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Tate conjecture for twisted Siegel modular threefolds

by

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1. Introduction. Let X be a smooth projective variety of dimension n defined over a number field F. For a prime number l, let $H^i_{\mathrm{et}}(X,\bar{\mathbb{Q}}_l)$ be the l-adic cohomology of $\bar{X}=X\times_F\bar{\mathbb{Q}}$. For K a number field, we denote $\Gamma_K:=\mathrm{Gal}(\bar{\mathbb{Q}}/K)$. The Galois group Γ_F acts on $H^i_{\mathrm{et}}(X,\bar{\mathbb{Q}}_l)$ by a representation $\rho_{i,l}$. For a finite extension E of F, the elements of $V^i(X,E):=(H^{2i}_{\mathrm{et}}(X,\bar{\mathbb{Q}}_l)(i))^{\Gamma_E}$ are called *Tate classes* defined over E (here $H^{2i}_{\mathrm{et}}(X,\bar{\mathbb{Q}}_l)(i)$ is the Tate twist, and $H^{2i}_{\mathrm{et}}(X,\bar{\mathbb{Q}}_l)=\{0\}$ for $i>2n$).

Let $U^i(X)$ be the $\mathbb{Q}$-linear space of algebraic subvarieties of X of codimension i. We have the l-adic cycle map

$$d_{i,l}:U^i(X)\otimes\bar{\mathbb{Q}}_l\to H^{2i}_{\mathrm{et}}(X,\bar{\mathbb{Q}}_l)(i).$$

The cohomology classes in the image of this map are said to be *algebraic.*

For each finite extension E of F, we denote by $U^i(X,E)$ the subspace of $d_{i,l}(U^i(X)\otimes\bar{\mathbb{Q}}_l)$ left fixed by Γ_E. The first part of the Tate conjecture [TA] states that

$$U^i(X,E)=V^i(X,E),$$

i.e. each Tate class is algebraic.

The L-function $L^i(s,X_{/F})$ attached to the representation $\rho_{i,l}$ converges for $\mathrm{Re}(s)>1+i/2$. The second part of the Tate conjecture [TA] states that for each finite extension E of F, the L-function $L^{2i}(s,X_{/E})$ has a meromorphic continuation to the entire complex plane and the order of the pole at $s=i+1$ is equal to

$$\dim_{\bar{\mathbb{Q}}_l}U^i(X,E).$$

The first part of the Tate conjecture for Siegel modular threefold was proved by Weissauer in [W1] and [W2]. Also the second part of the Tate conjecture for Siegel modular threefolds was proved in [W1] and [W2], but only for solvable number fields.

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Let $S_K := S_{G,K}$ be the Siegel modular threefold associated to $G = \mathrm{GSp}_4$ and to some open compact subgroup K of $G(\mathbb{A}_{\mathbb{Q},\mathrm{f}})$, where $\mathbb{A}_{\mathbb{Q},\mathrm{f}}$ is the finite part of the ring of adeles $\mathbb{A}_{\mathbb{Q}}$ of $\mathbb{Q}$. Then S_K is defined over $\mathbb{Q}$. We fix a rational prime p. In this paper we consider K of the form $K := K_p \times H$, where K_p is the set of elements of $G(\mathbb{Z}_p)$ that are congruent to 1 modulo p, and H is an open compact subgroup of $G(\mathbb{A}_{\mathbb{Q},p,\mathrm{f}})$ where $\mathbb{A}_{\mathbb{Q},p,\mathrm{f}}$ is the finite part of the ring of adeles $\mathbb{A}_{\mathbb{Q}}$ away from p. Then S_K is a quasi-projective variety defined over $\mathbb{Q}$.

The variety S_K has a natural action of $G(\mathbb{Z}/p\mathbb{Z})$ (see §2). For H sufficiently small this action is free. We fix such a small group H. Consider a continuous Galois representation $\phi : \Gamma_{\mathbb{Q}} \to G(\mathbb{Z}/p\mathbb{Z})$ and let S'_K be the variety defined over $\mathbb{Q}$ obtained from S_K via twisting by ϕ composed with the natural action of $G(\mathbb{Z}/p\mathbb{Z})$ on S_K (see §2 for details).

The surfaces S_K and S'_K become isomorphic over $\bar{\mathbb{Q}}$ and by descent we deduce that the first part of the Tate conjecture for the surface S_K over a given number field k is true if and only if it is true for the surface S'_K over the field k. But as we said above, from [W1] and [W2] we know that the first part of the Tate conjecture holds for S_K, and hence it also holds for S'_K.

In this article (see Theorem 5.1) we generalize the results in [W1] and [W2] and prove that if $M := \bar{\mathbb{Q}}^{\ker(\phi)}$ is a totally real field, then the L-function $L^{2i}(s, (S'_K)_{/k})$, for $i = 0, 1, 2, 3$, has a meromorphic continuation to the entire complex plane and satisfies a functional equation, and the order of the pole at $s = i + 1$ is equal to

$$\dim_{\bar{\mathbb{Q}}_l} U^i(S'_K, k)$$

if k is a totally real number field.

2. Twisted Siegel modular threefolds. Let $G := \mathrm{GSp}_4$ be the symplectic similitudes group over $\mathbb{Q}$ of rank 4. Then

$$\mathrm{GSp}_4(A) = \left\{ g \in \mathrm{GL}_4(A) \;\middle|\; {}^t g \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} g = \mu(g) \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \text{ for some } \mu(g) \in A^{\times} \right\}$$

for all $\mathbb{Q}$-algebras A, where I_2 is the identity matrix of rank 2. Let Sp_4 be the symplectic group over $\mathbb{Q}$ of rank 4. Then

$$\mathrm{Sp}_4(A) = \left\{ g \in \mathrm{GL}_4(A) \;\middle|\; {}^t g \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \right\}$$

for all $\mathbb{Q}$-algebras A.

Consider the morphism of $\mathbb{R}$-groups

$$h : \operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{\mathrm{m}} \to G_{\mathbb{R}}$$

given by

$$x + iy \mapsto \begin{pmatrix} xI_2 & yI_2 \\ -yI_2 & xI_2 \end{pmatrix}.$$

The stabilizer of h in $G(\mathbb{R})$ is $K_\infty = Z_\infty K_{\mathbb{R}}$, where Z_∞ is the center of $G(\mathbb{R})$, and $K_{\mathbb{R}}$ is a maximal compact subgroup of $\mathrm{Sp}_4(\mathbb{R})$.

For K a sufficiently small open compact subgroup of $G(\mathbb{A}_{\mathbb{Q},\mathrm{f}})$, let S_K be the smooth toroidal compactification of an open surface S_K^0 that satisfies

$$S_K^0(\mathbb{C}) = G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}})/K_\infty K,$$

which is a disjoint union of arithmetic quotients of the Siegel upper half plane of degree 2. Hence S_K has dimension 3, and is called a *Siegel modular threefold*. From [D], we know that S_K is defined over $\mathbb{Q}$.

Let p be a rational prime. Consider $K := K_p \times H$, where K_p is the set of elements of $G(\mathbb{Z}_p)$ that are congruent to 1 modulo p and H is an open compact subgroup of $G(\mathbb{A}_{\mathbb{Q},p,\mathrm{f}})$ where $\mathbb{A}_{\mathbb{Q},p,\mathrm{f}}$ is the finite part of the ring of adeles $\mathbb{A}_{\mathbb{Q}}$ away from p. Then it is well known (see for example Corollary 1.4.1.3 of [C]) that for H sufficiently small, the group $G(\mathbb{Z}/p\mathbb{Z})$ acts freely on S_K^0 and also on its compactification S_K. We fix such a small H. Then the action of $G(\mathbb{Z}/p)$ on

$$S_K^0(\mathbb{C}) = G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}})/K_\infty K$$

can be described in the following way: we have $G(\mathbb{Z}_p) \hookrightarrow G(\mathbb{A}_{\mathbb{Q}})$ by $\alpha \mapsto (1, \ldots, 1, \alpha, 1, \ldots, 1)$, α at the pth component. Using the isomorphism $G(\mathbb{Z}/p\mathbb{Z}) \cong G(\mathbb{Z}_p)/K_p$, the action of an element $g \in G(\mathbb{Z}_p)$ is given by right multiplication at the pth component.

We fix a continuous representation

$$\phi : \Gamma_{\mathbb{Q}} \to G(\mathbb{Z}/p\mathbb{Z}).$$

Let M be the finite Galois extension of $\mathbb{Q}$ defined by $M := (\bar{\mathbb{Q}})^{\ker(\phi)}$. We remark that $G(\mathbb{Z}/p\mathbb{Z})$ is not necessarily solvable, and thus M is not necessarily a solvable extension of $\mathbb{Q}$.

Let

$$S' = S_K \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(M).$$

The group $G(\mathbb{Z}/p\mathbb{Z})$ acts on S_K. Since $\phi : \mathrm{Gal}(M/\mathbb{Q}) \hookrightarrow G(\mathbb{Z}/p\mathbb{Z})$, the group $\mathrm{Gal}(M/\mathbb{Q})$ acts on S_K. We denote this last action by ϕ'. The Galois group $\mathrm{Gal}(M/\mathbb{Q})$ has a natural action on $\mathrm{Spec}(M)$ and we can descend via the quotient process S' to $S'_K/\mathrm{Spec}(\mathbb{Q})$ using the diagonal action

$$\mathrm{Gal}(M/\mathbb{Q}) \ni \sigma \mapsto \phi'(\sigma) \otimes \sigma$$

on S'. Thus, we obtain a quasi-projective variety $S'_K/\mathrm{Spec}(\mathbb{Q})$. This is the *twisted Siegel modular threefold* mentioned in the title.

3. Zeta function of twisted Siegel modular threefolds. Let K be a sufficiently small open compact subgroup of $G(\mathbb{A}_{\mathbb{Q},\mathrm{f}})$. Then we have a decomposition

$$H^{2i}_{\mathrm{et}}(S_K,\bar{\mathbb{Q}}_l) = IH^{2i}_{\mathrm{et}}(\bar{S}_K,\bar{\mathbb{Q}}_l) \oplus H^{2i}(S^{\infty}_K,\bar{\mathbb{Q}}_l)$$

where $IH^{2i}_{\mathrm{et}}(\bar{S}_K,\bar{\mathbb{Q}}_l)$ is the intersection cohomology of the Baily–Borel compactification $\bar{S}_K$ of S^0_K, and S^{∞}_K is the divisor at infinity (a finite set of cusps) such that $\bar{S}_K = S^0_K \cup S^{\infty}_K$, and is defined by

$$IH^{2i}_{\mathrm{et}}(\bar{S}_K,\bar{\mathbb{Q}}_l) := \mathrm{Im}(H^{2i}_{\mathrm{et}}(S_K,\bar{\mathbb{Q}}_l) \to H^{2i}_{\mathrm{et}}(S^0_K,\bar{\mathbb{Q}}_l)).$$

If l is a prime number, let $\mathbb{H}_K$ be the Hecke algebra generated by the bi-K-invariant $\bar{\mathbb{Q}}_l$-valued compactly supported functions on $G(\mathbb{A}_{\mathbb{Q},\mathrm{f}})$ under convolution. If $\Pi = \Pi_{\mathrm{f}} \otimes \Pi_{\infty}$ is an automorphic representation of $G(\mathbb{A}_{\mathbb{Q}})$, we denote by Π^K_{f} the space of K-invariants in Π_{f}. The Hecke algebra $\mathbb{H}_K$ acts on Π^K_{f}.

We have an action of the Hecke algebra $\mathbb{H}_K$ and an action of the Galois group $\Gamma_{\mathbb{Q}}$ on the intersection cohomology $IH^{2i}_{\mathrm{et}}(\bar{S}_K,\bar{\mathbb{Q}}_l)$ and these two actions commute. An automorphic representation Π of $G(\mathbb{A}_{\mathbb{Q}})$ is called *cohomological* if $H^*(G(\mathbb{R}),K_{\infty},\Pi_{\infty}) \neq 0$.

We know the following result (see [W1]):

PROPOSITION 3.1. *The representation of $\Gamma_{\mathbb{Q}} \times \mathbb{H}_K$ on the intersection cohomology $IH^{2i}_{\mathrm{et}}(\bar{S}_K,\bar{\mathbb{Q}}_l)$ is isomorphic to*

$$\bigoplus_{\Pi} \phi^i(\Pi_{\mathrm{f}}) \otimes \Pi^K_{\mathrm{f}},$$

where $\phi^i(\Pi_{\mathrm{f}})$ is a continuous representation of the Galois group $\Gamma_{\mathbb{Q}}$. The above sum is over cohomological automorphic representations $\Pi = \Pi_{\mathrm{f}} \otimes \Pi_{\infty}$ of $G(\mathbb{A}_{\mathbb{Q}})$ that occur in the discrete spectrum of $G(\mathbb{A}_{\mathbb{Q}})$ and the $\mathbb{H}_K$-representations Π^K_{f} are irreducible and mutually inequivalent.

The representation $\phi^i(\Pi_{\mathrm{f}})$ that appears in Proposition 3.1 is semisimple (see Theorem 1.1 and §1.7 of [W3]) and has dimension at most 2, is unramified outside some finite set of primes S which depends on K, is de Rham at l, and is crystalline at l if $l \notin S$.

We fix an isomorphism $\iota : \bar{\mathbb{Q}}_l \to \mathbb{C}$ and define the L-function

$$L^{2i}(s,S_K) := \prod_{\Pi} L(s,\phi^i(\Pi_{\mathrm{f}}))^{\dim \Pi^K_{\mathrm{f}}},$$

where

$$L(s,\phi^i(\Pi_{\mathrm{f}})) := \prod_q \det(1 - Nq^{-s}\iota(\phi^i(\Pi_{\mathrm{f}})(\mathrm{Frob}_q)^{I_q}))^{-1},$$

where Frob_q is a geometric Frobenius element at a finite rational prime q and I_q is an inertia group at q.

We consider the injective limit

$$V^i := \varinjlim_K IH^{2i}_{\mathrm{et}}(\bar{S}_K, \bar{\mathbb{Q}}_l) \cong \varinjlim_K \bigoplus_{\Pi} U^i(\Pi_{\mathrm{f}}) \otimes_{\bar{\mathbb{Q}}_l} \Pi_{\mathrm{f}}^K,$$

where $U^i(\Pi_{\mathrm{f}})$ is the space corresponding to $\phi^i(\Pi_{\mathrm{f}})$ (see Proposition 3.1 for notation).

Then the Π-component $V^i(\Pi)$ of V^i is isomorphic to $\phi^i(\Pi_{\mathrm{f}}) \otimes \Pi_{\mathrm{f}}$ as $\Gamma_{\mathbb{Q}} \times \mathbb{H}$-module. Taking the K-fixed vectors, we deduce that $V^i(\Pi)^K$ is isomorphic to $\phi^i(\Pi_{\mathrm{f}}) \otimes \Pi_{\mathrm{f}}^K$ as $\Gamma_{\mathbb{Q}} \times G(\mathbb{Z}/p\mathbb{Z})$-module. Since the varieties S_K and S'_K become isomorphic over $\bar{\mathbb{Q}}$, we have the isomorphism $IH^{2i}_{\mathrm{et}}(\bar{S}_K, \bar{\mathbb{Q}}_l) \cong IH^{2i}_{\mathrm{et}}(\bar{S}'_K, \bar{\mathbb{Q}}_l)$. The actions of $\Gamma_{\mathbb{Q}}$ on these cohomologies that give the expression of the zeta functions of these varieties are different. If we consider the component $V^i(\Pi)'$ that corresponds to Π of $IH^{2i}_{\mathrm{et}}(\bar{S}'_K, \bar{\mathbb{Q}}_l)$ (see the decomposition of Proposition 3.1), we find that $V^i(\Pi)'$ is isomorphic to $\phi^i(\Pi_{\mathrm{f}}) \otimes (\Pi_{\mathrm{f}}^K \circ \phi)$ as $\Gamma_{\mathbb{Q}}$-module. Hence we deduce the following result:

Proposition 3.2. *We have*

$$L^{2i}(s, S'_K) = \prod_{\Pi} L(s, \phi^i(\Pi_{\mathrm{f}}) \otimes (\Pi_{\mathrm{f}}^K \circ \phi)),$$

where Π is as in Proposition 3.1*.*

4. Meromorphic continuation. In this section we fix an automorphic representation Π as in Proposition 3.1.

We show the following result:

Theorem 4.1. *With the same notation as in* §3*, if F is a totally real number field, then there exists a totally real finite extension F' of F, which is Galois over $\mathbb{Q}$, such that $\phi^i(\Pi_{\mathrm{f}})|_{\Gamma_{F'}}$ is automorphic, i.e. $\phi^i(\Pi_{\mathrm{f}})|_{\Gamma_{F'}} \cong \rho_{\Pi'}$, where Π' is an automorphic representation of $GL_m(\mathbb{A}_{F'})$ and $\rho_{\Pi'}$ is the l-adic representation associated to Π'.*

Proof. We distinguish two cases (see [W1], [W2]):

(i) The representation $\phi^i(\Pi_{\mathrm{f}})|_{\Gamma_F}$ is trivial or a direct sum of one or two 1-dimensional Hecke characters, and thus Theorem 4.1 is obvious in this case, and the base change is actually arbitrary.

(ii) The representation $\phi^i(\Pi_{\mathrm{f}})|_{\Gamma_F}$ is irreducible of dimension 2, has τ-Hodge–Tate numbers 0 and 1 for any embedding $\tau : F \hookrightarrow \bar{\mathbb{Q}}$, and is totally odd, i.e. $\det \phi^i(\Pi_{\mathrm{f}})|_{\Gamma_F}(c) = -1$ for any complex conjugation c. Hence from Theorem A of [BGGT] (see the properties of $\phi^i(\Pi_{\mathrm{f}})$ after Proposition 3.1 above), we conclude the proof of Theorem 4.1. ∎

We denote $\omega := \Pi_{\mathrm{f}}^K \circ \phi$. We assume throughout this paper that the field $M := \bar{\mathbb{Q}}^{\ker(\phi)}$ is totally real. Thus $N := \bar{\mathbb{Q}}^{\ker(\omega)}$ is totally real.

Let k be a totally real number field. From Theorem 4.1 we deduce that there exists a Galois totally real field F' containing k and M such that $\phi^i(\Pi_{\mathrm{f}})|_{\Gamma_{F'}} \cong \rho_{\Pi'}$, where Π' is an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{F'})$.

From Brauer's theorem (see Theorems 16 and 19 of [SE]), we know that one can find subfields $F_j \subset F'$ with $\mathrm{Gal}(F'/F_j)$ solvable, characters $\chi_j : \mathrm{Gal}(F'/F_j) \to \bar{\mathbb{Q}}^\times$ and integers m_j such that the representation

$$\omega|_{\Gamma_k} : \mathrm{Gal}(F'/k) \to \mathrm{Gal}(Mk/k) \to \mathrm{GL}_N(\bar{\mathbb{Q}}_l)$$

can be written as $\omega|_{\Gamma_k} = \sum_{j=1}^u m_j \operatorname{Ind}_{\Gamma_{F_j}}^{\Gamma_k} \chi_j$ (a virtual sum). Then

$$L(s, (\phi^i(\Pi_{\mathrm{f}}) \otimes \omega)|_{\Gamma_k}) = \prod_{j=1}^u L(s, \phi^i(\Pi_{\mathrm{f}})|_{\Gamma_k} \otimes \operatorname{Ind}_{\Gamma_{F_j}}^{\Gamma_k} \chi_j)^{m_j}$$
$$= \prod_{j=1}^u L(s, \operatorname{Ind}_{\Gamma_{F_j}}^{\Gamma_k} (\phi^i(\Pi_{\mathrm{f}})|_{\Gamma_{F_j}} \otimes \chi_j))^{m_j} = \prod_{j=1}^u L(s, \phi^i(\Pi_{\mathrm{f}})|_{\Gamma_{F_j}} \otimes \chi_j)^{m_j}.$$

Since $\phi^i(\Pi_{\mathrm{f}})|_{\Gamma_{F'}}$ is automorphic and $\mathrm{Gal}(F'/F_j)$ is solvable, it follows easily that $\phi^i(\Pi_{\mathrm{f}})|_{\Gamma_{F_j}}$ is automorphic. Hence the function $L(s, (\phi^i(\Pi_{\mathrm{f}}) \otimes \omega)|_{\Gamma_k})$ has a meromorphic continuation to the entire complex plane and satisfies a functional equation because each function $L(s, \phi^i(\Pi_{\mathrm{f}})|_{\Gamma_{F_j}} \otimes \chi_j)$ has these properties.

5. Tate conjecture for twisted Siegel modular threefolds. Assume that k is a totally real field, and Π is an automorphic representation that appears in Proposition 3.1. Let $V^i(\Pi)'$ be the space considered in §3 just before Proposition 3.2.

Recall that in §4 we denoted $\omega := \Pi_{\mathrm{f}}^K \circ \phi$ and we assumed that $M := \bar{\mathbb{Q}}^{\ker(\phi)}$ is a totally real field and thus $N := \bar{\mathbb{Q}}^{\ker(\omega)}$ is also a totally real field.

Define

$$\mathbf{V}^i(\Pi, k) := \{x \in V^i(\Pi)' \mid (\phi^i(\Pi_{\mathrm{f}}) \otimes \omega)(a)x = \xi_l^{-i}(a)x \text{ for all } a \in \Gamma_k\},$$

where ξ_l is the l-adic cyclotomic character. The elements of $\mathbf{V}^i(\Pi, k)$ are called *Tate classes.*

We will prove the following result:

THEOREM 5.1. *Assume that $N := \bar{\mathbb{Q}}^{\ker(\omega)}$ and k are totally real fields. Then the order of the pole of the L-function $L(s, (\phi^i(\Pi_{\mathrm{f}}) \otimes \omega)|_{\Gamma_k})$ at $s = i+1$ is equal to* $\dim_{\bar{\mathbb{Q}}_l} \mathbf{V}^i(\Pi, k)$.

We consider

$$\mathbf{V}^i(\Pi, F_j) := \{x \in V^i(\Pi)' \mid (\phi^i(\Pi_{\mathrm{f}}) \otimes \chi_j)(a)x = \xi_l^{-i}(a)x \text{ for all } a \in \Gamma_{F_j}\}.$$

Since $\omega|_{\Gamma_k} = \sum_{j=1}^{u} m_j \operatorname{Ind}_{\Gamma_{F_j}}^{\Gamma_k} \chi_j$, in order to prove Theorem 5.1, it is sufficient to show the following result:

PROPOSITION 5.2. *For each i, the order of the pole of $L(s, \phi^i(\Pi_{\mathrm{f}})|_{\Gamma_{F_j}} \otimes \chi_j)$ at $s = i + 1$ is equal to $\dim_{\bar{\mathbb{Q}}_l} \mathbf{V}^i(\Pi, F_j)$.*

Proof. In case (i) (see the proof of Theorem 4.1 above), $\phi^i(\Pi_{\mathrm{f}})|_{\Gamma_{F_j}}$ is a direct sum of one-dimensional representations. So it is easy to see that the pole of $L(s, \phi^i(\Pi_{\mathrm{f}})|_{\Gamma_{F_j}} \otimes \chi_j)$ at $s = i + 1$ is equal to the dimension of the space of Tate classes $\mathbf{V}^i(\Pi, F_j)$ (so in case (i), Theorem 5.1 is true actually for any K and k). Hence we are done in case (i).

In case (ii), $\mathbf{V}^i(\Pi, F_j) = \emptyset$. Also in this case the automorphic l-adic representation $\phi^i(\Pi_{\mathrm{f}})|_{\Gamma_{F_j}}$ corresponds to a cuspidal representation Π_j of $\mathrm{GL}(2)/F_j$, and thus the function $L(s, \phi^i(\Pi_{\mathrm{f}})|_{\Gamma_{F_j}} \otimes \chi_j)$ has no pole or zero at $s = i + 1$. Hence we are done also in case (ii). ■

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