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The mean fourth power of real character sums

by

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1. Introduction. All real characters are given by the Kronecker symbol $\left(\frac{q}{n}\right)$, which gives a real character of modulus $|q|$. We denote by $S(X)$ the set of all real non-principal characters of modulus at most X.

The mean value estimate

$$\sum_{\chi\in S(X)}\Big|\sum_{n\le Y}\chi(n)\Big|^2 \ll XY\log^8 X$$

for real character sums was first proved by M. Jutila [J1] in 1973. This estimate is best possible up to the exponent of $\log X$. Several authors, including Jutila, have observed that the method of [J1] in fact yields the exponent two. The best known estimate for this mean square is due to M. V. Armon [Ar], where the exponent of $\log X$ is one.

In his paper [J2] Jutila made also the following conjecture concerning higher powers of the character sums.

Conjecture. *For all $k = 1, 2, \ldots$ and $X \ge 3$, $Y \ge 1$ the estimate*

$$S_k(X,Y) = \sum_{\chi\in S(X)}\Big|\sum_{n\le Y}\chi(n)\Big|^{2k} \le c_1(k)XY^k(\log X)^{c_2(k)}$$

holds with certain coefficients $c_1(k)$, $c_2(k)$ depending on k.

The purpose of this paper is to prove Jutila's conjecture in the case $k = 2$ in a slightly weaker form.

Theorem. *For $X \ge 3$ and $Y \ge 1$, we have*

$$S_2(X,Y) = \sum_{\chi\in S(X)}\Big|\sum_{n\le Y}\chi(n)\Big|^4 \ll_\varepsilon XY^2X^\varepsilon,$$

where the implied constant depends on ε.

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We shall first restrict the outer sum to primitive characters and the result is easy to generalize to all real characters afterwards.

The proof is quite easy when Y is "small" or "large" compared with X. We shall see that the critical size of Y is $X^{1/2+\varepsilon} \ll Y \ll X$. It is also clear that the n-sum can be restricted to $n \asymp N$.

The idea is to use the reflection principle (see [I, p. 122]). By a suitable smooth weight function, we can reformulate the sum approximately in an analytical form, and "reflect" it into a shorter sum, which is easier to estimate. In fact we get sums whose lengths depend only on X. For these shorter sums, and also in the case $Y \ll X^{1/2+\varepsilon}$, we use an estimate due to D. R. Heath-Brown for the mean square of real character sums (see Lemma 1 below).

Introducing the weight function, we make a certain error. The error must be sufficiently small, and to see this we need some theory of uniform distribution.

We let ε stand for an arbitrary small positive number and C for a sufficiently large constant, not necessarily the same at each occurrence. The symbol $\square$ is used to denote a square integer.

2. Preliminary lemmas. To estimate the mean value of real character sums we shall use the following estimate due to Heath-Brown [HB, Corollary 2].

LEMMA 1. *Let N, X be positive integers, and let $a_1, \ldots, a_n$ be arbitrary complex numbers. Let $S^*(X)$ denote the set of all real primitive characters of conductor at most X. Then*

$$\sum_{\chi \in S^*(X)} \Big| \sum_{n \le N} a_n \chi(n) \Big|^2 \ll_\varepsilon (XN)^\varepsilon (X+N) \sum_{n_1 n_2 = \square} |a_{n_1} a_{n_2}|,$$

where the implied constant depends on ε.

This result essentially implies Jutila's estimate when $a_n = 1$ for all n and $N \le X$, but it is more general. It can be used for estimating also the fourth powers of character sums.

Let u_n be a sequence of real numbers and $0 < \delta \le 1/2$. We denote by $Z(N, \delta)$ the number of those u_n whose distance from the nearest integer is at most δ, that is, $\|u_n\| \le \delta$, when $1 \le n \le N$. If the sequence u_n is uniformly distributed modulo one, then

$$\lim_{N \to \infty} \frac{1}{N} Z(N, \delta) = 2\delta$$

for every $0 < \delta \le 1/2$. Define

$$D(N, \delta) = Z(N, \delta) - 2\delta N.$$

The number $D(N,\delta)$ is related to the discrepancy of the sequence u_n. Therefore the following estimate [Mo, p. 8] holds:

$$(1)\qquad |D(N,\delta)| \le \frac{N}{L+1} + 2\sum_{l=1}^{L}\left(\frac{1}{L+1} + \min\left(2\delta, \frac{1}{\pi l}\right)\right)\Big|\sum_{n=1}^{N} e(lu_n)\Big|,$$

where L is arbitrary positive integer.

LEMMA 2. *Let* $u_n = \sqrt{N/(N_1+n)}$, *where* $N_1 \le N$, *and* $0 < \delta \le 1/2$. *Then*

$$(2)\qquad Z(N_1,\delta) \ll \delta N_1 \quad \textit{if} \quad N^{1/2}N_1^{-3/2} \le 2\delta.$$

And if

$$(3)\qquad 2\delta < N^{1/2}N_1^{-3/2} \le 1/2,$$

then

$$(4)\qquad Z(N_1,\delta) \ll \sqrt{N/N_1} + \log N,$$

and always

$$(5)\qquad Z(N_1,\delta) \ll \delta N_1 + \frac{N_1}{L} + \left(\frac{N}{N_1}\right)^{1/4} L^{1/2} + \frac{N_1^{5/4}}{N^{1/4}},$$

where L is any positive integer.

Proof. The difference of two successive terms u_n is $\asymp N^{1/2}/N_1^{3/2}$. If this is $\le 2\delta$, we get (2) by a simple combinatorial calculation.

Let us then assume (3). Let $D(N_1,\delta)$ be as above and choose $L = \lfloor N_1^{3/2}/(2N^{1/2})\rfloor$ in (1). Now we can apply the following well-known estimate [Ti, Lemmas 4.3 and 4.8] to the exponential sum in (1):

$$\sum_{0<n\le N_1} e(lu_n) = \int_{N_1}^{2N_1} e\left(l\sqrt{\frac{N}{x}}\right)dx + O(1) \ll \frac{N_1^{3/2}}{lN^{1/2}},$$

to obtain the estimate

$$Z(N_1,\delta) \ll \delta N_1 + \frac{N_1}{L} + \frac{\delta N_1^{3/2}}{N^{1/2}}\sum_{l=1}^{L}\frac{1}{l} \ll \delta N_1 + \sqrt{\frac{N}{N_1}} + \log N,$$

which proves (4).

The last estimate follows when we use the estimate [Ti, Th. 5.9]

$$\ll \left(\frac{N}{N_1}\right)^{1/4} l^{1/2} + \frac{N_1^{5/4}}{N^{1/4}l^{1/2}}$$

for the exponential sum in (1). ∎

Lemma 3. *For $N^{1/3} \le N_0 \le N$, we have*

$$\Sigma_2 = \sum_{\substack{N-N_0 \le m,n \le N+N_0 \\ mn=\square}} 1 \ll N_0 \log N.$$

Proof. Write $n = n_1 a^2$ and $m = n_1 b^2$, where n_1 is square-free. Then

$$\sqrt{\frac{N-N_0}{n_1}} \le a, b \le \sqrt{\frac{N+N_0}{n_1}}$$

and the length of the interval is $\ll N_0/\sqrt{n_1 N}$. We distinguish some cases depending on the size of the number n_1.

If $n_1 \le N_0^2/N$, then $N_0/\sqrt{n_1 N} \gg 1$ and the length of the interval can be used directly to estimate the number of the numbers a and b. In this case there are at most

$$\sum_{n_1 \le N_0^2/N} \frac{N_0}{\sqrt{n_1 N}} \cdot \frac{N_0}{\sqrt{n_1 N}} \ll \frac{N_0^2}{N} \log N \ll N_0 \log N$$

pairs of numbers m, n.

If the length of the interval where the numbers a and b lie is smaller than one, that is, $n_1 > N_0^2/N$, then we count only those numbers n_1 which really give some integers a and b, and at the same time a pair of numbers m, n.

Let $N_1 < n_1 \le 2N_1$. The case $N_1 \ll N^{1/3}$ is clear, since there cannot be more pairs m, n than there are numbers n_1 and $N^{1/3} \le N_0$. Therefore we can assume that $N_1 \ge (4N)^{1/3}$.

We apply Lemma 2 with $\delta = N_0/\sqrt{NN_1}$. If $N/(2N_0) \le N_1$, then (2) gives the estimate $\ll N_0\sqrt{N_1/N} \le N_0$, and from (4) we get the estimate $\ll \sqrt{N/N_1} \ll N_0$ when $N/(2N_0) > N_1$.

Since the sum $\sum_{N_0^2/N < n_1 \le N}$ can be divided into parts of the form $\sum_{A < n_1 \le 2A}$ which all are, as we have seen, at most $O(N_0)$, the lemma is proved. ∎

Lemma 4. *For $N^{3/5} \le N_0 \le N$, we have*

$$\Sigma_4 = \sum_{\substack{N-N_0 \le m,n,r,s \le N+N_0 \\ mnrs=\square}} 1 \ll N_0^2 N^{\varepsilon}.$$

Proof. The idea of the proof is as above. We write $nrs = k$, and then

$$\Sigma_4 \le \sum_{N-N_0 \le m \le N+N_0} \sum_{\substack{(N-N_0)^4/m \le k \le (N+N_0)^4/m \\ mk=\square}} \mathrm{d}_3(k),$$

where d_3 is a divisor function in the standard notation. If $m = m_1 m_2^2$ with m_1 square-free, then k must be of the shape $m_1 x^2$. There are $\ll N_0\sqrt{N/m_1}$ numbers x, and m_2 lies in an interval of length $\ll \sqrt{N/m_1} \cdot N_0/N \ll$

$N_0/\sqrt{m_1 N}$. The cases $m_1 \le N_0^2/N$ and $m_1 \asymp M_1 \ge N/(2N_0)$ are similar to the previous lemma and the case $N_0^2/N < M_1 < N/(2N_0)$ follows from (5) by choosing $L = \lfloor N_0^2 M_1^{3/2}/N^{3/2} \rfloor$; note that $L \ge 1$ by our assumption on N_0. ■

LEMMA 5. *Let χ_q be a primitive real character modulo q and let $a = \frac{1}{2}(1-\chi_q(-1))$. The Dirichlet L-function satisfies the functional equation*

$$L(s,\chi_q) = \psi(s,\chi_q)L(1-s,\chi_q),$$

where

$$\psi(s,\chi_q) = 2^s \frac{G(\chi_q)}{i^a\sqrt{\pi q}}\left(\frac{\pi}{q}\right)^{s-1/2} \Gamma(1-s)\sin\frac{\pi}{2}(s+a),$$

and $G(\chi_q)$ is a Gaussian sum. Furthermore

$$\psi(s,\chi_q) \ll (q|s|)^{1/2-\sigma}, \tag{6}$$

when $\sigma \le 1/2$.

Proof. It is well-known that the L-function has the above functional equation. To verify the bound of the ψ-function, we only need some estimates for the Γ-function, since $|G(\chi_q)/(i^a\sqrt{\pi q})| = 1$.

By Stirling's formula we get the following well known estimates:

$$|\Gamma(s)| = \sqrt{2\pi}\, t^{\sigma-1/2}e^{-(\pi/2)t}(1+O(1/t)) \tag{7}$$

where σ is bounded and $t \to \infty$, and

$$\frac{\Gamma'}{\Gamma}(s) = \log s - \frac{1}{2s} + O\left(\frac{1}{|s|^2}\right),$$

where $|\arg s| \le \pi - \delta$, $|s| \ge \delta > 0$.

We can now write, for $\sigma \ge 1/2$,

$$\begin{aligned}
\log\left|\frac{\Gamma(\sigma+it)}{\Gamma(1/2+it)}\right| &= \operatorname{Re}\int_{1/2}^{\sigma} \frac{\Gamma'}{\Gamma}(u+it)\,du \\
&= \operatorname{Re}\int_{1/2}^{\sigma}\left(\log(u+it) - \frac{1}{2(u+it)} + O\left(\frac{1}{u^2+t^2}\right)\right)du \\
&= \frac{1}{2}\int_{1/2}^{\sigma}\log(u^2+t^2)\,du - \frac{1}{2}\log\frac{|\sigma+it|}{|1/2+it|} + O(1) \\
&< \frac{1}{2}\log[(\sigma^2+t^2)^{\sigma-1/2}] + O(1).
\end{aligned}$$

The above estimates give

$$|\Gamma(\sigma+it)| = \left|\Gamma\left(\frac{1}{2}+it\right)\right|\cdot\left|\frac{\Gamma(\sigma+it)}{\Gamma(1/2+it)}\right| \ll e^{-(\pi/2)t}|s|^{\sigma-1/2},$$

when $\sigma \ge 1/2$. Since $\sin \frac{\pi}{2}(s+a) \ll e^{(\pi/2)t}$, we have $\psi(s,\chi_q) \ll (q|s|)^{1/2-\sigma}$ when $\sigma \le 1/2$. ∎

3. Proof of the Theorem. We first prove the desired estimate if χ is restricted to primitive characters, that is, $\chi \in S^*(X)$.

Using the classical Pólya–Vinogradov estimate, we see that the case $Y \gg X$ is clear. If we first square out the sum and then use Lemma 1, we see that also the case $Y \ll X^{1/2+\varepsilon}$ is clear.

To estimate the sum when $X^{1/2+\varepsilon} \ll Y \ll X$ we use the reflection principle. We start with the familiar formula

$$e^{-x} = (2\pi i)^{-1} \int_{(c)} \Gamma(s) x^{-s}\, ds, \qquad x, c > 0.$$

Making the substitutions $x = Y^h$ and $s = w/h$, where $h > 1$, we get

$$e^{-Y^h} = (2\pi i)^{-1} \int_{(c)} \Gamma\left(1 + \frac{w}{h}\right) Y^{-w} w^{-1}\, dw.$$

Now let $Y = n/N$. Multiplying both sides by $\chi(n)$ and summing over n, we have

$$\sum_{n=1}^{\infty} \chi(n) e^{-(n/N)^h} = (2\pi i)^{-1} \int_{(c)} \Gamma\left(1 + \frac{s}{h}\right) \frac{N^s}{s} L(s,\chi)\, ds.$$

Consider the sum

$$\sum_{\chi \in S^*(X)} \Big| \sum_{N < n \le M} \chi(n) \Big|^4, \tag{8}$$

where $X^{1/2+\varepsilon} \ll N \asymp M \ll X$ and $S^*(X)$ as in Lemma 1. It is clear that the desired estimate for this sum implies the same estimate for the sum $S_2(X,Y)$ when $X^{1/2+\varepsilon} \ll Y \ll X$. We start with the weighted sum

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \chi(n) (e^{-(n/M)^h} - e^{-(n/N)^h}) \\ &= (2\pi i)^{-1} \int_{(c)} \Gamma\left(1 + \frac{s}{h}\right) \frac{M^s - N^s}{s} L(s,\chi)\, ds. \end{aligned}$$

Let us then move the integration line to $\sigma = -\varepsilon$, and use the functional equation for the L-function. We can cut the integration line at $|t| = T$, where $T = Ch \log X$, with a small error, since the Γ-function makes the integrand small when $|t| > T$. Now we divide the L-series, and hence the integral, into two parts $\sum_{n \le K}$ and $\sum_{n > K}$ where $K = CX^{1/2} \log^3 X$. Then we fix $h = (CM \log X)/X^{1/2}$, and move the integration line in the first integral to $\sigma = 1/2$, and in the second integral to $\sigma = -h/2$. By the choice of the parameters h, K and T we see that all the integrals over the horizontal segments are $o(1)$. So we have

$$S = (2\pi i)^{-1} \int\limits_{1/2-iT}^{1/2+iT} \Gamma\left(1+\frac{s}{h}\right)\frac{M^s-N^s}{s}\psi(s,\chi)\sum_{n\le K}\chi(n)n^{s-1}\,ds$$
$$+ (2\pi i)^{-1} \int\limits_{-h/2-iT}^{-h/2+iT} \Gamma\left(1+\frac{s}{h}\right)\frac{M^s-N^s}{s}\psi(s,\chi)\sum_{n>K}\chi(n)n^{s-1}\,ds + o(1).$$

The second integral above is also small. Indeed, since $h/2 \le |s| \ll T = Ch\log X$ and estimates (6) and (7) are valid, this integral is

$$\ll \int\limits_{-T}^{T} e^{-\frac{\pi}{2h}|t|}\left(\frac{MK}{q|s|}\right)^{-h/2} q^{1/2}|s|^{-1/2}\,dt \ll \left(\frac{C}{\log X}\right)^{h/2}\left(\frac{q}{h}\right)^{1/2} = o(1),$$

where K and h are as above and h is at least $\log X$. So

$$S = (2\pi i)^{-1} \int\limits_{1/2-iT}^{1/2+iT} \Gamma\left(1+\frac{s}{h}\right)\frac{M^s-N^s}{s}\psi(s,\chi)\sum_{n\le K}\chi(n)n^{s-1}\,ds + o(1).$$

We write $\phi(s) = |\Gamma(1+s/h)s^{-1}|$ noting that $\int_{1/2-iT}^{1/2+iT}\phi(s)\,|ds| \ll \log T$. Using the Schwarz inequality twice we get

$$|S|^4 \ll M^2\log^2 T \int\limits_{1/2-iT}^{1/2+iT}\phi(s)\,|ds| \int\limits_{1/2-iT}^{1/2+iT}\phi(s)\Big|\sum_{n\le K}\chi(n)n^{s-1}\Big|^4\,|ds| + 1$$
$$\ll M^2\log^3 T \int\limits_{1/2-iT}^{1/2+iT}\phi(s)\Big|\sum_{n\le K}\chi(n)n^{s-1}\Big|^4\,|ds| + 1$$
$$= M^2\log^3 T \int\limits_{1/2-iT}^{1/2+iT}\phi(s)\Big|\sum_{n\le K^2}c(n)\chi(n)n^{s-1}\Big|^2\,|ds| + 1,$$

where $c(n) \le \mathrm{d}(n)$.

For given s we can use the same estimation as in the case $Y \ll X^{1/2+\varepsilon}$. Now Heath-Brown's estimate is applied with $a_n = c(n)/n^{1/2+it}$, and

$$\sum_{nm=\Box}|a_n a_m| \ll \sum_{n\le K^2}\frac{\mathrm{d}^2(n^2)}{n} \ll K^{\varepsilon}.$$

The mean value of the character sum is therefore $\ll_\varepsilon X^{\varepsilon}(X+K^2)$.

So the sum of $|S|^4$ over primitive characters is

$$\begin{aligned}&\ll_\varepsilon M^2\log^3 T\int\limits_{1/2-iT}^{1/2+iT}\phi(s)X^\varepsilon(X+K^2)\,|ds|\\ &\ll_\varepsilon M^2\log^4 T\,X^\varepsilon(X+K^2)\ll_\varepsilon XM^2X^\varepsilon.\end{aligned}$$

Next, consider the error caused by the smoothing. The difference of the original sum and the smoothed sum is

$$\sum_{N<n\le M}\chi(n)-\sum_{n=1}^{\infty}\chi(n)(e^{-(n/M)^h}-e^{-(n/N)^h})=\sum_{n=1}^{\infty}w(n)\chi(n),$$

where $w(n)$ is small unless $|n-N|\ll N_0$ or $|n-M|\ll N_0$, and $N_0=C\frac{N}{h}\log N\asymp X^{1/2}$.

Let us estimate the sum

$$\sum_{\chi\in S^*(X)}\Big|\sum_{|n-N|\ll N_0}w(n)\chi(n)\Big|^4,$$

where $w(n)$ is as above. Squaring out the character sum and applying the result of Heath-Brown we get

$$\sum_{\chi\in S^*(X)}\Big|\sum_{|n-N^2|\ll NN_0}a_n\chi(n)\Big|^2\ll_\varepsilon X^\varepsilon(X+N^2)\sum_{nm=\square}|a_na_m|,$$

where

$$\sum_{nm=\square}|a_na_m|\ll\sum_{\substack{N-N_0\ll n,m,r,s\ll N+N_0\\ nmrs=\square}}1.$$

By Lemma 4 the last sum is $\ll N_0^2N^\varepsilon$, when $X^{1/2+\varepsilon}\ll N\le X^{3/4}$, so in this case the error is at most $\ll_\varepsilon XN^2X^\varepsilon$.

When $X^{3/4}\le N\ll X$, we can first estimate trivially the square of the character sum and then apply Heath-Brown's estimate to obtain

$$\ll N_0^2\sum_{\chi\in S^*(X)}\Big|\sum_{|n-N|\ll N_0}w(n)\chi(n)\Big|^2\ll_\varepsilon X^{1+\varepsilon}(X+N)\sum_{nm=\square}|a_na_m|,$$

where

$$\sum_{nm=\square}|a_na_m|\ll\sum_{\substack{N-N_0\ll n,m\ll N+N_0\\ nm=\square}}1.$$

And Lemma 3 gives the estimate

$$\ll_\varepsilon X^{2+\varepsilon}N_0N^\varepsilon\ll_\varepsilon X^{5/2+\varepsilon},$$

which is $\ll_\varepsilon XN^2X^\varepsilon$, when $X^{3/4}\le N\ll X$.

The above results gives the desired estimate for primitive characters. But it is easy to generalize the same estimate to all real characters (see for example [Ar]). So the Theorem is proved.

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