Small prime solutions of linear ternary equations

by

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1. Introduction. In this paper, we consider the size of small solutions of the following diophantine equation (1.1) in prime variables p_j:

$$(1.1) \qquad a_1 p_1 + a_2 p_2 + a_3 p_3 = b.$$

In particular, we estimate the numerical value of a relevant constant in the upper bound for small prime solutions of (1.1).

Let a_1, a_2, a_3 be any integers such that

$$(1.2) \qquad a_1 a_2 a_3 \neq 0 \quad \text{and} \quad (a_1, a_2, a_3) := \gcd(a_1, a_2, a_3) = 1.$$

Let b be any integer satisfying

$$(1.3) \quad b \equiv a_1 + a_2 + a_3 \ (\mathrm{mod}\ 2) \quad \text{and} \quad (b, a_i, a_j) = 1 \quad \text{for } 1 \leq i < j \leq 3.$$

Conditions (1.2) and (1.3) are plainly necessary in our investigation, for otherwise, the equation (1.1) will either be insolvable or be reduced to fewer than three prime variables. The problem of bounds for small prime solutions p_1, p_2, p_3 of the equation (1.1) was first considered by A. Baker in connection with his now well known work [1] on the solvability of certain diophantine inequalities involving primes. Baker's investigation raised immediately the problem of obtaining the best possible upper bound for small prime solutions. In [7] the following was proved:

THEOREM 0. *Assume the conditions* (1.2) *and* (1.3). *If not all* a_1, a_2, a_3 *are of the same sign, then there is an effective absolute constant* $B > 0$ *such that the equation* (1.1) *has a prime solution* p_1, p_2, p_3 *satisfying*

$$(1.4) \qquad \max_{1 \leq j \leq 3} p_j \leq 3|b| + \max\{3, |a_1|, |a_2|, |a_3|\}^B.$$

Obviously, B is the only relevant constant in (1.4). It is easy to see ([8], p. 125) that B must be larger than 1. The infimum $\mathcal{B}$ of all possible values

of the constant B in (1.4) is now called the *Baker constant*. Plainly, the determination of $\mathcal{B}$ will completely settle the Baker problem on the bound for small prime solutions of the equation (1.1).

It was shown in [7] that Theorem 0 contains the well known Linnik Theorem on the smallest prime in an arithmetic progression, namely, for any positive integers l, q with $l \leq q$ and $(l, q) = 1$, the smallest prime $P(l, q)$ in the arithmetic progression $l + kq$ satisfies $P(l, q) \ll q^L$ where L is an absolute positive constant. The infimum $\mathcal{L}$ of all values of L is called the *Linnik constant*. It was shown in [8] that $\mathcal{B} \geq \mathcal{L}$. Many authors (see Table 1 in [5]) investigated the numerical bounds for $\mathcal{L}$. The first numerical result for $\mathcal{B}$ was obtained by Choi [2]: $\mathcal{B} \leq 4190$. In [9] Liu and Wang proved that $\mathcal{B} \leq 45$. In [3] Choi, Liu and Tsang proved that $\mathcal{B} \leq 4$ under the Generalized Riemann Hypothesis. In the present paper we prove that $\mathcal{B} \leq 38$:

THEOREM 1. *Assume the conditions* (1.2) *and* (1.3). *If not all* a_1, a_2, a_3 *are of the same sign, then there is an absolute constant* $C > 0$ *such that the equation* (1.1) *has a prime solution* p_1, p_2, p_3 *satisfying*

$$(1.5) \qquad \max_{1 \leq j \leq 3} |a_j| p_j \leq C \max\{|b|, (\max\{|a_1|, |a_2|, |a_3|\})^{38}\}.$$

That is, $\mathcal{B} \leq 38$.

It should be noted that the constant 38 can be reduced a little by the method of [3].

2. Some lemmas. As usual, let $\chi \pmod{q}$ and $\chi_0 \pmod{q}$ denote a character and the principal character modulo q respectively. $L(s, \chi)$ denotes a Dirichlet L-function, and ε and ε_j are small positive numbers. Let

$$(2.1) \qquad \Pi(s) := \prod_{q \leq Q} \prod_{\chi \pmod{q}}^{*} L(s, \chi).$$

We consider the zero-free regions of the function $\Pi(s)$ in the region $|\mathrm{Im}(s)| \leq C, 1/2 \leq \mathrm{Re}(s) \leq 1$, where Q is a given sufficiently large positive number, C is any positive constant, and the $*$ indicates that the product $\prod^{*}$ is over all primitive characters $\chi \pmod{q}$. We put

$$(2.2) \qquad \mathcal{L} := \log Q.$$

We begin by proving that for sufficiently large Q there exists an absolute positive constant k such that none of the functions $L(s, \chi)$ with characters $\chi \pmod{q}$, $q \leq Q$, have zeros in the rectangles

$$1 - 10/\mathcal{L} \leq \sigma \leq 1, \quad 10^k C < |t| \leq 10^{k+1} C.$$

As in Lemma 6.1 of [3], this is a consequence of the estimate

$$\sum_{q \leq Q} \sum_{\chi \,(\mathrm{mod}\, q)} N(\sigma, T, \chi) \ll_\varepsilon (Q^2 T)^{(2+\varepsilon)(1-\sigma)} \quad (4/5 \leq \sigma \leq 1, T \geq 1, \varepsilon > 0)$$

due to Jutila. Let $\sigma = 1 - 10/\mathcal{L}$. Jutila's bound implies that

$$\sum_{q \leq Q} \sum_{\chi \,(\mathrm{mod}\, q)} N(\sigma, \mathcal{L}, \chi) \ll_\varepsilon (Q^2 \mathcal{L})^{(2+\varepsilon)(1-\sigma)} \ll (Q^2 \mathcal{L})^{3(1-\sigma)} = O(1)$$

providing that Q is sufficiently large. Similarly to Lemma 6.1 of [3], on choosing an appropriate k, we deduce the conclusion.

We proceed to number certain of the characters $\chi \,(\mathrm{mod}\, q)$ and zeros $\varrho = \beta + i\gamma$ of $L(s, \chi)$ as follows. Let ϱ denote any zero of $\Pi(s)$ in the rectangle

$$(2.3) \qquad R = \{s = \sigma + it : 1 - 10\mathcal{L}^{-1} \leq \sigma \leq 1, \ |t| \leq 10^k C\}.$$

Denote by ϱ_1 one of the above zeros for which β is maximal, and let χ_1 be a corresponding primitive character in (2.1) such that $L(\varrho_1, \chi_1) = 0$. Now, remove $L(s, \chi_1)$ and $L(s, \overline{\chi}_1)$ from (2.1), and choose ϱ_2 to be one of the zeros of $\Pi(s)(L(s, \chi_1)L(s, \overline{\chi}_1))^{-1}$ in R, for which β is maximal. We take χ_2 to be a primitive character in (2.1) for which $L(\varrho_2, \chi_2) = 0$. Then by arguments similar to [5] we see that if a primitive character $\chi \neq \chi_1, \overline{\chi}_1$, then every zero ϱ of $L(s, \chi)$ satisfies

$$(2.4) \qquad \mathrm{Re}(\varrho) \leq \mathrm{Re}(\varrho_2) \quad \text{or} \quad |\mathrm{Im}(\varrho)| > 10^{k+1} C.$$

For convenience of notation we shall set

$$(2.5) \qquad \varrho_k = \beta_k + i\gamma_k, \qquad \beta_k = 1 - \mathcal{L}^{-1}\lambda_k.$$

Suppose that $L(s, \chi_1)$ has a zero $\varrho' \neq \varrho_1$ in the rectangle R, given by (2.3) (or a repeated zero $\varrho' = \varrho_1$); in case χ_1 is real and ϱ_1 is complex we choose such a zero $\varrho' \neq \overline{\varrho}_1$ with $\mathrm{Re}(\varrho')$ maximal, and put

$$\varrho' = \beta' + i\gamma', \qquad \beta' = 1 - \mathcal{L}^{-1}\lambda'$$

in analogy to the above.

LEMMA 1. *For any constant $C > 0$, there exists a $K(C) > 0$ depending on C only such that if $Q \geq K(C)$, then the function $\Pi(s)$ defined by (2.1) has at most one zero in the region $\sigma \geq 1 - 0.364/\mathcal{L}, |t| \leq 10^k C$. Such a zero $\widetilde{\beta}$, if it exists, is real and simple, and corresponds to a non-principal, real, primitive character $\widetilde{\chi}$ to a modulus $\widetilde{r} \leq Q$. $\widetilde{\beta}$ is called the Siegel zero or the exceptional zero.*

This is Proposition 2.3 of [9].

LEMMA 2. *For any constant $C > 0$, there exists a $K(C) > 0$ depending on C only such that if $Q \geq K(C)$, then the function $\Pi(s)$ defined by (2.1)*

has at most two zeros in the region $\sigma \geq 1 - 0.504/\mathcal{L}, |t| \leq 10^k C$. *Moreover, the bounds in Table 1 can be applied.*

Table 1. The lower bound for λ_2

λ_1	λ_2	λ_1	λ_2
0.348	0.587	0.45	0.530
0.36	0.578	0.48	0.516
0.40	0.555	0.504	0.504

This is Lemma 2.5 of [9].

LEMMA 3. *If the exceptional zero $\widetilde{\beta}$ in Lemma 1 does indeed exist, then for any constant c with $0 < c < 1$ and for any small $\varepsilon > 0$ there is a $K(c,\varepsilon) > 0$ depending on c and ε only such that for any zero $\varrho = \beta + i\gamma \neq \widetilde{\beta}$ (corresponding to $\chi \pmod{q}$) of the function $\Pi(s)$ defined by (2.1) we have*

$$(2.6) \qquad \beta \leq 1 - \min\left\{ \frac{c}{6}, \frac{(1-c)(8/9 - \varepsilon)}{\log([\widetilde{r},q]|\gamma|)} \log\left(\frac{(1-c)(8/9 - \varepsilon)}{(1-\widetilde{\beta})\log([\widetilde{r},q]|\gamma|)} \right) \right\},$$

if $[\widetilde{r},q]|\gamma| > K(c,\varepsilon)$. Moreover, for any positive ε there exists a constant $c(\varepsilon) > 0$ depending on ε only such that

$$(2.7) \qquad 1 - 0.364/\mathcal{L} \leq \widetilde{\beta} \leq 1 - c(\varepsilon)\widetilde{r}^{-\varepsilon}.$$

Proof. By Lemma 2.4 of [6], if χ is a non-principal character modulo q, and $\phi = 3/8$, then for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$-\operatorname{Re}\left(\frac{L'}{L}(s,\chi) \right) \leq - \sum_{|1+it-\varrho| \leq \delta} \operatorname{Re}\left(\frac{1}{s-\varrho} \right) + \left(\frac{\phi}{2} + \varepsilon \right)\mathcal{L}$$

uniformly for

$$1 + \frac{1}{\mathcal{L}\log\mathcal{L}} \leq \sigma \leq 1 + \frac{\log\mathcal{L}}{\mathcal{L}}$$

providing that q is sufficiently large, where $\mathcal{L} = \log q(|t|+2)$. As in Lemmas 8.1, 8.5, 8.6 of [5], we can get Principle 2 of [5], p. 266, with $c_2 \leq 8/9$ providing that q is large enough and δ is small enough. Similarly to Lemma 2.6 of [9], the assertion follows.

Let

$$(2.8) \qquad \alpha = 1 - \lambda/\mathcal{L} \quad \text{for } 0.364 \leq \lambda \leq \log\log\mathcal{L},$$

$$(2.9) \qquad D := \{s = \sigma + it : \alpha \leq \sigma \leq 1 - 0.364/\mathcal{L}, \ |t| \leq C\},$$

$$(2.10) \qquad N(\chi, \alpha, C) := \text{number of zeros of } L(s,\chi) \text{ in } D,$$

$$(2.11) \qquad N^*(\alpha, Q, C) := \sum_{q \leq Q} \sideset{}{^*}\sum_{\chi \pmod{q}} N(\chi, \alpha, C),$$

where $\sum^*$ denotes that the sum is over primitive characters.

LEMMA 4. *For any absolute constant $C > 0$, let $\alpha = 1 - \lambda/\mathcal{L}$. Then for $Q \geq K(C)$ which is a positive constant depending on C only, we have*

$$N^*(\alpha, Q, C) \leq \begin{cases} 8.86706 C_1(\lambda) e^{4.31403\lambda}, & 0.504 < \lambda \leq 0.696, \\ 26.93 C_2(\lambda) e^{4.28374\lambda}, & 0.696 < \lambda \leq 1, \\ 50.36 C_3(\lambda) e^{3.753506\lambda}, & 1 < \lambda \leq 2, \\ 167.67 C_4(\lambda) e^{3.116796\lambda}, & 2 < \lambda \leq 6, \\ 42.54(\lambda + 35.385) C_5(\lambda) e^{2.87538\lambda}, & 6 < \lambda \leq \log\log \mathcal{L}, \end{cases}$$

where

$$C_1(\lambda) = \lambda^{-1}\left(1 - e^{-4.31403\lambda} \frac{e^{3.15402\lambda} - e^{2.32002\lambda}}{0.834\lambda}\right),$$

$$C_2(\lambda) = \lambda^{-1}\left(1 - e^{-4.28374\lambda} \frac{e^{3.19253\lambda} - e^{2.42653\lambda}}{0.766\lambda}\right),$$

$$C_3(\lambda) = \lambda^{-1}\left(1 - e^{-3.753506\lambda} \frac{e^{2.747904\lambda} - e^{2.160104\lambda}}{0.58\lambda}\right),$$

$$C_4(\lambda) = \lambda^{-1}\left(1 - e^{-3.116796\lambda} \frac{e^{2.223794\lambda} - e^{1.869794\lambda}}{0.354\lambda}\right),$$

$$C_5(\lambda) = \lambda^{-1}\left(1 - e^{-2.87538\lambda} \frac{e^{2.07176\lambda} - e^{1.92136\lambda}}{0.1504\lambda}\right).$$

This is Lemma 3.1 of [9].

3. The circle method. From now on, we let N be a sufficiently large positive number, and let

(3.1) $\theta := 1/(38/3 - 20\varepsilon_1), \quad Q := N^\theta, \quad T := Q^{5/2+3\varepsilon_1}, \quad \tau := N^{-1}Q^{1+\varepsilon_1},$

where ε_1 is a fixed sufficiently small positive number. For $1 \leq j \leq 3$, let

(3.2) $$N_j := N|a_j|^{-1}, \quad N_j' := N(4|a_j|)^{-1},$$

(3.3) $$B := \max\{|a_1|, |a_2|, |a_3|\}.$$

We always assume

(3.4) $$B^{3+3\varepsilon_1} \ll Q.$$

Denote by $\Lambda(n)$ the von Mangoldt function, and define, for any real y,

(3.5) $$S_j(y) := \sum_{N_j' < n \leq N_j} \Lambda(n)e(a_j n y),$$

(3.6) $$I(b) := \sum_{(n_1, n_2, n_3)} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3),$$

where $\sum_{(n_1, n_2, n_3)}$ denotes the summation over all triples (n_1, n_2, n_3) satisfying $\sum_{1 \leq j \leq 3} a_j n_j = b$ and $N_j' < n_j \leq N_j, 1 \leq j \leq 3$. For any integers

h and q with $1 \leq h \leq q \leq Q$ and $(h, q) = 1$, let $m(h, q)$ be the interval $[(h - \tau)/q, (h + \tau)/q]$. Let $\mathcal{M}$ be the union of these mutually disjoint intervals and $\mathcal{M}'$ be the complement of $\mathcal{M}$ in $[\tau, 1 + \tau]$. By (3.5) and (3.6) we have

$$(3.7) \qquad I(b) = \left\{ \int_{\mathcal{M}} + \int_{\mathcal{M}'} \right\} e(-bx) \prod_{j=1}^{3} S_j(x) \, dx =: I_1(b) + I_2(b), \quad \text{say,}$$

where $I_1(b)$ and $I_2(b)$ are the integrals on $\mathcal{M}$ and $\mathcal{M}'$ respectively. For any integer n and any character $\chi \pmod{q}$, denote the Gaussian sum by

$$G(n, \chi) := \sum_{l=1}^{q} \chi(l) e(nl/q) \quad \text{and let} \quad G(n, q) := G(n, \chi_0).$$

If $x \in m(h, q)$, write

$$x = h/q + \eta, \qquad (h, q) = 1, \qquad |\eta| \leq \tau q^{-1}.$$

From now on, we put $L := \log N$. For any real η and any $\chi \pmod{q}$ with $q \leq Q, 1 \leq j \leq 3$, let

$$I_j(\eta) := \int_{N_j'}^{N_j} e(a_j \eta x) \, dx, \qquad \widetilde{I}_j(\eta) := \int_{N_j'}^{N_j} x^{\widetilde{\beta}-1} e(a_j \eta x) \, dx,$$

$$(3.8)$$

$$I_j(\chi, \eta) := \int_{N_j'}^{N_j} e(a_j \eta x) \sum_{|\gamma| \leq T}{}' x^{\varrho-1} dx,$$

where $\widetilde{\beta}$ is the possible Siegel zero in Lemma 1 and the corresponding character is $\widetilde{\chi} \pmod{\widetilde{r}}$, $\sum_{|\gamma| \leq T}'$ denotes the summation over all zeros $\varrho = \beta + i\gamma$ of $L(s, \chi)$ satisfying $\varrho \neq \widetilde{\beta}, |\gamma| \leq T$ and $1/2 \leq \beta \leq 1$.

LEMMA 5. *Let*

$$(3.9) \qquad \mathcal{G}_j(h, q, \eta) := \sum_{\chi \pmod{q}} G(a_j h, \overline{\chi}) I_j(\chi, \eta),$$

$$(3.10) \qquad H_j(h, q, \eta) := G(a_j h, q) I_j(\eta) - \delta(q) G(a_j h, \widetilde{\chi}\chi_0) \widetilde{I}_j(\eta) - \mathcal{G}_j(h, q, \eta),$$

where $\delta(q) = 1$ if $\widetilde{r} \mid q$ and $\delta(q) = 0$ otherwise. Then for any $x \in m(h, q)$,

$$S_j(x) = \varphi(q)^{-1} H_j(h, q, \eta) + O((1 + N|\eta|) N_j q^{1/2} T^{-1} L^2).$$

This is Lemma 4.2 of [9].

LEMMA 6. *For any $x \geq 1$ and $y \geq 2$, let*

$$N(\alpha, x, y) := \sum_{q \leq x} \sum_{\chi \pmod{q}}{}^{*} \sum_{\substack{\varrho = \beta + i\gamma \\ |\gamma| \leq y, \, \beta \geq \alpha}} 1,$$

where $\varrho = \beta + i\gamma$ is any non-trivial zero of $L(s, \chi)$. Then

$$(3.11) \qquad N(\alpha, x, y) \ll (x^2 y^{6/5})^{(20/9+\varepsilon)(1-\alpha)} \qquad \text{for } 1/2 \leq \alpha \leq 1,$$

$$(3.12) \qquad N(\alpha, x, y) \ll (x^2 y)^{(2+\varepsilon)(1-\alpha)} \qquad \text{for } 11/14 \leq \alpha \leq 1.$$

See Theorems 1 and 2 of [4].

LEMMA 7. *Under the notations and conditions of* (3.1) *to* (3.4), *we have for* $1 \leq j \leq 3$,

$$\Sigma_1 := \sum_{q \leq Q} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \sum_{|\gamma| \leq T} N_j^{\beta-1} \ll 1.$$

Proof. In view of $Q \geq B^3$ and $\theta = 1/(38/3 - 20\varepsilon_1)$, we have for $1 \leq j \leq 3$,

$$(3.13) \qquad N_j \geq Q^{38/3 - 20\varepsilon_1 - 1/3}.$$

As in Lemma 4.5 of [9], note that $T = Q^{2.5+3\varepsilon_1}$; by Lemma 6, the assertion follows.

LEMMA 8. *Under the conditions of Lemma 7, for* $1 \leq j \leq 3$ *we have*

$$S_{1,j} := \sum_{q \leq Q} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \sum_{|\gamma| \leq T} \left(\int_{-\tau/q}^{\tau/q} \left| \int_{N_j'}^{N_j} x^{\varrho-1} e(a_j x \eta) \, dx \right|^2 d\eta \right)^{1/2} \ll N^{1/2} |a_j|^{-1}.$$

Proof. Similarly to Lemma 4.6 of [9], this follows from Lemma 7.

LEMMA 9. *Let* $I_1(b)$ *be defined as in* (3.7). *Under the assumptions of Lemma 7, we have*

$$I_1(b) = \sum_{q \leq Q} \varphi^{-3}(q) \sideset{}{'}\sum_{h=1}^{q} e\left(-\frac{b}{q} h \right) \int_{-\tau/q}^{\tau/q} e(-b\eta) \prod_{j=1}^{3} H_j(h, q, \eta) \, d\eta + O(\Omega_1),$$

where $\sum_{h=1}^{q'}$ *is the summation over all* $1 \leq h \leq q$, $(h, q) = 1$, *and*

$$(3.14) \qquad \Omega_1 := N^2 Q^{2.5+\varepsilon_1} |a_1 a_2 a_3|^{-1} T^{-1} L^2.$$

Proof. As in Lemma 4.7 of [9], the assertion follows from Lemma 8.

4. The estimation of M_1 and M_3. Multiplying out the product $\prod_{j=1}^{3} H_j(h, q, \eta)$ in Lemma 9, we get 27 terms (if $\widetilde{\beta}$ exists). They are grouped into the following three categories:

- $\mathcal{J}_1$: the term $\prod_{j=1}^{3} G(a_j h, q) I_j(\eta)$;
- $\mathcal{J}_2$: 19 terms (if $\widetilde{\beta}$ exists), each has at least one $\mathcal{G}_j(h, q, \eta)$ as a factor;
- $\mathcal{J}_3$: the 7 terms remaining (if $\widetilde{\beta}$ exists).

For $i = 1, 2, 3$, define

(4.1) $$M_i := \sum_{q \leq Q} \varphi^{-3}(q) \sideset{}{'}\sum_{h=1}^{q} e\left(-\frac{b}{q}h\right)$$

$$\times \int_{-\tau/q}^{\tau/q} e(-b\eta)\{\text{sum of the terms in } \mathcal{J}_i\} \, d\eta.$$

Then by Lemma 9 we get

(4.2) $$I_1(b) = M_1 + M_2 + M_3 + O(\Omega_1),$$

if one assumes (3.4). For the estimation of M_1 and M_3, we need the following notations. For any positive integer q, define

(4.3) $$A(q) := \varphi^{-3}(q) \sideset{}{'}\sum_{h=1}^{q} e\left(-\frac{b}{q}h\right) \prod_{j=1}^{3} G(a_j h, q).$$

By Lemma 4.1 of [7], $A(q)$ is multiplicative. For any prime p, put

(4.4) $$s(p) := 1 + A(p).$$

Let r_1, r_2, r_3 be any positive integers and denote by $[r_1, r_2, r_3]$ the least common multiple of r_1, r_2, r_3. For any primitive characters $\chi_j \pmod{r_j}$ $(1 \leq j \leq 3)$ and $[r_1, r_2, r_3] \mid q$, define

(4.5) $$Z(q) := Z(q; \chi_1, \chi_2, \chi_3) = \sideset{}{'}\sum_{h=1}^{q} e\left(-\frac{b}{q}h\right) \prod_{j=1}^{3} G(a_j h, \chi_j \chi_0),$$

where χ_0 is the principal character modulo q, and

(4.6) $$\mathcal{D} := \{(x_1, x_2) : (4|a_j|)^{-1} \leq x_j \leq |a_j|^{-1}, \ j = 1, 2, 3\},$$

where $x_3 = a_3^{-1}(bN^{-1} - a_1 x_1 - a_2 x_2)$.

LEMMA 10. *Let* χ *be a character modulo* q, *induced by a primitive character* χ^* *modulo* q^*, $q = q_1 q_2, (q_2, q^*) = 1, p \mid q_1 \Rightarrow p \mid q^*$. *Then*

$$G(m, \chi)$$
$$= \overline{\chi}^*\left(\frac{m}{(m,q)}\right)\chi^*\left(\frac{q}{q^*(m,q)}\right)\mu\left(\frac{q}{q^*(m,q)}\right)\varphi(q)\varphi^{-1}\left(\frac{q}{(m,q)}\right)G(1, \chi^*)$$

if $q^* = q_1/(m, q_1)$; *otherwise* $G(m, \chi) = 0$.

This is Lemma 1.5 of [10].

LEMMA 11. *Let* $Z(q)$ *and* $s(p)$ *be defined as in* (4.5) *and* (4.4) *respectively, and let* $r = [r_1, r_2, r_3]$. *Then under the conditions of* (1.2) *and* (1.3),

we have

(4.7)
$$\prod_p s(p) \gg 1,$$

and

(4.8)
$$\sum_{\substack{q \le Q \\ r|q}} \varphi^{-3}(q)|Z(q)| \le 2.140782 \prod_p s(p).$$

In particular, if two of r_1, r_2, r_3 are 1, for example, $r_1 = r_2 = 1, r_3 = r$, then

(4.9)
$$\sum_{\substack{q \le Q \\ r|q}} \varphi^{-3}(q)|Z(q)| \ll \frac{r}{\varphi(r)\varphi\left(\frac{r}{(a_1,r)}\right)\varphi\left(\frac{r}{(a_2,r)}\right)} \prod_p s(p).$$

If one of r_1, r_2, r_3 is 1, for example, $r_1 = 1, [r_2, r_3] = r$, then

(4.10)
$$\sum_{\substack{q \le Q \\ r|q}} \varphi^{-3}(q)|Z(q)| \ll Q^\varepsilon \frac{(a_1, r)}{\sqrt{r}} \prod_p s(p).$$

Proof. (4.7) and (4.8) are Lemma 5.2 of [9]. Now we consider (4.9). By the proof of Lemma 4.6 of [7] we have

(4.11)
$$\sum_{\substack{q \le Q \\ r|q}} \varphi^{-3}(q)|Z(q)| = \varphi^{-3}(r)|Z(r)| \sum_{\substack{q \le Q/r \\ (q,r)=1}} |A(q)|$$

$$\ll \varphi^{-3}(r)|Z(r)| \prod_{p \nmid r}(1 + |A(p)|).$$

If $r_1 = r_2 = 1, r_3 = r$, then by (4.5), for $(h, r) = 1$ we have

(4.12)
$$Z(r) = G(a_1, \chi_0)G(a_2, \chi_0)G(a_3, \chi_3)G(-b, \overline{\chi}_3).$$

By Lemma 1.2 of [10] we have

$$|G(a_1, \chi_0)| \le \varphi(r)\varphi^{-1}\left(\frac{r}{(a_1, r)}\right), \quad |G(a_2, \chi_0)| \le \varphi(r)\varphi^{-1}\left(\frac{r}{(a_2, r)}\right).$$

By Lemma 1.6 of [10] we have

$$|G(a_3, \chi_3)| \le \sqrt{r}, \quad |G(-b, \overline{\chi}_3)| \le \sqrt{r}.$$

Hence by (4.11) and (4.12),

$$\varphi^{-3}(r)|Z(r)| \ll \frac{r}{\varphi(r)\varphi\left(\frac{r}{(a_1,r)}\right)\varphi\left(\frac{r}{(a_2,r)}\right)}.$$

As in Lemma 5.2 of [9], (4.9) follows.

For (4.10), if $r_1 = 1, [r_2, r_3] = r$, then by (4.5) we have

(4.13)
$$Z(r) = G(a_1, \chi_0)G(a_2, \chi_2\chi_0)G(a_3, \chi_3\chi_0)G(-b, \overline{\chi}_2\overline{\chi}_3\chi_0).$$

Let $(r_2, r_3) = r_4, r_2 = r_4 r_2', r_3 = r_4 r_3'$. Then $(r_2', r_3') = 1$. Let $r_2' = r_5 r_5'$, $(r_5, r_5') = 1, (r_4, r_5') = 1, p \mid r_5 \Rightarrow p \mid r_4, r_3' = r_6 r_6', (r_6, r_6') = 1, (r_4, r_6') = 1,$ $p \mid r_6 \Rightarrow p \mid r_4$. Then $r = r_4 r_5 r_5' r_6 r_6'$. By Lemma 1.2 of [10] we have

$$(4.14) \qquad |G(a_1, \chi_0)| \le \varphi(r) \varphi^{-1} \left(\frac{r}{(a_1, r)} \right).$$

By Lemma 10, if $(a_2, r_2 r_6) = r_6$, then

$$(4.15) \qquad |G(a_2, \chi_2 \chi_0)| \le \varphi(r) \varphi^{-1} \left(\frac{r}{(a_2, r)} \right) \sqrt{r_2},$$

otherwise $|G(a_2, \chi_2 \chi_0)| = 0$. Similarly, if $(a_3, r_3 r_5) = r_5$, then

$$(4.16) \qquad |G(a_3, \chi_3 \chi_0)| \le \varphi(r) \varphi^{-1} \left(\frac{r}{(a_3, r)} \right) \sqrt{r_3},$$

otherwise $|G(a_3, \chi_3 \chi_0)| = 0$. We may put

$$(4.17) \qquad \overline{\chi}_2 = \overline{\chi}_4^{(1)} \overline{\chi}_5 \overline{\chi}_5', \qquad \overline{\chi}_3 = \overline{\chi}_4^{(2)} \overline{\chi}_6 \overline{\chi}_6',$$

where $\chi_i \bmod r_i$ and $\chi_i' \bmod r_i'$ are primitive; thus we have

$$(4.18) \qquad \overline{\chi}_2 \overline{\chi}_3 \chi_0 = \overline{\chi}_4 \overline{\chi}_5 \overline{\chi}_6 \overline{\chi}_5' \overline{\chi}_6',$$

where $\chi_4 = \chi_4^{(1)} \chi_4^{(2)}$. Hence by Lemma 10 we have

$$(4.19) \quad \begin{aligned} & |G(-b, \overline{\chi}_2 \overline{\chi}_3 \chi_0)| \\ & \le |G(-b, \overline{\chi}_4 \overline{\chi}_5 \overline{\chi}_6) G(-b, \overline{\chi}_5') G(-b, \overline{\chi}_6')| \\ & \le \sqrt{r_5'} \sqrt{r_6'} |G(-b, \overline{\chi}_4 \overline{\chi}_5 \overline{\chi}_6)| \\ & \le \sqrt{r_5' r_6'} \varphi(r_4 r_5 r_6) \varphi^{-1} \left(\frac{r_4 r_5 r_6}{(b, r_4 r_5 r_6)} \right) \sqrt{\frac{r_4 r_5 r_6}{(b, r_4 r_5 r_6)}} \le \sqrt{r_5' r_6'} r_4 r_5 r_6. \end{aligned}$$

Hence

$$(4.20) \quad \varphi^{-3}(r) |Z(r)| \le \frac{r^2 \sqrt{r_5 r_6}}{\varphi^2(r) \varphi \left(\frac{r}{(a_1, r)} \right)} \le Q^{\varepsilon^2} \frac{(a_1, r) \sqrt{r_5 r_6}}{r} \le Q^\varepsilon \frac{(a_1, r)}{\sqrt{r}}.$$

So (4.10) follows.

LEMMA 12. *Let M_1 be defined as in (4.1). If not all the a_j's are of the same sign and $N \ge 3|b|$, then*

$$(4.21) \qquad M_1 = M_0 + O(\Omega_2),$$

where

$$(4.22) \qquad M_0 := N^2 |a_3|^{-1} \left(\prod_p s(p) \right) \int_{\mathcal{D}} dx_1 \, dx_2,$$

with $\mathcal{D}$ given by (4.6), *and*

$$(4.23) \quad M_0 \gg N^2 |a_1 a_2 a_3|^{-1} \prod_p s(p), \quad \Omega_2 := N^2 Q^{-1+\varepsilon_1/4} |a_1 a_2 a_3|^{-1}.$$

See Lemmas 5.3 and 7.4 of [9].

LEMMA 13. *If the exceptional zero $\widetilde{\beta}$ defined as in Lemma 1 exists, put*

$$(4.24) \qquad\qquad \omega := (1 - \widetilde{\beta})\mathcal{L}.$$

Then

$$M_1 + M_3 \geq \begin{cases} 20\omega^3 M_0 + O(\widetilde{r}\Omega_2 + N^2 Q^{-1-\varepsilon_1}) & \text{if } 0.34 < \omega \leq 0.364, \\ 24.3\omega^3 M_0 + O(\widetilde{r}\Omega_2 + N^2 Q^{-1-\varepsilon_1}) & \text{if } 0.326 < \omega \leq 0.34, \\ 27.3\omega^3 M_0 + O(\widetilde{r}\Omega_2 + N^2 Q^{-1-\varepsilon_1}) & \text{if } 0.306 < \omega \leq 0.326, \\ 32.55\omega^3 M_0 + O(\widetilde{r}\Omega_2 + N^2 Q^{-1-\varepsilon_1}) & \text{if } 0.23 < \omega \leq 0.306, \\ 68.5\omega^3 M_0 + O(\widetilde{r}\Omega_2 + N^2 Q^{-1-\varepsilon_1}) & \text{if } 0.2 < \omega \leq 0.23, \\ 95.7\omega^3 M_0 + O(\widetilde{r}\Omega_2 + N^2 Q^{-1-\varepsilon_1}) & \text{if } 0.1 < \omega \leq 0.2, \\ 355\omega^3 M_0 + O(\widetilde{r}\Omega_2 + N^2 Q^{-1-\varepsilon_1}) & \text{if } 0.066 < \omega \leq 0.1, \\ 600\omega^3 M_0 + O(\widetilde{r}\Omega_2 + N^2 Q^{-1-\varepsilon_1}) & \text{if } 0.006 < \omega \leq 0.066, \\ 1680\omega^3 M_0 + O(\widetilde{r}\Omega_2 + N^2 Q^{-1-\varepsilon_1}) & \text{if } 0.0025 < \omega \leq 0.006, \\ 1791\omega^3 M_0 + O(\widetilde{r}\Omega_2 + N^2 Q^{-1-\varepsilon_1}) & \text{if } 10^{-6} < \omega \leq 0.0025, \\ 1876\omega^3 M_0 + O(\widetilde{r}\Omega_2 + N^2 Q^{-1-\varepsilon_1}) & \text{if } \omega \leq 10^{-6}. \end{cases}$$

Proof. By Lemma 1 we have $\omega \leq 0.364$. We only prove the lemma for $0.34 < \omega \leq 0.364$. Noting $N'_j \geq 4^{-1} Q^{38/3 - 20\varepsilon_1 - 1/3}$, as in Lemma 5.5 of [9] we have for $1 \leq j \leq 3$,

$$(4.25) \qquad\qquad 1 - N'^{\widetilde{\beta}-1}_j \geq 2.7164\omega.$$

Hence the lemma follows.

LEMMA 14. *We have $M_1 + M_3 = M_0 + O(\Omega_2 + N^2 \widetilde{r}^{-1} \log^3 L + N^2 Q^{-1-\varepsilon_1})$.*

This is Lemma 5.6 of [9].

5. Further estimates on triple sums

LEMMA 15. *Let ε_2 be a fixed sufficiently small positive constant and $Q > K(\varepsilon_2)$, a positive constant depending on ε_2 only. Suppose that the exceptional zero $\widetilde{\beta}$ exists and satisfies $\omega \leq \varepsilon_2$. Then for $1 \leq j \leq 3$,*

$$\Sigma_2 := \sum_{q \leq Q} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \sideset{}{'}\sum_{|\gamma| \leq Q^{1+\varepsilon_1} q^{-1}} N_j^{\beta-1} \ll \varepsilon_2^{1/2} \omega^3.$$

Proof. Noting that $N_j \geq Q^{38/3 - 20\varepsilon_1 - 1/3}$, as in Lemma 6.1 of [9], by Lemmas 3 and 6 we may take

$$\eta(Q) := \frac{1 - \varepsilon_3}{(9/4)\mathcal{L}} \log\left(\frac{1 - \varepsilon_3}{(9/4)\omega}\right),$$

where $\varepsilon_3 = \varepsilon_3(c, \varepsilon, \varepsilon_1)$ is a positive constant depending on $c, \varepsilon, \varepsilon_1$ only. And ε_3 becomes sufficiently small if c, ε and ε_1 are chosen to be sufficiently small. Hence the lemma follows.

LEMMA 16. *Under the notations of Lemma* 15, *for any positive constant* $C > 0$ *let* $Q > K(C, \varepsilon_2)$, *a positive constant depending on* C *and* ε_2 *only. If* (i) $\widetilde{\beta}$ *does not exist or* (ii) $\widetilde{\beta}$ *exists and satisfies* $\omega > \varepsilon_2$, *then we have for* $1 \le j \le 3$,

$$\Sigma_3 := \sum_{q \le Q} \sum_{\chi \,(\mathrm{mod}\, q)}^{*} \sideset{}{'}\sum_{|\gamma| \le C} N_j'^{\beta - 1}$$

$$\le \begin{cases} 0.3511263 & \text{if } \widetilde{\beta} \text{ does not exist,} \\ 0.083376 & \text{if } 0.306 < \omega \le 0.364, \\ 0.0302676 & \text{if } 0.2 < \omega \le 0.306, \\ 0.002229681 & \text{if } 0.066 < \omega \le 0.2, \\ 0.964773 \cdot 10^{-6} & \text{if } 0.0025 < \omega \le 0.066, \\ 0.29853221 \cdot 10^{-16} & \text{if } 10^{-6} < \omega \le 0.0025, \\ 0.001\omega^3 & \text{if } \omega \le 10^{-6}. \end{cases}$$

Proof. We first prove the lemma under the assumption that the exceptional zero $\widetilde{\beta}$ does not exist. By Lemma 1 and in view of the bounds for λ in Lemma 2, we can write

$$\Sigma_3 = N_j'^{-1/2} N^*(1/2, Q, C)$$

$$+ \left\{ \int_{1/2}^{11/14} + \int_{11/14}^{1 - \mathcal{L}^{-1} \log \log \mathcal{L}} + \int_{1 - \mathcal{L}^{-1} \log \log \mathcal{L}}^{1 - 6/\mathcal{L}} + \int_{1 - 6/\mathcal{L}}^{1 - 2/\mathcal{L}} + \int_{1 - 2/\mathcal{L}}^{1 - 1/\mathcal{L}} + \int_{1 - 1/\mathcal{L}}^{1 - 0.696/\mathcal{L}} \right.$$

$$+ \left. \int_{1 - 0.696/\mathcal{L}}^{1 - 0.504/\mathcal{L}} + \int_{1 - 0.504/\mathcal{L}}^{1 - 0.364/\mathcal{L}} \right\} N^*(\alpha, Q, C) N_j'^{\alpha - 1} \log N_j' \, d\alpha$$

$$=: \sum_{l=1}^{9} C_l, \quad \text{say.}$$

As in Lemma 6.2 of [9], by Lemmas 4 and 6 we have

$$(5.1) \qquad\qquad\qquad \Sigma_3 \le 0.3511263.$$

Now we come to estimating Σ_3 under the assumption that $\widetilde{\beta}$ exists. We separate the arguments in six cases according to the value of the upper bound for ω: $10^{-6}, 0.0025, 0.066, 0.2, 0.306$ and 0.364. We only prove the case $\varepsilon_2 < \omega \le 10^{-6}$, other cases can be proved in the same way as in Lemma 6.2 of [9]. If $\omega \le 10^{-6}$, then by noting $\omega \ge \varepsilon_2$, for any zero $\varrho = \beta + i\gamma \,(\ne \widetilde{\beta})$, we have

$$(5.2) \qquad\qquad\qquad \beta \le 1 - \eta(Q),$$

where $\eta(Q)$ is defined in Lemma 15. Hence

(5.3) $\quad \Sigma_3 \le (\log \mathcal{L})^{-6}\omega^3$

$$+ \int_{1-\mathcal{L}^{-1}\log\log\mathcal{L}}^{1-\eta(Q)} N^*(\alpha, Q, C)Q^{(38/3-20\varepsilon_1-1/3)(\alpha-1)}\log N \, d\alpha.$$

To estimate the integral, we consider two cases according as $\eta(Q) \ge 6/\mathcal{L}$ or not. When $\eta(Q) \ge 6/\mathcal{L}$, we get $\omega \le (4/9)\exp(-27/2-\varepsilon)$. Thus by Lemma 4 the integral in (5.3) can be estimated as

$$\le \int_{\eta(Q)\mathcal{L}}^{\log\log\mathcal{L}} (38/3)(42.54)(1+35.385/6)\exp(2.87538\lambda)Q^{-(38/3-21\varepsilon_1-1/3)\lambda\mathcal{L}^{-1}} \, d\alpha$$

$$\le 394\left(\frac{(9/4)\omega}{1-\varepsilon_3}\right)^{(1-\varepsilon_3)((38/3-21\varepsilon_1-1/3-2.87538)4)/9}$$

$$\le 394\left(\frac{(9/4)\omega}{1-\varepsilon_3}\right)^{4.2} \le 0.0005\omega^3.$$

When $\eta(Q) \le 6/\mathcal{L}$, we get $\omega > (9/4+\varepsilon_3)^{-1}\exp(-27/2-\varepsilon_3)$. Thus in view of $C_4 \le 0.873 \cdot 10^{-22}$, the integral in (5.3) is

(5.4) $\quad \le \dfrac{0.873 \cdot 10^{-22}}{((9/4+\varepsilon_3)^{-1}\exp(-27/2-\varepsilon_3))^3}\omega^3$

$$+ \int_{1-6/\mathcal{L}}^{1-\eta(Q)} N^*(\alpha, Q, C)Q^{(38/3-20\varepsilon_1-1/3)(\alpha-1)}\log N \, d\alpha$$

$$\le 0.0004\omega^3 + \int_{1-6/\mathcal{L}}^{1-\eta(Q)} N^*(\alpha, Q, C)Q^{(38/3-20\varepsilon_1-1/3)(\alpha-1)}\log N \, d\alpha.$$

By page 374 of [9] we have $\beta \le 1 - 4.55/\mathcal{L}$; by Lemma 4 the last integral in (5.4) is

$$\le \int_{\eta(Q)\mathcal{L}}^{6} (38/3)(167.67)(4.55)^{-1}\exp(3.116796\lambda)Q^{(38/3-20\varepsilon_1-1/3)(-\lambda/\mathcal{L})} \, d\lambda$$

$$\le 52\left(\frac{(9/4)\omega}{1-\varepsilon_3}\right)^{4.09} \le 0.0005\omega^3.$$

Hence $\Sigma_3 \le 0.001\omega^3$.

LEMMA 17. *Under the notations of Lemma 14, for any positive constant $C > 0$ let $Q > K(C, \varepsilon_2)$, a positive constant depending on C and ε_2 only. Then we have for $1 \le j \le 3$,*

$$\Sigma_4 := \sum_{q \le B^{1+\varepsilon_2}} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \sideset{}{'}\sum_{|\gamma| \le C} N_j'^{\beta-1} \le 0.366 \cdot 10^{-5},$$

$$\Sigma_5 := \sum_{q \le B^{2+\varepsilon_2}} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \sideset{}{'}\sum_{|\gamma| \le C} N_j'^{\beta-1} \le 0.012922.$$

Proof. This follows in the same way as in the case where the exceptional zero $\widetilde{\beta}$ does not exist in Lemma 16.

6. The estimation of M_2

LEMMA 18. *For any absolute constant $C > 1$, we have for $1 \le j \le 3$,*

$$S_{2,j} := \sum_{q \le Q} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \sum_{C \le |\gamma| \le T} \left(\int_{-\tau/q}^{\tau/q} \left| \int_{N_j'}^{N_j} x^{\varrho-1} e(a_j x\eta)\, dx \right|^3 d\eta \right)^{1/3}$$
$$\ll C^{-1/6} N^{2/3} |a_j|^{-1}.$$

Proof. This follows in the same way as Lemma 7.1 of [9].

LEMMA 19. *We have for $1 \le j \le 3$,*

$$S_{3,j} := \sum_{q \le Q} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \sideset{}{'}\sum_{|\gamma| \le T} \left(\int_{-\tau/q}^{\tau/q} \left| \int_{N_j'}^{N_j} x^{\varrho-1} e(a_j x\eta)\, dx \right|^3 d\eta \right)^{1/3} \ll N^{2/3} |a_j|^{-1}.$$

Proof. Just as in Lemma 7.2 of [9], this follows from Lemma 7.

LEMMA 20. *For the $\varepsilon_2 > 0$ given as in Lemma 15, let $Q > K(\varepsilon_2)$, a positive constant depending on ε_2 only. If $\widetilde{\beta}$ exists and satisfies $\omega \le \varepsilon_2$, then $S_{3,j}$ defined as in Lemma 19 can be estimated further as $S_{3,j} \ll \varepsilon_2^{1/2} \omega^3 N^{2/3} |a_j|^{-1}$.*

Proof. This follows from Lemma 15 as in Lemma 7.3 of [9].

Now we come to estimating M_2. We consider two cases according as the exceptional zero $\widetilde{\beta}$ exists or not.

If $\widetilde{\beta}$ does not exist, then there are 7 terms in the integrand of M_2 and they are the first, fourth and sixth type in case (I) of p. 376 of [9]. For M_{21}, as for M_{25} of [9], we get

$$M_{21} = \sum_{r_3 \le Q} \sideset{}{^*}\sum_{\chi_3 \,(\mathrm{mod}\, r_3)} \sideset{}{'}\sum_{|\gamma_3| \le C} \sum_{\substack{q \le Q \\ r_3 | q}} \varphi^{-3}(q) Z(q; \chi_0, \chi_0, \overline{\chi}_3)$$

$$\times \int_{-\infty}^{\infty} e(-b\eta) \left(\prod_{j=1}^{2} \int_{N_j'}^{N_j} e(a_j x\eta)\, dx \right) \left(\int_{N_3'}^{N_3} x^{\varrho_3-1} e(a_3 x\eta)\, dx \right) d\eta$$

$$+ O\left(N^2 |a_1 a_2 a_3|^{-1} C^{-1/6} \prod_{p} s(p) \right).$$

By Lemma 11 we get

$$M_{21} = \sum_{r_3 \leq B^{1+2\varepsilon_1}} \sideset{}{^*}\sum_{\chi_3 \pmod{r_3}} \sideset{}{'}\sum_{|\gamma_3| \leq C} \sum_{\substack{q \leq Q \\ r_3 | q}} \varphi^{-3}(q) Z(q; \chi_0, \chi_0, \overline{\chi}_3)$$

$$\times \int_{-\infty}^{\infty} e(-b\eta) \Big(\prod_{j=1}^{2} \int_{N'_j}^{N_j} e(a_j x \eta)\, dx \Big) \Big(\int_{N'_3}^{N_3} x^{\varrho_3 - 1} e(a_3 x \eta)\, dx \Big) d\eta$$

$$+ O\Big(N^2 |a_1 a_2 a_3|^{-1} C^{-1/6} \prod_p s(p) \Big) + O\Big(N^2 |a_1 a_2 a_3|^{-1} Q^{-\varepsilon_1} \prod_p s(p) \Big).$$

Hence we have

$$|M_{21}| \leq N^2 |a_3|^{-1} (2.140782)(\Sigma_4) \Big(\prod_p s(p) \Big) \int_{\mathcal{D}} dx_1\, dx_2$$

$$+ o\Big(N^2 |a_1 a_2 a_3|^{-1} \prod_p s(p) \Big)$$

$$\leq (2.140782 + \varepsilon)(\Sigma_4) M_0.$$

Similarly, we have

$$|M_{24}| \leq (2.140782 + \varepsilon)(\Sigma_5)^2 M_0,$$
$$|M_{26}| \leq (2.140782 + \varepsilon)(\Sigma_3)^3 M_0,$$

providing that C is large enough. Consequently,

$$(6.1) \quad M_2 \leq (2.140782 + \varepsilon)$$
$$\times (3(0.366 \cdot 10^{-5}) + 3(0.012922)^2 + (0.3511263)^3) M_0$$
$$\leq 0.096 M_0.$$

If the exceptional zero exists, then as on page 376 of [9], there are 19 terms in the integrand of M_2 and they are of 6 types; the treatment of these six types is quite similar to be above, so we have

$$|M_{21}| \leq (2.140782 + \varepsilon)(\Sigma_4) M_0,$$
$$|M_{22}| \leq (2.140782 + \varepsilon)(\Sigma_5) M_0,$$
$$|M_{23}| \leq (2.140782 + \varepsilon)(\Sigma_3) M_0,$$
$$|M_{24}| \leq (2.140782 + \varepsilon)(\Sigma_5)^2 M_0,$$
$$|M_{25}| \leq (2.140782 + \varepsilon)(\Sigma_3)^2 M_0,$$
$$|M_{26}| \leq (2.140782 + \varepsilon)(\Sigma_3)^3 M_0,$$

$$(6.2) \quad M_2 \leq (2.140782 + \varepsilon)$$
$$\times (3(\Sigma_4) + 6(\Sigma_5) + 3(\Sigma_3) + 3(\Sigma_5)^2 + 3(\Sigma_3)^2 + (\Sigma_3)^3) M_0.$$

Note that trivially we have

$$\Sigma_4 \leq \Sigma_5 \leq \Sigma_3.$$

By Lemmas 16 and 17 one can deduce the following.

LEMMA 21. *For the $\varepsilon_2 > 0$ given as in Lemma 15, let $Q > K(\varepsilon_2)$, a positive constant depending on ε_2 only. Under the assumptions in Lemma 12, we have*

$$M_2 \leq \begin{cases} 0.096 & \text{if } \widetilde{\beta} \text{ does not exist,} \\ 0.749 & \text{if } 0.306 < \omega \leq 0.364, \\ 0.368 & \text{if } 0.2 < \omega \leq 0.306, \\ 0.0431 & \text{if } 0.066 < \omega \leq 0.2, \\ 0.2479 \cdot 10^{-4} & \text{if } 0.0025 < \omega \leq 0.066, \\ 0.767 \cdot 10^{-15} & \text{if } 10^{-6} < \omega \leq 0.0025, \\ 0.05\omega^3 & \text{if } \varepsilon_2 < \omega \leq 10^{-6}. \end{cases}$$

7. Proof of Theorem 1

LEMMA 22. *Let $I_2(b)$ be defined as in (3.7). Then*

$$I_2(b) \ll N^2 Q^{-1/2} |a_1 a_2 a_3|^{-1/2} L^5.$$

This is Lemma 8.1 of [9].

LEMMA 23. *Let $I_1(b)$ be defined as in (3.7) and θ be given as in (3.1). Under the assumptions of Lemma 12 and (3.4), we have $I_1(b) \gg \omega^3 M_0$.*

Proof. Just as in Lemma 8.2 of [9], this follows from Lemmas 9, 12, 13, 14, 21, and (4.2).

Proof of Theorem 1. By (3.7), Lemmas 22, 23, 12, and noting $\omega \geq Q^{-\varepsilon}$, we get

$$I(b) = I_1(b) + I_2(b) \gg \omega^3 N^2 |a_1 a_2 a_3|^{-1}$$

providing that $Q \gg B^{3+3\varepsilon_1}$, that is, $N \gg B^{(3+3\varepsilon_1)(38/3-20\varepsilon_1)} \gg B^{38-22\varepsilon_1-60\varepsilon_1^2}$ with ε_1 small enough. Noting the assumptions of Lemma 12, the proof of Theorem 1 is complete.

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