil$. It is convenient in what follows to assume that $1/h \in \mathbb{N}$. We begin with an array of known associated exponents δ_{jh} $(0 \le j \le J)$. Thus, we have the associated exponents $\delta_4 = 0$ and $\delta_s = \frac{1}{2}s - 3$ $(s \ge 8)$ which follow from Hua's lemma (see [22, Lemma 2.5]). Making use also of the associated exponent $\delta_6 = 1/4$ due to Vaughan [21, Theorem 4.4], one may apply convexity to deliver the associated exponents

$$\delta_s = \max\bigl\{0, \tfrac{1}{8}(s-4), \tfrac{3}{8}s - 2, \tfrac{1}{2}s - 3\bigr\}.$$

For the interesting values of j with $4 < jh < 8$, one may now calculate new associated exponents δ_{jh} by means of Lemmata 4.2, 5.1–5.3 and 6.1. Here, we note that associated exponents δ_s are related to admissible exponents Δ_s by means of the relation $\delta_s = \frac{1}{2}s - 3 + \Delta_s$. Should any of these new associated exponents be superior to the old ones, they may be substituted into the array of values δ_{jh}. By iterating this process for $4/h < j < 8/h$, one derives new associated exponents converging to some set of limiting values.

We summarise the formulae delivered by the above-cited lemmata as follows.

(i) *Method* $A_s(t,\gamma)$. We apply Lemma 4.2 for $\gamma = lh$ and $t = mh$ with $0 \le l \le (4h)^{-1}$ and

$$\frac{2jh - 6 + 8lh}{1 + 2lh} \le mh \le \frac{2jh - 4}{1 + 2lh}. \tag{7.1}$$

Thus one finds that the exponent δ'_{jh} is associated, where

$$\delta'_{jh} = \delta_{jh-2}(1-\theta) + \tfrac{1}{2}(jh-2)\theta, \tag{7.2}$$

in which $\theta = \max\{0, \min\{\theta_0, 1/3\}\}$, and

$$\theta_0 = \frac{2jh - 4 - mh + 2(jh-2)\delta_{mh} - 2mh\delta_{jh-2}}{6jh - 12 + mh - 4(lh)(mh) + 2(jh-2)\delta_{mh} - 2mh\delta_{jh-2}}.$$

(ii) *Method* $B_6(t)$. We apply Lemma 5.1 for $t = mh$ with $4 < mh \le 8$. Thus, when $\delta_{mh} \le \frac{1}{6}(mh-4)$, we find that the exponent δ'_6 is associated, where

$$\delta'_6 = 2\max\left\{\frac{8 - mh + 8\delta_{mh}}{24 + mh + 8\delta_{mh}}, \frac{\delta_6}{4+\delta_6}\right\}.$$

(iii) *Method* C_s. First, if i is the integer for which $\frac{4}{3}(j-2/h) \in (i, i+1]$, then convexity provides the associated exponent

$$\delta_{4(jh-2)/3} = \big(i+1-\tfrac{4}{3}(j-2/h)\big)\delta_{ih} + \big(\tfrac{4}{3}(j-2/h)-i\big)\delta_{(i+1)h}.$$

Next, Lemma 5.2 shows the exponent δ'_{jh} given by (7.2) to be associated, where

$$\theta_0 = \frac{1+3\delta_{4(jh-2)/3}-4\delta_{jh-2}}{9+3\delta_{4(jh-2)/3}-4\delta_{jh-2}}.$$

(iv) *Process* $L_s(t)$. We apply Lemma 5.3 for $t = mh$ with $1 \le m \le 1/h$. Thus one finds that the exponent δ'_{jh} is associated, where $\delta'_{jh} = \frac{1}{2}(\delta_{(j+m)h} + \delta_{(j-m)h})$.

(v) *Process* W_s. We apply Lemma 6.1. Thus one finds that $\delta'_{jh} = \frac{1}{2}jh-3$ is an associated exponent whenever δ_{jh-mh} is associated and satisfies

$$3-\tfrac{1}{2}(j-m)h+\delta_{jh-mh} < \tfrac{1}{8}mh.$$

We wrote a straightforward computer program to implement this iterative process. Our language of choice was the QB64 implementation of QuickBasic, running on a Windows Surface Pro3 in Windows 8.1 (Intel Core i3 processor at 1.5 GHz). All parameters were stored using double-precision variables. The most time consuming method to apply is process $A_s(t,\gamma)$, since there are many possible choices for $t = mh$ and $\gamma = lh$ to test. It is apparent that γ should be chosen as small as possible consistent with the constraint (7.1). However, applying process $A_s(t,\gamma)$ for each eligible value of $s = jh$ $(4 < s < 8)$ nonetheless has running time with order of growth h^{-2}. This limited our computation, in the first instance, to a step size of $h \ge 10^{-4}$.

Experimentation with this iteration makes it apparent that certain of the processes dominate the others for different values of s. By refining the program to select dominant processes for different ranges of s, the running time is vastly improved to order of growth h^{-1}. Note that the array size limit effective for QB64 on the platform employed was at least $2\cdot 10^8$. Thus, final computations with step size $h = 10^{-6}$ were feasible for $4 < s \le 6.5$, and step size $h = 10^{-5}$ throughout $4 < s \le 8$, this being limited only by running-time considerations rather than memory limitations. We summarise below the parameters associated with these dominant processes.

(i) $4 < s \le 5$. Process $A_s(2s-4,0)$, so that δ'_s is determined according to Corollary 4.3. Thus δ'_{jh} is given by (7.2) with

$$\theta_0 = \frac{\delta_{2jh-4}}{4+\delta_{2jh-4}}.$$

(ii) $5 < s \le 5.6462$. Process $A_s(6,0)$, so that δ'_s is determined according to Corollary 4.4. Thus δ'_{jh} is given by (7.2) with

$$\theta_0 = \frac{jh - 5 + (jh-2)\delta_6}{3jh - 3 + (jh-2)\delta_6}.$$

(iii) $5.6462 < s < 6$. Process $L_s(t)$, linear interpolation between $\delta_{5.6462}$ and δ_6.

(iv) $s = 6$. Process $B_6(5.392938)$.

(v) $6 < s \le 6.081$. Process $L_s(t)$, linear interpolation between δ_6 and $\delta_{6.081}$.

(vi) $6.081 < s \le 6.3395$. Process $A_s\big(6, \frac{1}{2}(s-6)\big)$, so that δ'_s is determined according to Corollary 4.5. Thus δ'_{jh} is given by (7.2) with

$$\theta_0 = \frac{jh - 5 + (jh-2)\delta_6 - 6\delta_{jh-2}}{33 - 3jh + (jh-2)\delta_6 - 6\delta_{jh-2}}.$$

(vii) $6.3395 < s \le 6.5$. Process $L_s(t)$, linear interpolation between $\delta_{6.3395}$ and $\delta_{6.5}$.

(viii) $6.5 < s \le 7.06$. Processes C_s and $L_s(t)$.

(ix) $7.06 < s < 8$. Processes W_s and $L_s(t)$.

Some additional discussion seems warranted concerning the robustness of these computations. The first point to make is that, while the above restricted iteration may not be guaranteed to deliver optimal estimates, the exponents that it delivers will at least be legitimate associated exponents. Thus the exponents presented in Table 1 in the introduction may be considered upper bounds for optimal associated exponents. In this context, it is worth noting that we experimented with adjustments to the step size h, and found no improvement in the first eight digits of the decimal expansions of the computed values of δ_s, even when h varied from 10^{-4} to 10^{-6}.

The second point concerns the stability of the iteration. There is a potential danger in iterations involving large numbers of cycles that round-off errors may accumulate, leading to substantial cumulative errors and even to unstable iterative processes. In our computations, we exercised some caution concerning this issue by artificially inflating the newly computed associated exponents by adding a small positive quantity τ at the end of each iteration. Thus, with $\tau = 10^{-9}$, we replaced the newly computed associated exponent δ_s by $\delta_s + \tau$. This has the effect of slightly weakening our exponents, though round-off errors (which in double-precision arithmetic are very much smaller) are swamped by this cushion of numerical security. This device has the effect of permitting some control on the number of decimal digits reliably computed.

We now interpret these computations in the context of the conclusions presented in the introduction. First, Theorem 1.2 follows from the computed associated exponent $\delta_t = 0.14963020$ for $t = 5.392938$ that in turn follows from the computations underlying Table 1 via convexity, and the upper bound (5.2) of Lemma 5.1. Next, the exponent $\Delta_{7.1} = 0.06131437$ is admissible, according to Theorem 1.5 and the associated Table 1. Then it follows from Lemma 6.1 that

$$\int_0^1 |f(\alpha; P, R)|^u \, d\alpha \ll P^{u-3}$$

whenever $u > 7.1 + 8\Delta_{7.1} = 7.59051\ldots$. This establishes Theorem 1.4. Finally, the proof of Theorem 1.1 is a standard consequence of Theorem 1.2, following an application of Cauchy's inequality. The proof of [26, Theorem 1.1] to be found in the final phases of [26, §2] shows, for example, that whenever δ_6 is an associated exponent, then $N(X) \gg X^{1-\delta_6/3-\varepsilon}$. The conclusion of Theorem 1.1 therefore follows on making use of the associated exponent $\delta_6 = 0.24871567$. Note also that Theorem 1.5 for $s = 4$ follows from [12].

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