ACTA ARITHMETICA
XCVI.1 (2000)

A lattice point problem associated with two polynomials

by

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1. Introduction. B. Lichtin [5] proves the following result: *Let* $g_\nu \in \mathbb{R}[x,y]$, $\nu = 1,2$, *be two polynomials which are nondegenerate with respect to their polygon at infinity and hypoelliptic on* $[1,\infty)^2$. *Define*

$$R(A_1,A_2) := \#\{(x,y) \in \mathbb{N}^2 \mid g_\nu(x,y) \le A_\nu,\ \nu = 1,2\}$$

for $A_1, A_2 > 0$. *There are finitely many sets*

$$\mathcal{R}_j = \{(A_1,A_2) \in [1,\infty)^2 \mid A_1^{b_j} \le A_2 \le A_1^{B_j}\}$$

with $0 \le b_j < B_j \le \infty$ *and* $[1,\infty)^2 = \bigcup_j \mathcal{R}_j$, *polynomials* $p_j(u,v) \in \mathbb{R}[u,v]$ *which are positive outside some compact subset of* $[1,\infty)^2$ *and constants* $u_j, v_j > 0$ *so that the following asymptotics holds*: *If* $\mathcal{R}_\infty \subseteq \mathcal{R}_j$ *is an unbounded connected semialgebraic set with* $\operatorname{dist}((A_1,A_2),\partial\mathcal{R}_j) \to \infty$ *as* $(A_1,A_2) \to (\infty,\infty)$ *in* $\mathcal{R}_\infty$ *then with some* $\Theta > 0$,

$$R(A_1,A_2) = A_1^{u_j} A_2^{v_j} p_j(\log A_1, \log A_2) + O(A_1^{u_j-\Theta} A_2^{v_j-\Theta})$$

as $(A_1,A_2) \to (\infty,\infty)$ *in* $\mathcal{R}_\infty$.

Lichtin gives an explicit description of $\mathcal{R}_j$, u_j, v_j in terms of the region of analyticity of some Dirichlet series which is associated with the polynomials g_ν. It is the aim of this paper to derive a much sharper asymptotic expansion of $R(A_1,A_2)$ under conditions on the polynomials g_ν which are in some sense complementary to those of Lichtin.

Let

$$g_\nu(x,y) = \sum_{i+j \le d_\nu} a_{ij}^{(\nu)} x^i y^j \in \mathbb{Z}[x,y], \qquad \nu = 1,2,$$

be polynomials with nonnegative integer coefficients and $a_{d_\nu 0}^{(\nu)} a_{0 d_\nu}^{(\nu)} \ne 0$. Define

$$\widetilde{g}_\nu(x,y) = \sum_{i+j = d_\nu} a_{ij}^{(\nu)} x^i y^j \in \mathbb{Z}[x,y]$$

2000 *Mathematics Subject Classification*: Primary 11P21.
Key words and phrases: lattice points, polynomials.

and let the functions $\widetilde{y}_\nu : [0,\widetilde{\xi}_\nu] \to \mathbb{R}_0^+$ and $\widetilde{x}_\nu : [0,\widetilde{\eta}_\nu] \to \mathbb{R}_0^+$ be implicitly defined by

$$\begin{aligned}\widetilde{g}_\nu(x,\widetilde{y}_\nu(x)) = 1 \quad &\text{for } 0 \le x \le \widetilde{\xi}_\nu,\ \widetilde{y}_\nu(\widetilde{\xi}_\nu) = 0,\\ \widetilde{g}_\nu(\widetilde{x}_\nu(y),y) = 1 \quad &\text{for } 0 \le y \le \widetilde{\eta}_\nu,\ \widetilde{x}_\nu(\widetilde{\eta}_\nu) = 0.\end{aligned}$$

Assume that the rational function

$$\widetilde{g}_1(1,y)^{d_2}/\widetilde{g}_2(1,y)^{d_1} \tag{1.1}$$

is not constant. Assume further that $\widetilde{g}_\nu$ is not of the form

$$a_{0d_\nu}^{(\nu)}(y-bx)^{d_\nu} \quad \text{or} \quad a_{d_\nu 0}^{(\nu)}(x-cy)^{d_\nu} \tag{1.2}$$

with some $b,c \in \mathbb{R}$. This paper is devoted to the proof of

THEOREM 1.1. *There is a decomposition* $0 = C_{-1}^* < C_0^* < \ldots < C_m^* < C_{m+1}^* = \infty$ *and constants* $K,\varepsilon > 0$ *with the property*: *For* $0 \le \mu \le m+1$ *and* $A_1, A_2 \ge K$ *with* $C := A_2 A_1^{-d_2/d_1} \in (C_{\mu-1}^* + \min\{A_1,A_2\}^{-\varepsilon}, C_\mu^* - \min\{A_1,A_2\}^{-\varepsilon})$ *we have*

$$\begin{aligned}R(A_1,A_2) = \operatorname{area}(\{(x,y)\in\mathbb{R}^2 \mid x,y\ge 0,\ g_\nu(x,y)\le A_\nu,\ \nu=1,2\})\\ + T_1^{(\mu)}(A_1) + T_2^{(\mu)}(A_2) + U_1^{(\mu)}(A_1) + U_2^{(\mu)}(A_2)\\ - \tfrac{1}{2}\min\{A_1^{1/d_1}\widetilde{\xi}_1, A_2^{1/d_2}\widetilde{\xi}_2\} - \tfrac{1}{2}\min\{A_1^{1/d_1}\widetilde{\eta}_1, A_2^{1/d_2}\widetilde{\eta}_2\}\\ + O(A_1^{46/(73d_1)}(\log A_1)^{315/146})\\ + O(A_2^{46/(73d_2)}(\log A_2)^{315/146}).\end{aligned}$$

The representation

$$\begin{aligned}T_\nu^{(\mu)}(A_\nu) = A_\nu^{(1/d_\nu)(1-1/(p_{\mu,\nu}+2))}\sum_{l=1}^{L_\nu^{(\mu)}} H_{\nu l}^{(\mu)}(A_\nu^{1/d_\nu})\\ + O(A_\nu^{(1/d_\nu)(1-1/(p_{\mu,\nu}+2)-1/(p_{\mu,\nu}+2)^2)}\log A_\nu)\end{aligned}$$

holds with $p_{\mu,\nu}\in\mathbb{N}$ *and* $H_{\nu l}^{(\mu)}$ *periodic functions which are given by absolutely convergent Fourier series. Furthermore, with some* $q_{\mu,\nu}\in\mathbb{N}$,

$$U_\nu^{(\mu)}(A_\nu) = O_\delta(A_\nu^{(1/d_\nu)(1-1/q_{\mu,\nu}+\delta)})$$

for each $\delta > 0$. $T_\nu^{(\mu)}$ (*resp.* $U_\nu^{(\mu)}$) *can only appear if there is a zero of* $\widetilde{y}_\nu''$ *or* $\widetilde{x}_\nu''$ *at which* $\widetilde{y}_\nu'$ *or* $\widetilde{x}_\nu'$ *is rational* (*resp. irrational*). $p_{\mu,\nu}$ (*resp.* $q_{\mu,\nu}$) *is at most* $d_\nu - 2$.

The proof consists of three parts. In a combinatorial part the set in which the lattice points are counted is dissected into finitely many subsets. Each of these has a boundary which is described by one single algebraic curve. These local problems can be treated in just the same way as in [9]. That is the analytical part which uses exponential sum estimates. The last section is concerned with the question whether the requirements of this local analysis can be met by imposing suitable conditions on the relative position of A_1 and A_2. This part is again combinatorial in nature.

In principle it is possible to give an explicit description of C_μ^*, $T_\nu^{(\mu)}$, $U_\nu^{(\mu)}$ and ε in terms of $\widetilde{y}_\nu$, $\widetilde{x}_\nu$ and the coefficients of the g_ν. In Sections 4, 5 and 6 this is done for the local lattice point asymptotics. In the global asymptotics an explicit description of all the constants and functions would make it necessary to distinguish between a lot of cases. Therefore I chose the above formulation which lays emphasis on the structure of the asymptotic formula and not on explicit calculations.

In the following $\varepsilon > 0$ will be a sufficiently small constant, $K > 0$ a sufficiently large constant and $K_j > 0$, $j \in \mathbb{N}$, constants depending only on the g_ν. Furthermore $\psi(x) := x - [x] - 1/2$.

I would like to thank the referee for a suggestion which led to an improvement of the paper.

2. Reduction to algebraic boundary curves. Define an auxiliary function $h : [0, \widetilde{\xi}_1] \to \mathbb{R}^+$ by $h(x) := \widetilde{g}_2(x, \widetilde{y}_1(x))$.

LEMMA 2.1. *The function h is not constant.*

Proof. Assume $h(x) = \lambda \in \mathbb{R}^+$ for each $x \in [0, \widetilde{\xi}_1]$. Then for $0 < x < \widetilde{\xi}_1$,

$$\begin{aligned}\lambda^{d_1}\widetilde{g}_1(1, x^{-1}\widetilde{y}_1(x))^{d_2} &= \lambda^{d_1}x^{-d_1d_2}\widetilde{g}_1(x, \widetilde{y}_1(x))^{d_2} = h(x)^{d_1}x^{-d_1d_2}\\ &= x^{-d_1d_2}\widetilde{g}_2(x, \widetilde{y}_1(x))^{d_1} = \widetilde{g}_2(1, x^{-1}\widetilde{y}_1(x))^{d_1}.\end{aligned}$$

With $\widetilde{y}_1(x) \to \widetilde{y}_1(0) > 0$ as $x \to 0+0$ it follows that $x^{-1}\widetilde{y}_1(x) \to \infty$ as $x \to 0+0$. Therefore the identity $\lambda^{d_1}\widetilde{g}_1(1, y)^{d_2} = \widetilde{g}_2(1, y)^{d_1}$ holds for infinitely many y and consequently it holds as a polynomial identity. This contradicts the assumption on (1.1). ∎

Define the homogeneous polynomials of degree d_ν

$$g_\nu(\tau, x, y) := \sum_{i+j\le d_\nu} a_{ij}^{(\nu)}\tau^{d_\nu-i-j}x^iy^j \in \mathbb{Z}[\tau, x, y].$$

Define

$$\begin{aligned} j_0^{(\nu)} &:= \min\{1 \le j \le d_\nu \mid a_{d_\nu-j,j}^{(\nu)} \ne 0\},\\ i_0^{(\nu)} &:= \min\{1 \le i \le d_\nu \mid a_{i,d_\nu-i}^{(\nu)} \ne 0\}.\end{aligned}$$

For $0 \le \tau < (a_{00}^{(\nu)})^{-1/d_\nu}$ the functions $y_\nu(\tau, \cdot) : [0, \xi_\nu(\tau)] \to \mathbb{R}_0^+$ and $x_\nu(\tau, \cdot) : [0, \eta_\nu(\tau)] \to \mathbb{R}_0^+$ are implicitly defined by

$$\begin{aligned} g_\nu(\tau, x, y_\nu(\tau, x)) &= 1, \quad 0 \le x \le \xi_\nu(\tau),\ y_\nu(\tau, \xi_\nu(\tau)) = 0,\\ g_\nu(\tau, x_\nu(\tau, y), y) &= 1, \quad 0 \le y \le \eta_\nu(\tau),\ x_\nu(\tau, \eta_\nu(\tau)) = 0.\end{aligned}$$

Both are strictly decreasing and inverse to each other. For $0 \le x < \xi_\nu(\tau)$,

$$
\begin{aligned}
(2.1)\quad & g_{\nu y}(\tau, x, y_\nu(\tau,x)) \\
& \ge \max\{a^{(\nu)}_{d_\nu - j_0^{(\nu)}, j_0^{(\nu)}} j_0^{(\nu)} x^{d_\nu - j_0^{(\nu)}} y_\nu(\tau,x)^{j_0^{(\nu)}-1}, a^{(\nu)}_{0 d_\nu} d_\nu y_\nu(\tau,x)^{d_\nu - 1}\} > 0.
\end{aligned}
$$

Consequently, y_ν is C^∞ on an open neighbourhood of

$$\{(\tau, x) \mid 0 \le \tau < (a^{(\nu)}_{00})^{-1/d_\nu},\ 0 \le x < \xi_\nu(\tau)\}$$

by the implicit function theorem. If $j_0^{(\nu)} = 1$ then (2.1) is also valid for $x = \xi_\nu(\tau)$ and consequently y_ν is C^∞ on an open neighbourhood of

$$\{(\tau, x) \mid 0 \le \tau < (a^{(\nu)}_{00})^{-1/d_\nu},\ 0 \le x \le \xi_\nu(\tau)\}.$$

We have $\eta_\nu(\tau) = y_\nu(\tau, 0)$ for $0 \le \tau < (a^{(\nu)}_{00})^{-1/d_\nu}$. Consequently, η_ν is C^∞ in this interval and $\eta_\nu(\tau) = \widetilde{\eta}_\nu + O(\tau)$ as $\tau \to 0$. Analogous results are valid for x_ν.

For $A_1, A_2 > \max\{a^{(1)}_{00}, a^{(2)}_{00}, 1\}$ define $f_{A_1,A_2} : [0, \varrho_{A_1,A_2}] \to \mathbb{R}_0^+$ by

$$
\begin{aligned}
f_{A_1,A_2}(x) &:= \min\{A_1^{1/d_1} y_1(A_1^{-1/d_1}, A_1^{-1/d_1} x), A_2^{1/d_2} y_2(A_2^{-1/d_2}, A_2^{-1/d_2} x)\}, \\
\varrho_{A_1,A_2} &:= \min\{A_1^{1/d_1} \xi_1(A_1^{-1/d_1}), A_2^{1/d_2} \xi_2(A_2^{-1/d_2})\}.
\end{aligned}
$$

Then the following equivalence holds for $x, y \ge 0$:

$$
\begin{aligned}
(2.2)\quad & g_\nu(x,y) \le A_\nu,\ \nu = 1,2 \\
& \Leftrightarrow g_\nu(A_\nu^{-1/d_\nu}, A_\nu^{-1/d_\nu} x, A_\nu^{-1/d_\nu} y) \le 1,\ \nu = 1,2 \\
& \Leftrightarrow A_\nu^{-1/d_\nu} x \le \xi_\nu(A_\nu^{-1/d_\nu}),\ A_\nu^{-1/d_\nu} y \le y_\nu(A_\nu^{-1/d_\nu}, A_\nu^{-1/d_\nu} x),\ \nu = 1,2 \\
& \Leftrightarrow x \le \varrho_{A_1,A_2},\ y \le f_{A_1,A_2}(x).
\end{aligned}
$$

Now it is clear that $R(A_1, A_2)$ is the number of lattice points below the graph of f_{A_1,A_2}. The function $\delta : (0, (\widetilde{\xi}_1 \widetilde{\xi}_2^{-1})^{d_2}] \to \mathbb{R}$ with

$$\delta(C) := \min\{\widetilde{y}_1(x) - C^{1/d_2} \widetilde{y}_2(C^{-1/d_2} x) \mid 0 \le x \le C^{1/d_2} \widetilde{\xi}_2\}$$

is continuous with $\lim_{C\to 0} \delta(C) = \widetilde{y}_1(0) > 0$ and $\delta((\widetilde{\xi}_1 \widetilde{\xi}_2^{-1})^{d_2}) \le \widetilde{y}_1(\widetilde{\xi}_1) - \widetilde{\xi}_1 \widetilde{\xi}_2^{-1} \widetilde{y}_2(\widetilde{\xi}_2) = 0$. Choose $0 < C_0 \le (\widetilde{\xi}_1 \widetilde{\xi}_2^{-1})^{d_2}$ minimal with $\delta(C_0) = 0$ and fix $0 < x_0 < x_1 < C_0^{1/d_2} \widetilde{\xi}_2$ with $h'(x_0) \ne 0$. Set $C_1 := (x_1 \widetilde{\xi}_2^{-1})^{d_2} < C_0$. Then for $0 < C \le C_1$, $0 \le x \le C^{1/d_2} \widetilde{\xi}_2$ we have

$$\widetilde{y}_1(x) - C^{1/d_2} \widetilde{y}_2(C^{-1/d_2} x) \ge \delta(C) \ge \min\{\delta(C') \mid 0 \le C' \le C_1\} =: K_1 > 0.$$

So for $A_1, A_2 \ge K$, $C := A_2 A_1^{-d_2/d_1} \le C_1$, $0 \le x \le \varrho_{A_1,A_2}$, with $\tau_\nu := A_\nu^{-1/d_\nu}$, we have

$$A_2^{1/d_2} y_2(\tau_2, \tau_2 x) \le A_1^{1/d_1} C^{1/d_2} \widetilde{y}_2(C^{-1/d_2} \tau_1 x) \le A_1^{1/d_1} (\widetilde{y}_1(\tau_1 x) - K_1).$$

In particular $\widetilde{y}_1(\tau_1 x) \geq K_1 > 0$ and so $\tau_1 x \leq \widetilde{\xi}_1 - K_3$ with some constant $K_3 > 0$. Now y_1 is C^∞ on an open neighbourhood of $[0, \tau_0] \times [0, \widetilde{\xi}_1 - K_3]$ for sufficiently small τ_0. The mean value theorem gives $y_1(\tau_1, \tau_1 x) = \widetilde{y}_1(\tau_1 x) + O(\tau_1)$ and so

$$A_2^{1/d_2} y_2(\tau_2, \tau_2 x) \leq A_1^{1/d_1}(\widetilde{y}_1(\tau_1 x) - K_1) \leq A_1^{1/d_1} y_1(\tau_1, \tau_1 x)$$

if K is sufficiently large. It follows that

$$f_{A_1,A_2}(x) = A_2^{1/d_2} y_2(\tau_2, \tau_2 x). \tag{2.3}$$

Furthermore from $x_1 < \widetilde{\xi}_1$ it follows that

$$\begin{aligned} A_2^{1/d_2} \xi_2(\tau_2) &= A_2^{1/d_2}(\widetilde{\xi}_2 + O(\tau_2)) \leq A_1^{1/d_1} C_1^{1/d_2} \widetilde{\xi}_2 + O(1) \\ &= A_1^{1/d_1}(x_1 + O(\tau_1)) \leq A_1^{1/d_1} \xi_1(\tau_1) \end{aligned}$$

and consequently

$$\varrho_{A_1,A_2} = A_2^{1/d_2} \xi_2(\tau_2). \tag{2.4}$$

Set $C_2 := \max\{(2\widetilde{\xi}_1 \widetilde{\xi}_2^{-1})^{d_2}, (2\widetilde{y}_1(0)\widetilde{y}_2(\widetilde{\xi}_2/2)^{-1})^{d_2}\}$. Then for $C \geq C_2$, $0 \leq x \leq \widetilde{\xi}_1$ we have

$$\widetilde{y}_1(x) \leq \widetilde{y}_1(0) \leq C^{1/d_2} \widetilde{y}_2(\widetilde{\xi}_2/2)/2 \leq C^{1/d_2}(\widetilde{y}_2(C^{-1/d_2} x) - \widetilde{y}_2(\widetilde{\xi}_2/2)/2).$$

For $A_1, A_2 \geq K$, $C := A_2 A_1^{-d_2/d_1} \geq C_2$, $0 \leq x \leq \varrho_{A_1,A_2}$, it follows that

$$A_1^{1/d_1} y_1(\tau_1, \tau_1 x) \leq A_1^{1/d_1} \widetilde{y}_1(\tau_1 x) \leq A_1^{1/d_1} C^{1/d_2}(\widetilde{y}_2(C^{-1/d_2}\tau_1 x) - \widetilde{y}_2(\widetilde{\xi}_2/2)/2).$$

In particular $\widetilde{y}_2(C^{-1/d_2}\tau_1 x) \geq \widetilde{y}_2(\widetilde{\xi}_2/2)/2 > 0$ and so $\tau_2 x \leq \widetilde{\xi}_2 - K_4$ for some constant $K_4 > 0$. Therefore

$$A_1^{1/d_1} y_1(\tau_1, \tau_1 x) \leq A_2^{1/d_2}(\widetilde{y}_2(\tau_2 x) + O(\tau_2)) = A_2^{1/d_2} y_2(\tau_2, \tau_2 x)$$

for sufficiently large K and so

$$f_{A_1,A_2}(x) = A_1^{1/d_1} y_1(\tau_1, \tau_1 x). \tag{2.5}$$

Furthermore

$$\begin{aligned} A_1^{1/d_1} \xi_1(\tau_1) &= A_1^{1/d_1}(\widetilde{\xi}_1 + O(\tau_1)) \leq A_2^{1/d_2} C_2^{-1/d_2} \widetilde{\xi}_1 + O(1) \\ &\leq A_2^{1/d_2} \widetilde{\xi}_2/2 + O(1) \leq A_2^{1/d_2} \xi_2(\tau_2) \end{aligned}$$

and consequently

$$\varrho_{A_1,A_2} = A_1^{1/d_1} \xi_1(\tau_1). \tag{2.6}$$

It remains to analyse the range $C_1 \leq C \leq C_2$. The functions $\widetilde{y}_\nu$ can be continued holomorphically to regions of the form

$$G_\nu := \{z \in \mathbb{C} \mid -\varepsilon < \Re z < \widetilde{\xi}_\nu + \varepsilon,\ |\Im z| < \varepsilon,\ z \notin [\widetilde{\xi}_\nu, \infty)\}$$

with at most an ordinary algebraic singularity of order $\le d_\nu$ at $\widetilde{\xi}_\nu$. This follows from general theorems on algebraic functions (for example [1], Chapter 8.2). Therefore h' is holomorphic on G_1 with at most an algebraic pole at $\widetilde{\xi}_1$ and consequently in $(0,\widetilde{\xi}_1)$ it has only a finite number of zeros $\xi_1 < \ldots < \xi_{r-1}$ with $r \in \mathbb{N}$. Define $\xi_0 := 0$, $\xi_r := \widetilde{\xi}_1$. Let $n_\varrho \in \mathbb{N}_0$ be the order of the zero ξ_ϱ of h'. For each $1 \le \varrho \le r$ the function $h\lceil[\xi_{\varrho-1},\xi_\varrho]$ is strictly monotonic and continuous. Therefore it has an inverse $k_\varrho : [a_\varrho, b_\varrho] \to [\xi_{\varrho-1},\xi_\varrho]$ which is strictly monotonic and continuous. For $A_1, A_2 \ge K$, $C := A_2 A_1^{-d_2/d_1} \in [C_1, C_2]$, we have

$$
\begin{gathered}
x_0 < x_1 < C_0^{1/d_2}\widetilde{\xi}_2 \le \widetilde{\xi}_1, \\
C^{-1/d_2} x_0 \le C_1^{-1/d_2} x_0 = x_0 x_1^{-1}\widetilde{\xi}_2 =: x_2 < \widetilde{\xi}_2
\end{gathered}
\tag{2.7}
$$

and consequently

$$
\begin{aligned}
A_1^{-1/d_1}\varrho_{A_1,A_2} &= \min\{\xi_1(\tau_1), C^{1/d_2}\xi_2(\tau_2)\} \\
&= \min\{\widetilde{\xi}_1 + O(\tau_1), C^{1/d_2}(\widetilde{\xi}_2 + O(\tau_2))\} \ge x_0
\end{aligned}
$$

for sufficiently large K. Furthermore

$$
\begin{aligned}
y_0 &:= y_{0,A_1,A_2} := A_1^{-1/d_1} f_{A_1,A_2}(A_1^{1/d_1}x_0) \\
&= \min\{\widetilde{y}_1(x_0) + O(\tau_1), C^{1/d_2}(\widetilde{y}_2(C^{-1/d_2}x_0) + O(\tau_2))\}.
\end{aligned}
$$

So there are constants y_1, y_2 with

$$
\begin{gathered}
y_0 \le \widetilde{y}_1(x_0) + O(\tau_1) \le y_1 < \widetilde{\eta}_1, \\
C^{-1/d_2} y_0 \le \widetilde{y}_2(C^{-1/d_2}x_0) + O(\tau_2) \\
\le \widetilde{y}_2(C_2^{-1/d_2}x_0) + O(\tau_2) \le y_2 < \widetilde{\eta}_2
\end{gathered}
\tag{2.8}
$$

for sufficiently large K.

Lemma 2.2. *Let $1 \le \varrho \le r$, $\xi_{\varrho-1} \le \overline{x} \le \xi_\varrho$, $\overline{x} < \widetilde{\xi}_1$, $0 < \overline{\overline{x}} < \widetilde{\xi}_2$, $\overline{C}_2 > \overline{C}_1 > 0$. There are constants $\varepsilon, K > 0$ with the property: For $A_1, A_2 \ge K$, $C := A_2 A_1^{-d_2/d_1} \in [\overline{C}_1, \overline{C}_2]$, $|C - a_\varrho|$, $|C - b_\varrho| \ge A_1^{-\varepsilon}$, $x \in [\xi_{\varrho-1}, \overline{x}]$, $x \le C^{1/d_2}\overline{\overline{x}}$, and $|x - k_\varrho(C)| \ge A_1^{-50/(73d_1)}$ if $C \in [a_\varrho, b_\varrho]$ then*

$$
f_{A_1,A_2}(A_1^{1/d_1}x) = \begin{cases} A_1^{1/d_1} y_1(\tau_1, x), & h(x) < C, \\ A_2^{1/d_2} y_2(\tau_2, C^{-1/d_2}x), & h(x) \ge C. \end{cases}
\tag{2.9}
$$

Proof. From the mean value theorem it follows that, with ζ between $\widetilde{y}_1(x)$ and $C^{1/d_2}\widetilde{y}_2(C^{-1/d_2}x)$,

$$
\begin{aligned}
|h(x) - C| &= |\widetilde{g}_2(x, \widetilde{y}_1(x)) - C\widetilde{g}_2(C^{-1/d_2}x, \widetilde{y}_2(C^{-1/d_2}x))| \\
&= |\widetilde{y}_1(x) - C^{1/d_2}\widetilde{y}_2(C^{-1/d_2}x)| \cdot |\widetilde{g}_{2y}(x,\zeta)| \\
&\ll |\widetilde{y}_1(x) - C^{1/d_2}\widetilde{y}_2(C^{-1/d_2}x)|.
\end{aligned}
\tag{2.10}
$$

Assume first $C \in [a_\varrho, b_\varrho]$. Define $y := k_\varrho(C) \in [\xi_{\varrho-1}, \xi_\varrho]$. Assume $y > (\xi_{\varrho-1} + \xi_\varrho)/2$.

Case 1: $\varrho = r$, $y > (\overline{x}+\xi_\varrho)/2$. The monotonicity of $h\restriction[\xi_{\varrho-1}, \xi_\varrho]$ together with (2.10) gives

$$|\widetilde{y}_1(x) - C^{1/d_2}\widetilde{y}_2(C^{-1/d_2}x)| \gg |h(x) - C| = |h(x) - h(y)| \\ \geq |h(\overline{x}) - h((\overline{x} + \widetilde{\xi}_1)/2)| \gg 1. \tag{2.11}$$

Case 2: $\varrho < r$ *or* $y \leq (\overline{x} + \xi_\varrho)/2$. Taylor's formula gives

$$h(x) - C = h(x) - h(y) = h'(y)(x - y) + O(|x - y|^2).$$

In the case $\varrho = r$, $(\xi_{\varrho-1} + \xi_\varrho)/2 < y \leq (\overline{x} + \xi_\varrho)/2$, we have

$$|h'(y)| \asymp 1 \asymp |y - \xi_\varrho|^{n_\varrho}.$$

In the case $\varrho < r$ the function h' has a zero of order n_ϱ at ξ_ϱ and is nonzero on $[(\xi_{\varrho-1} + \xi_\varrho)/2, \xi_\varrho)$. This gives $|h'(y)| \asymp |y - \xi_\varrho|^{n_\varrho}$ again. So in Case 2 we have

$$\frac{|h(x) - C|}{|x - k_\varrho(C)|} = |h'(y) + O(|x - y|)| \asymp |y - \xi_\varrho|^{n_\varrho} \tag{2.12}$$

if $|x - y| \leq \delta|y - \xi_\varrho|^{n_\varrho}$. Here $\delta > 0$ depends only on $h\restriction[\xi_{\varrho-1}, \xi_\varrho]$.

Case 2.1: $|x - y| \leq \delta|y - \xi_\varrho|^{n_\varrho}$. With (2.10) it follows that

$$|\widetilde{y}_1(x) - C^{1/d_2}\widetilde{y}_2(C^{-1/d_2}x)| \gg |x - k_\varrho(C)| \cdot |y - \xi_\varrho|^{n_\varrho}. \tag{2.13}$$

Case 2.2: $|x-y| > \delta|y-\xi_\varrho|^{n_\varrho}$. Then $x \notin [x_1, x_2]$, $x_{1/2} := y \mp \delta|y-\xi_\varrho|^{n_\varrho}$. In the case $x < x_1$ it follows from the monotonicity of $h\restriction[\xi_{\varrho-1}, \xi_\varrho]$ that

$$|h(x) - h(y)| \geq |h(x_1) - h(y)| \gg |x_1 - k_\varrho(C)| \cdot |y - \xi_\varrho|^{n_\varrho} = \delta|y - \xi_\varrho|^{2n_\varrho}$$

as in (2.12). The same is true in the case $x > x_2$. So in Case 2.2 we have

$$|\widetilde{y}_1(x) - C^{1/d_2}\widetilde{y}_2(C^{-1/d_2}x)| \gg |y - \xi_\varrho|^{2n_\varrho}. \tag{2.14}$$

In the case $\varrho < r$ Taylor's formula gives

$$h(y) - h(\xi_\varrho) = \frac{h^{(n_\varrho+1)}(\xi_\varrho)}{(n_\varrho + 1)!}(y - \xi_\varrho)^{n_\varrho+1}(1 + O(|y - \xi_\varrho|)).$$

With $\delta' > 0$ depending only on $h\restriction[\xi_{\varrho-1}, \xi_\varrho]$ it follows that, for $|y - \xi_\varrho| \leq \delta'$,

$$|y - \xi_\varrho|^{n_\varrho+1} \asymp |h(y) - h(\xi_\varrho)| = |C - h(\xi_\varrho)|,$$

and for $|y - \xi_\varrho| > \delta'$,

$$|y - \xi_\varrho|^{n_\varrho+1} \geq \delta'^{n_\varrho+1} \gg 1 \gg |C - h(\xi_\varrho)|.$$

The last estimate is true also in the case $\varrho = r$, $y \leq (\overline{x} + \xi_\varrho)/2$.

Taking $\varepsilon := 23/(73d_1)$ it follows in Case 2 from (2.13) and (2.14) and $h(\xi_\varrho) \in \{a_\varrho, b_\varrho\}$ that

$$\begin{aligned}(2.15)\quad &|\widetilde{y}_1(x) - C^{1/d_2}\widetilde{y}_2(C^{-1/d_2}x)|\\ &\quad \gg \min\{|x - k_\varrho(C)| \cdot |y - \xi_\varrho|^{n_\varrho}, |y - \xi_\varrho|^{2n_\varrho}\}\\ &\quad \gg \min\{A_1^{-50/(73d_1)}|C - h(\xi_\varrho)|^{n_\varrho/(n_\varrho+1)}, |C - h(\xi_\varrho)|^{2n_\varrho/(n_\varrho+1)}\}\\ &\quad \gg \min\{A_1^{-50/(73d_1)-\varepsilon n_\varrho/(n_\varrho+1)}, A_1^{-2\varepsilon n_\varrho/(n_\varrho+1)}\} \gg A_1^{-\kappa}\end{aligned}$$

with some constant $0 < \kappa < 1/d_1$. By (2.11) this estimate also holds in Case 1. Under the assumption $y \le (\xi_{\varrho-1} + \xi_\varrho)/2$ the same arguments can be simplified somewhat.

In the case $C \notin [a_\varrho, b_\varrho]$ it follows from (2.10) that

$$|\widetilde{y}_1(x) - C^{1/d_2}\widetilde{y}_2(C^{-1/d_2}x)| \gg \min\{|a_\varrho - C|, |b_\varrho - C|\} \gg A_1^{-\varepsilon},$$

which also gives (2.15).

The equivalence

$$\begin{aligned}(2.16)\quad &\widetilde{y}_1(x) \ge C^{1/d_2}\widetilde{y}_2(C^{-1/d_2}x)\\ &\qquad\qquad \Leftrightarrow\ \widetilde{g}_2(C^{-1/d_2}x, C^{-1/d_2}\widetilde{y}_1(x)) \ge 1 \ \Leftrightarrow\ h(x) \ge C\end{aligned}$$

follows from the definitions. As $x \le \overline{x} < \widetilde{\xi}_1$, $C^{-1/d_2}x \le \overline{\overline{x}} < \widetilde{\xi}_2$ it follows from the mean value theorem that

$$y_1(\tau_1, x) = \widetilde{y}_1(x) + O(\tau_1), \qquad y_2(\tau_2, C^{-1/d_2}x) = \widetilde{y}_2(C^{-1/d_2}x) + O(\tau_2).$$

If $h(x) < C$ then by (2.16) and (2.15),

$$\begin{aligned}y_1(\tau_1, x) - C^{1/d_2}y_2(\tau_2, C^{-1/d_2}x) &= \widetilde{y}_1(x) - C^{1/d_2}\widetilde{y}_2(C^{-1/d_2}x) + O(\tau_1)\\ &\le -K_4A_1^{-\kappa} + K_5A_1^{-1/d_1} < 0\end{aligned}$$

for sufficiently large K where $K_4, K_5 > 0$ are constants independent of A_1, A_2. If $h(x) \ge C$ the same reasoning gives

$$y_1(\tau_1, x) - C^{1/d_2}y_2(\tau_2, C^{-1/d_2}x) > 0.$$

From this (2.9) follows. ∎

3. Decomposition of the lattice point set. For $A_1, A_2 \ge K$, $C := A_2A_1^{-d_2/d_1} \le C_1$ it follows from (2.2), (2.3) and (2.4) that

$$R(A_1, A_2) = \#\{(x, y) \in \mathbb{N}^2 \mid x \le A_2^{1/d_2}\xi_2(\tau_2),\ y \le A_2^{1/d_2}y_2(\tau_2, \tau_2x)\}.$$

If $C \ge C_2$ it follows from (2.5) and (2.6) that

$$R(A_1, A_2) = \#\{(x, y) \in \mathbb{N}^2 \mid x \le A_1^{1/d_1}\xi_1(\tau_1),\ y \le A_1^{1/d_1}y_1(\tau_1, \tau_1x)\}.$$

So in these cases the problem of evaluating $R(A_1, A_2)$ asymptotically involves only one polynomial. This situation was investigated in [9]. There the error estimates are more precise than in this paper and an explicit description of the terms T_ν and U_ν is possible. Therefore in the remainder of this paper the range $C \in [C_1, C_2]$ is investigated. Then

$$R(A_1,A_2) = R^\dagger(A_1,A_2) + R^\#(A_1,A_2) - [A_1^{1/d_1} y_{0,A_1,A_2}][A_1^{1/d_1} x_0] \tag{3.1}$$

with

$$R^\dagger(A_1,A_2) = \#\{(x,y)\in\mathbb{N}^2 \mid x \le A_1^{1/d_1} x_0,\ y \le f_{A_1,A_2}(x)\},$$
$$R^\#(A_1,A_2) = \#\{(x,y)\in\mathbb{N}^2 \mid y \le A_1^{1/d_1} y_{0,A_1,A_2},\ x \le f^{-1}_{A_1,A_2}(y)\}.$$

From $h'(x_0) \neq 0$ it follows that there is some $1 \le \varrho_0 \le r$ with $\xi_{\varrho_0-1} < x_0 < \xi_{\varrho_0}$. Take $\varrho = \varrho_0$, $x = \overline{x} = x_0$, $\overline{\overline{x}} = x_2$, $\overline{C}_1 = C_1$, $\overline{C}_2 = C_2$ in Lemma 2.2. From (2.7) it follows that for $A_1, A_2 \ge K$, $C := A_2 A_1^{-d_2/d_1} \in [C_1, C_2]$, $|C - a_{\varrho_0}|$, $|C - b_{\varrho_0}| \ge A_1^{-\varepsilon}$ and $|x_0 - k_{\varrho_0}(C)| \ge A_1^{-50/(73d_1)}$ if $C \in [a_{\varrho_0}, b_{\varrho_0}]$ then

$$y_{0,A_1,A_2} = \begin{cases} y_1(\tau_1, x_0) = \widetilde{y}_1(x_0) + O(\tau_1), & C > h(x_0), \\ C^{1/d_2} y_2(\tau_2, C^{-1/d_2} x_0) = C^{1/d_2} \widetilde{y}_2(C^{-1/d_2} x_0) + O(\tau_1), & C \le h(x_0). \end{cases} \tag{3.2}$$

From $k'_{\varrho_0}(h(x_0)) = h'(x_0)^{-1} \neq 0$ it follows that for $C \in [a_{\varrho_0}, b_{\varrho_0}]$ by Taylor's theorem we have

$$C - h(x_0) = h(k_{\varrho_0}(C)) - h(x_0) = h'(x_0)(k_{\varrho_0}(C) - x_0)(1 + O(|k_{\varrho_0}(C) - x_0|))$$

and consequently $|C - h(x_0)| \asymp |k_{\varrho_0}(C) - x_0|$ for $k_{\varrho_0}(C)$ near x_0. In the opposite case the same holds by the monotonicity of $h\restriction[\xi_{\varrho_0-1}, \xi_{\varrho_0}]$. Therefore there are constants $\varepsilon_0, K > 0$ so that (3.2) holds for $A_1, A_2 \ge K$, $C := A_2 A_1^{-d_2/d_1} \in [C_1, C_2]$, $|C - a_{\varrho_0}|$, $|C - b_{\varrho_0}|$, $|C - h(x_0)| \ge A_1^{-\varepsilon_0}$.

$R^\dagger(A_1,A_2)$ and $R^\#(A_1,A_2)$ are defined in the same way but with x and y interchanged. The only asymmetry is that x_0 is constant whereas y_{0,A_1,A_2} depends on A_1, A_2. Therefore the following notation is introduced which covers both cases: Let $\overline{C}_2 > \overline{C}_1 > 0$ and $0 < \overline{z} < \widetilde{\xi}_1$, $0 < \overline{\overline{z}} < \widetilde{\xi}_2$. Let $z : [K, \infty)^2 \to \mathbb{R}^+$ be a function with

$$z(A_1,A_2) \le \overline{z}, \qquad C^{-1/d_2} z(A_1,A_2) \le \overline{\overline{z}}$$

for $A_1, A_2 \ge K$, $C := A_2 A_1^{-d_2/d_1} \in [\overline{C}_1, \overline{C}_2]$. Define

$$R^*(A_1,A_2) := \#\{(x,y)\in\mathbb{N}^2 \mid x \le A_1^{1/d_1} z(A_1,A_2),\ y \le f_{A_1,A_2}(x)\}.$$

From (2.7) and (2.8) it follows that both $R^\dagger$ and $R^\#$ are of this type. Then $R^*(A_1,A_2) = \sum_{\varrho=1}^{r} R^*_\varrho(A_1,A_2)$ with

$$R^*_\varrho(A_1,A_2) := \#\{(x,y)\in\mathbb{Z}^2 \mid A_1^{1/d_1}\xi_{\varrho-1} < x \le A_1^{1/d_1} \min\{\xi_\varrho, z(A_1,A_2)\}, \\ 0 < y \le f_{A_1,A_2}(x)\}.$$

The following lemma gives a reduction of R^*_ϱ to the case where only one algebraic curve is involved.

LEMMA 3.1. *Let* $1 \le \varrho \le r$. *Then for* $A_1, A_2 \ge K$, $C := A_2 A_1^{-d_2/d_1} \in [\overline{C}_1, \overline{C}_2]$, $|C - a_\varrho|$, $|C - b_\varrho| \ge A_1^{-\varepsilon}$ *we have*:

- *in the case* $C \in [a_\varrho, b_\varrho]$, $h\restriction[\xi_{\varrho-1}, \xi_\varrho]$ *increasing with* $z_\varrho(A_1, A_2) := \min\{k_\varrho(C), z(A_1, A_2)\}$:

$$\begin{aligned} R^*_\varrho(A_1, A_2) = \#\{(x, y) \in \mathbb{Z}^2 \mid A_1^{1/d_1}\xi_{\varrho-1} < x \le A_1^{1/d_1} z_\varrho(A_1, A_2),\\ 0 < y \le A_1^{1/d_1} y_1(\tau_1, \tau_1 x)\}\\ + \#\{(x, y) \in \mathbb{Z}^2 \mid A_1^{1/d_1} k_\varrho(C) < x \le A_1^{1/d_1} \min\{\xi_\varrho, z(A_1, A_2)\},\\ 0 < y \le A_2^{1/d_2} y_2(\tau_2, \tau_2 x)\} + O(A_1^{46/(73d_1)}); \end{aligned}$$

- *in the case* $C \in [a_\varrho, b_\varrho]$, $h\restriction[\xi_{\varrho-1}, \xi_\varrho]$ *decreasing*:

$$\begin{aligned} R^*_\varrho(A_1, A_2) = \#\{(x, y) \in \mathbb{Z}^2 \mid A_1^{1/d_1}\xi_{\varrho-1} < x \le A_1^{1/d_1} z_\varrho(A_1, A_2),\\ 0 < y \le A_2^{1/d_2} y_2(\tau_2, \tau_2 x)\}\\ + \#\{(x, y) \in \mathbb{Z}^2 \mid A_1^{1/d_1} k_\varrho(C) < x \le A_1^{1/d_1} \min\{\xi_\varrho, z(A_1, A_2)\},\\ 0 < y \le A_1^{1/d_1} y_1(\tau_1, \tau_1 x)\} + O(A_1^{46/(73d_1)}); \end{aligned}$$

- *in the case* $C < a_\varrho$:

$$\begin{aligned} R^*_\varrho(A_1, A_2) = \#\{(x, y) \in \mathbb{Z}^2 \mid A_1^{1/d_1}\xi_{\varrho-1} < x \le A_1^{1/d_1} \min\{\xi_\varrho, z(A_1, A_2)\},\\ 0 < y \le A_2^{1/d_2} y_2(\tau_2, \tau_2 x)\}; \end{aligned}$$

- *in the case* $C > b_\varrho$:

$$\begin{aligned} R^*_\varrho(A_1, A_2) = \#\{(x, y) \in \mathbb{Z}^2 \mid A_1^{1/d_1}\xi_{\varrho-1} < x \le A_1^{1/d_1} \min\{\xi_\varrho, z(A_1, A_2)\},\\ 0 < y \le A_1^{1/d_1} y_1(\tau_1, \tau_1 x)\}. \end{aligned}$$

Proof. Only the case $z(A_1, A_2) > \xi_{\varrho-1}$ is of interest. Assume first $C \in [a_\varrho, b_\varrho]$, $h\restriction[\xi_{\varrho-1}, \xi_\varrho]$ increasing. For $\xi_{\varrho-1} \le x \le \min\{\xi_\varrho, z(A_1, A_2)\}$, $|x - k_\varrho(C)| \ge \tau_1^{50/73}$ it follows from Lemma 2.2 with $\overline{x} := \min\{\xi_\varrho, \overline{z}\}$, $\overline{\overline{x}} := \overline{\overline{z}}$ that

$$f_{A_1,A_2}(A_1^{1/d_1} x) = \begin{cases} A_2^{1/d_2} y_2(\tau_2, C^{-1/d_2} x), & x > k_\varrho(C),\\ A_1^{1/d_1} y_1(\tau_1, x), & x < k_\varrho(C). \end{cases}$$

Consequently,

$$\begin{aligned} &R^*_\varrho(A_1, A_2)\\ &\quad = \#\{(x, y) \in \mathbb{Z}^2 \mid A_1^{1/d_1}\xi_{\varrho-1} < x \le A_1^{1/d_1} \min\{k_\varrho(C) - \tau_1^{50/73}, z(A_1, A_2)\},\\ &\qquad 0 < y \le A_1^{1/d_1} y_1(\tau_1, \tau_1 x)\} \end{aligned}$$

$$+\#\{(x,y)\in\mathbb{Z}^2 \mid A_1^{1/d_1}\max\{\xi_{\varrho-1},k_\varrho(C)-\tau_1^{50/73}\}<x \\ \le A_1^{1/d_1}\min\{k_\varrho(C)+\tau_1^{50/73},z(A_1,A_2),\xi_\varrho\},\ 0<y\le f_{A_1,A_2}(x)\} \\ +\#\{(x,y)\in\mathbb{Z}^2 \mid A_1^{1/d_1}(k_\varrho(C)+\tau_1^{50/73})<x\le A_1^{1/d_1}\min\{\xi_\varrho,z(A_1,A_2)\}, \\ 0<y\le A_2^{1/d_2}y_2(\tau_2,\tau_2x)\}.$$

For $\max\{\xi_{\varrho-1},k_\varrho(C)-\tau_1^{50/73}\}\le x\le\min\{\xi_\varrho,k_\varrho(C)+\tau_1^{50/73},z(A_1,A_2)\}$ we have $x=k_\varrho(C)+O(\tau_1^{50/73})$, $k_\varrho(C)\le z(A_1,A_2)+\tau_1^{50/73}$ and by Taylor's theorem

$$\begin{aligned} y_1(\tau_1,x)&=\widetilde{y}_1(k_\varrho(C))+O(\tau_1^{50/73}),\\ y_2(\tau_2,C^{-1/d_2}x)&=\widetilde{y}_2(C^{-1/d_2}k_\varrho(C))+O(\tau_2^{50/73}). \end{aligned} \tag{3.3}$$

Furthermore

$$\begin{aligned} \widetilde{g}_2(C^{-1/d_2}k_\varrho(C),C^{-1/d_2}\widetilde{y}_1(k_\varrho(C)))&=C^{-1}\widetilde{g}_2(k_\varrho(C),\widetilde{y}_1(k_\varrho(C)))\\ &=C^{-1}h(k_\varrho(C))=1 \end{aligned}$$

and consequently

$$\widetilde{y}_2(C^{-1/d_2}k_\varrho(C))=C^{-1/d_2}\widetilde{y}_1(k_\varrho(C)). \tag{3.4}$$

This gives

$$\begin{aligned} f_{A_1,A_2}(A_1^{1/d_1}x)&=A_1^{1/d_1}\widetilde{y}_1(k_\varrho(C))+O(A_1^{23/(73d_1)})\\ &=A_2^{1/d_2}\widetilde{y}_2(C^{-1/d_2}k_\varrho(C))+O(A_2^{23/(73d_2)}). \end{aligned} \tag{3.5}$$

It follows that

$$\begin{aligned} &R_\varrho^*(A_1,A_2)\\ &=\#\{(x,y)\in\mathbb{Z}^2 \mid A_1^{1/d_1}\xi_{\varrho-1}<x\le A_1^{1/d_1}z_\varrho(A_1,A_2),\\ &\qquad 0<y\le A_1^{1/d_1}y_1(\tau_1,\tau_1x)\}\\ &\quad-\sum_{A_1^{1/d_1}\max\{k_\varrho(C)-\tau_1^{50/73},\xi_{\varrho-1}\}<x\le A_1^{1/d_1}z_\varrho(A_1,A_2)}[A_1^{1/d_1}y_1(\tau_1,\tau_1x)]\\ &\quad+\sum_{A_1^{1/d_1}\max\{\xi_{\varrho-1},k_\varrho(C)-\tau_1^{50/73}\}<x\le A_1^{1/d_1}z_\varrho(A_1,A_2)}[f_{A_1,A_2}(x)]\\ &\quad+\sum_{A_1^{1/d_1}\max\{k_\varrho(C)-\tau_1^{50/73},z_\varrho(A_1,A_2)\}<x\le A_1^{1/d_1}\min\{k_\varrho(C)+\tau_1^{50/73},z(A_1,A_2),\xi_\varrho\}}[f_{A_1,A_2}(x)]\\ &\quad+\#\{(x,y)\in\mathbb{Z}^2 \mid A_1^{1/d_1}k_\varrho(C)<x\le A_1^{1/d_1}\min\{\xi_\varrho,z(A_1,A_2)\},\\ &\qquad 0<y\le A_2^{1/d_2}y_2(\tau_2,\tau_2x)\}\\ &\quad-\sum_{A_1^{1/d_1}k_\varrho(C)<x\le A_1^{1/d_1}\min\{k_\varrho(C)+\tau_1^{50/73},\xi_\varrho,z(A_1,A_2)\}}[A_2^{1/d_2}y_2(\tau_2,\tau_2x)]. \end{aligned}$$

By (3.3) and (3.5) the difference between the first and the second sum is $O(A_1^{23/(73d_1)} A_1^{1/d_1} \tau_1^{50/73}) = O(A_1^{46/(73d_1)})$. The difference between the third and the fourth sum is

$$O(A_2^{23/(73d_2)} A_1^{1/d_1} \tau_1^{50/73}) \\ + \sum_{A_1^{1/d_1} \max\{k_\varrho(C)-\tau_1^{50/73}, z_\varrho(A_1,A_2)\} < x \le A_1^{1/d_1} z_\varrho(A_1,A_2)} [f_{A_1,A_2}(x)].$$

The sum is zero and the error term is $O(A_1^{46/(73d_1)})$. The case of $h\restriction[\xi_{\varrho-1},\xi_\varrho]$ decreasing is handled in the same way.

Assume now $C < a_\varrho$. For $\xi_{\varrho-1} \le x \le \min\{\xi_\varrho, z(A_1,A_2)\}$ we have $h(x) \ge a_\varrho > C$. From Lemma 2.2 it follows that $f_{A_1,A_2}(A_1^{1/d_1}x) = A_2^{1/d_2} y_2(\tau_2, C^{-1/d_2}x)$. This proves the conclusion of the theorem. The case $C > b_\varrho$ is handled in the same way. ■

4. The case of irrational slope. In this and the next section the following general situation is investigated:

Let $\tau_0 > 0$, $d \in \mathbb{N}$, $a, b \in \mathbb{R}$, *and let* $f : U \to \mathbb{R}$ *be* C^∞ *on the open neighbourhood* U *of* $\{(\tau, x) \mid 0 \le \tau \le \tau_0,\ a \le x \le b\}$. *Define* $\widetilde{f} := f(0,\cdot)$. *For* $A \ge \tau_0^{-d}$, $A^{1/d}a \le x \le A^{1/d}b$, *define* $f_A(x) := A^{1/d} f(A^{-1/d}, A^{-1/d}x)$. *Let* $a, b : [0,\tau_0] \to \mathbb{R}$ *be functions with* $a(\tau) = a + O(\tau)$, $b(\tau) = b + O(\tau)$ *as* $\tau \to 0$.

The argument of this section follows the general line of Müller–Nowak [6]. The main difference is that the estimates are uniform in the variable τ. This gives rise to additional complications.

LEMMA 4.1. *Let* $I \subseteq \mathbb{R}$ *be an interval and* $f \in C^\infty(I)$ *with* $\widetilde{g}_\nu(x, f(x)) = 1$ *for* $x \in I$. *Then* $f^{(k)} \not\equiv 0$ *for each* $k \in \mathbb{N}_0$. *In particular* $\widetilde{y}_\nu^{(k)} \not\equiv 0$ *on* $[0, \widetilde{\xi}_\nu)$ *and* $\widetilde{x}_\nu^{(k)} \not\equiv 0$ *on* $[0, \widetilde{\eta}_\nu)$ *for each* $k \in \mathbb{N}_0$.

Proof. Surely f is not constant. If $f^{(k)} \equiv 0$ is assumed for some $k \in \mathbb{N}_0$ then let k be minimal with this property. Then $k \ge 2$ and $f(x) = \sum_{\kappa=0}^{k-1} b_\kappa x^\kappa$ on I with coefficients $b_\kappa \in \mathbb{C}$ and $b_{k-1} \ne 0$. Then

$$\sum_{i+j=d_\nu} a_{ij}^{(\nu)} x^i \Big(\sum_{\kappa=0}^{k-1} b_\kappa x^\kappa \Big)^j = 1.$$

For $k \ge 3$ this gives the contradiction $1 = a_{0d_\nu}^{(\nu)} (b_{k-1} x^{k-1})^{d_\nu} +$ monomials of lower order. Therefore $k = 2$ and

$$\sum_{m=0}^{d_\nu} x^m \sum_{\substack{0 \le \iota \le j \le d_\nu \\ d_\nu - \iota = m}} a_{d_\nu - j, j}^{(\nu)} \binom{j}{\iota} b_1^{j-\iota} b_0^\iota = 1.$$

This gives $a^{(\nu)}_{0d_\nu} b_0^{d_\nu} = 1$ and therefore $b_0 \neq 0$, and for $1 \le m \le d_\nu$,

$$0 = \sum_{j=d_\nu - m}^{d_\nu} a^{(\nu)}_{d_\nu - j, j} \binom{j}{d_\nu - m} b_1^{j-(d_\nu - m)} b_0^{d_\nu - m}$$
$$= b_0^{d_\nu - m} \frac{1}{(d_\nu - m)!} \cdot \frac{\partial^{d_\nu - m}}{\partial y^{d_\nu - m}} \widetilde{g}_\nu(1, b_1).$$

Therefore b_1 is a zero of $\widetilde{g}_\nu(1,y)$ with multiplicity at least d_ν and so $\widetilde{g}_\nu(1,y) = a^{(\nu)}_{0d_\nu}(y-b_1)^{d_\nu}$. Since the coefficients of $\widetilde{g}_\nu$ are real this would imply $b_1 \in \mathbb{R}$ and $\widetilde{g}_\nu(x,y) = a^{(\nu)}_{0d_\nu}(y - b_1 x)^{d_\nu}$ contrary to assumption (1.2). ∎

LEMMA 4.2. *Let $x_0 \in [0, \widetilde{\xi}_\nu)$ with $\widetilde{y}''_\nu(x_0) = 0$. Then $\widetilde{y}'_\nu(x_0)$ is algebraic over $\mathbb{Q}$.*

Proof. Twofold differentiation of $\widetilde{g}_\nu(x, \widetilde{y}_\nu(x)) = 1$ shows $(x_0, \widetilde{y}_\nu(x_0))$ is a zero of $k := \widetilde{g}_\nu - 1$ and $l := \widetilde{g}_{\nu xx}\widetilde{g}^2_{\nu y} - 2\widetilde{g}_{\nu xy}\widetilde{g}_{\nu x}\widetilde{g}_{\nu y} + \widetilde{g}_{\nu yy}\widetilde{g}^2_{\nu x}$. Now l is not zero because otherwise $\widetilde{y}''_\nu \equiv 0$ contrary to Lemma 4.1. Let $0 \neq b(x) \in \mathbb{Z}[x]$ be the leading coefficient of l as a polynomial in y with coefficients in $\mathbb{Z}[x]$. Let $R(x) \in \mathbb{Z}[x]$ be the resultant of k and l with respect to y. Then $R(x_0) = 0$ or $b(x_0) = 0$ because k has leading coefficient $a^{(\nu)}_{0d_\nu} \neq 0$ with respect to y (van der Waerden [10], p. 104). If $R \neq 0$ then $x_0 \in \overline{\mathbb{Q}}$. Then $k(x_0, \cdot) \neq 0$ has algebraic coefficients and therefore its zero $\widetilde{y}_\nu(x_0)$ is algebraic. Consequently,

$$\widetilde{y}'_\nu(x_0) = -\widetilde{g}_{\nu x}(x_0, \widetilde{y}_\nu(x_0))\widetilde{g}_{\nu y}(x_0, \widetilde{y}_\nu(x_0))^{-1} \in \overline{\mathbb{Q}}.$$

Now the assumption $R = 0$ will be proved contradictory. Then for each $z \in \mathbb{C}$ with $b(z) \neq 0$ the polynomials $k(z, \cdot)$ and $l(z, \cdot)$ would have a common zero. The discriminant $D_k(x)$ of k with respect to y is not zero because $D_k(0) = (-1)^{d_\nu - 1}(d_\nu a^{(\nu)}_{0d_\nu})^{d_\nu} \neq 0$. Let $x_1 \in \mathbb{R}$ with $D_k(x_1) \neq 0$, $b(x_1) \neq 0$. From general theorems on algebraic functions it follows that there is an open disk $U \subseteq \mathbb{C}$ with centre x_1 where b and D_k have no zeros and on which there exist d_ν holomorphic branches $\widetilde{w}_1, \ldots, \widetilde{w}_{d_\nu}$ of the algebraic function which is defined by $k(z,w) = 0$. Twofold differentiation of $k(z, \widetilde{w}_j(z)) = 0$ gives $\widetilde{w}''_j(z) = -(l/\widetilde{g}^3_{\nu y})(z, \widetilde{w}_j(z))$ on U. Here $\widetilde{g}_{\nu y}(z, \widetilde{w}_j(z)) \neq 0$ for $z \in U$ because $k(z, \cdot)$ has only simple zeros. We have $b(z) \neq 0$ and $\widetilde{w}_1(z), \ldots, \widetilde{w}_{d_\nu}(z)$ are the zeros of $k(z, \cdot)$. Therefore by assumption there is some $1 \le j(z) \le d_\nu$ with $l(z, \widetilde{w}_{j(z)}(z)) = 0$ and consequently $\widetilde{w}''_{j(z)}(z) = 0$. From the identity theorem it follows that $\widetilde{w}''_j(z) = 0$ on U for some $1 \le j \le d_\nu$. This contradicts Lemma 4.1. ∎

The following lemma is used for parts of the boundary curve on which the curvature does not vanish.

LEMMA 4.3. *Let $\widetilde{f}''$, $\widetilde{f}'''$ be zerofree on $[a,b]$. Then for $A \ge \tau_0^{-d}$, $a \le a' < b' \le b$ we have*

$$\sum_{A^{1/d}a' < n \le A^{1/d}b'} \psi(f_A(n)) \ll A^{46/(73d)}(\log A)^{315/146}.$$

Proof. Define $\tau := A^{-1/d}$, $M := [A^{1/d}(b-a)]$, $T := A^{1/d}M$. For $A^{1/d}a \le x \le A^{1/d}b$, $0 \le \tau \le \tau_0$ the mean value theorem gives

$$f_A''(x) = A^{-1/d} f_{xx}(\tau, \tau x) = A^{-1/d}(\widetilde{f}''(\tau x) + O(\tau)).$$

$\widetilde{f}''$ is zerofree and consequently $|\widetilde{f}''(\tau x)| \asymp 1$. For large A this gives $|f_A''(x)| \asymp A^{-1/d} \asymp TM^{-3}$ and similarly $|f_A'''(x)| \asymp TM^{-4}$. Let $c \in [a,b]$, $M_c := M + [A^{1/d}(c-a)]$. Define $h(x) := f_A(x + [A^{1/d}a] - M)$, $x \in [M, 2M]$. Then $|h''(x)| \asymp TM^{-3}$, $|h'''(x)| \asymp TM^{-4}$, $M \le M_c \le 2M$, $T^{1/2} \asymp M$, and the discrete Hardy–Littlewood method in the form of [2], Theorem 18.2.2, gives

$$\sum_{A^{1/d}a < n \le A^{1/d}c} \psi(f_A(n)) = \sum_{M \le n \le M_c} \psi(h(n)) + O(1) \ll A^{46/(73d)}(\log A)^{315/146}$$

uniformly in A and c. In this theorem $f(\tau,\cdot)$ is assumed to be independent of τ. This is not an essential assumption as was pointed out in [7], Theorem B. Choosing $c = b'$ and $c = a'$ and subtracting proves the lemma. ∎

The next lemma is used for parts of the boundary curve which do not come too close to points of vanishing curvature.

LEMMA 4.4. *Let $\widetilde{f}'$, $\widetilde{f}''$, $\widetilde{f}'''$ be zerofree on $(a,b]$. Let $\mu \in \mathbb{N}$ with $\widetilde{f}^{(k)}(a) = 0$ for $2 \le k \le \mu+1$ and $\widetilde{f}^{(\mu+2)}(a) \ne 0$. Let $0 < \lambda < (\mu+1)^{-1}$ and $\lambda_0 := \min\{20(83\mu+103)^{-1}, \lambda\}$. Then for $A \ge \tau_0^{-d}$, $a + \tau^{\lambda_0} \le b' \le b$ we have*

$$\sum_{A^{1/d}(a+\tau^{\lambda}) < n \le A^{1/d}b'} \psi(f_A(n)) \ll A^{46/(73d)}(\log A)^{315/146} + A^{(\lambda\mu+1)/(2d)}.$$

Proof. Let $a < c < b$. For $0 < \tau \le \tau_0$, $a + \tau^{\lambda} \le x \le c$, $k = 2, 3$, Taylor's theorem gives

$$\frac{\partial^k f}{\partial x^k}(\tau, x) = \frac{\widetilde{f}^{(\mu+2)}(a)}{(\mu+2-k)!}(x-a)^{\mu+2-k}(1 + O(|c-a| + \tau^{1-\lambda(\mu+2-k)})),$$

$$f_x(\tau, x) = \widetilde{f}'(a) + \frac{\widetilde{f}^{(\mu+2)}(a)}{(\mu+1)!}(x-a)^{\mu+1}(1 + O(|c-a| + \tau^{1-\lambda(\mu+1)})).$$

Fixing c close to a gives, with $1 - \lambda(\mu+1) > 0$,

$$|f_{xx}(\tau, x)| \asymp |x-a|^{\mu}, \qquad |f_{xxx}(\tau, x)| \asymp |x-a|^{\mu-1}.$$

In the case $\widetilde{f}'(a) \ne 0$ we have $|f_x(\tau, x)| \asymp 1$ and with constants $K_1, K_2 > 0$,

$$\begin{aligned}|f_x(\tau,x) f_{xxx}(\tau,x) - 3 f_{xx}(\tau,x)^2| &\ge |x-a|^{\mu-1}(K_1 - K_2|x-a|^{\mu+1}) \\ &\gg |x-a|^{\mu-1}|f_x(\tau,x)|\end{aligned}$$

for c sufficiently close to a. In the case $\widetilde{f}'(a) = 0$ we have $|f_x(\tau,x)| \asymp |x-a|^{\mu+1}$ and

$$\begin{aligned}|f_x(\tau,x)f_{xxx}(\tau,x) - 3f_{xx}(\tau,x)^2| \\ &= \widetilde{f}^{(\mu+2)}(a)^2|x-a|^{2\mu}\left|\frac{\mu-3(\mu+1)}{(\mu+1)!\mu!} + O(|c-a| + \tau^{1-\lambda(\mu+1)})\right| \\ &\asymp |x-a|^{\mu-1}|f_x(\tau,x)|.\end{aligned}$$

For $A^{1/d}(a+\tau^{\lambda}) \le x \le A^{1/d}c$ this gives

$$\begin{gathered}|f_A''(x)| \asymp A^{-1/d}|A^{-1/d}x-a|^{\mu}, \quad |f_A'''(x)| \asymp A^{-2/d}|A^{-1/d}x-a|^{\mu-1}, \\ |f_A'(x)f_A'''(x) - 3f_A''(x)^2| \gg |f_A'(x)|A^{-2/d}|A^{-1/d}x-a|^{\mu-1}.\end{gathered} \tag{4.1}$$

Define $M_0 := A^{1/d}\tau^{\lambda_0}$, $M_J := A^{1/d}(c-a)$, $J := [\log A]$, $B := (M_J/M_0)^{1/J}$. Then $B = e^{\lambda_0/d} + o(1)$ as $A \to \infty$. For $1 \le j \le J$ define $M_j := M_0 B^j$ and

$$g_j(x) := f_A(x + [2M_{j-1} - M_j] + [A^{1/d}a]) \quad \text{on } [M_j - M_{j-1}, 2(M_j - M_{j-1})].$$

Then

$$\begin{aligned}S_1 &:= \sum_{A^{1/d}(a+\tau^{\lambda_0})<n\le A^{1/d}\min\{b',c\}} \psi(f_A(n)) \\ &= \sum_{j=1}^{J} \sum_{M_j - M_{j-1} < n \le M_j'(b')} \psi(g_j(n)) + O(J)\end{aligned}$$

with

$$M_j(b') := \min\{2(M_j - M_{j-1}), A^{1/d}(b'-a) - 2M_{j-1} + M_j\} \le 2(M_j - M_{j-1}).$$

For $1 \le j \le J$ define $T := A^{-(\mu+1)/d}M_{j-1}^{\mu+3}$, $M := M_j - M_{j-1}$. For $x \in [M, 2M]$ it follows that

$$\begin{gathered}|g_j''(x)| \asymp TM^{-3}, \quad |g_j'''(x)| \asymp TM^{-4}, \\ |g_j'(x)g_j'''(x) - 3g_j''(x)^2| \gg TM^{-4}|g_j'(x)|.\end{gathered}$$

From the choice of λ_0 it follows that $MT^{-83/146} \ll 1$ and $T^{1/2}M^{-1} \ll 1$. Theorem 18.2.2 in [2] gives $S_1 \ll A^{46/(73d)}(\log A)^{315/146}$. Lemma 4.3 gives

$$S_2 := \sum_{A^{1/d}c<n\le A^{1/d}b'} \psi(f_A(n)) \ll A^{46/(73d)}(\log A)^{315/146}.$$

By (4.1), f_A'' and f_A''' are zerofree on $[A^{1/d}(a+\tau^{\lambda}), A^{1/d}(a+\tau^{\lambda_0})]$ and consequently f_A'' is monotonic. From van der Corput's theorem (Krätzel [3], Theorem 2.3) it follows that

$$S_3 := \sum_{A^{1/d}(a+\tau^{\lambda})<n\le A^{1/d}(a+\tau^{\lambda_0})} \psi(f_A(n))$$
$$\ll \int_{A^{1/d}(a+\tau^{\lambda})}^{A^{1/d}(a+\tau^{\lambda_0})} A^{-1/(3d)}|A^{-1/d}x-a|^{\mu/3}\,dx + (A^{-1/d}\tau^{\lambda\mu})^{-1/2}$$
$$\ll A^{46/(73d)} + A^{(1+\lambda\mu)/(2d)}$$

using the special choice of λ_0. ■

The main result of this section is

PROPOSITION 4.5. *Let $\widetilde{f}'(a)$ be an algebraic irrational, $\widetilde{f}'$, $\widetilde{f}''$, $\widetilde{f}'''$ zerofree on $(a,b]$ and $\mu\in\mathbb{N}$ with $\widetilde{f}^{(k)}(a)=0$ for $2\le k\le\mu+1$ and $\widetilde{f}^{(\mu+2)}(a)\ne0$. Let $\delta>0$. Then for $A\ge\tau_0^{-d}$ and $a+\tau^{20/(83\mu+103)}\le b'\le b$ we have*

$$\sum_{A^{1/d}a(A^{-1/d})<n\le A^{1/d}b'} \psi(f_A(n)) \ll A^{(1/d)(1-1/\mu+\delta)} + A^{46/(73d)}(\log A)^{315/146}.$$

Proof. Define $\lambda:=1/(3\mu+2)$. For $h\in\mathbb{N}$ define

$$S(h) := \sum_{A^{1/d}(a+\tau^{1/\mu-\delta})<n\le A^{1/d}(a+\tau^{\lambda})} e(hf_A(n)).$$

From Krätzel [3], Theorem 1.8 for $s=2$ it follows that for arbitrary $H>0$,

$$S := \sum_{A^{1/d}(a+\tau^{1/\mu-\delta})<n\le A^{1/d}(a+\tau^{\lambda})} \psi(f_A(n)) \ll A^{(1-\lambda)/d}H^{-1} + \sum_{h\ge1}\min\left\{\frac{H^2}{h^3},\frac{1}{h}\right\}|S(h)|. \tag{4.2}$$

Let $\mathcal{C}_1$ be the arc of the circle with radius $r:=CA^{1/d}(\tau^{\lambda}-\tau^{1/\mu-\delta})$ which starts at

$$P_1 := (A^{1/d}(a+\tau^{1/\mu-\delta}), f_A(A^{1/d}(a+\tau^{1/\mu-\delta})))$$

and proceeds clockwise to its endpoint

$$P_2 := (A^{1/d}(a+\tau^{\lambda}), f_A(A^{1/d}(a+\tau^{\lambda})))$$

and whose centre M lies below the line (P_1P_2). Here $C\ge1$ is a constant which is fixed later. The $\ll$-constants below are independent of C.

Let 2α be the angle under which $\mathcal{C}_1$ is seen from its centre M and $\mp\beta$ ($\beta\ge0$) the angle between (P_1P_2) and the horizontal axis. In the following the upper resp. lower sign is valid whenever (P_1P_2) has negative resp. positive slope. From the mean value theorem it follows that

$$|\mp\tan\beta| = |f_A'(\zeta)| \ll 1 \quad \text{with } A^{1/d}(a+\tau^{1/\mu-\delta})<\zeta<A^{1/d}(a+\tau^{\lambda}).$$

Consequently, there is a constant $\varepsilon_0 > 0$ with $0 \le \beta \le \pi/2 - \varepsilon_0$. Therefore

$$|P_1P_2| = (\cos\beta)^{-1}(A^{1/d}(a+\tau^{\lambda}) - A^{1/d}(a+\tau^{1/\mu-\delta})) < 2r$$

for sufficiently large $C \ge 1$. For such C the arc $\mathcal{C}_1$ exists in the form described above. From $\sin\alpha = (2C\cos\beta)^{-1}$ it follows that $\alpha \asymp C^{-1}$. Choose $C \ge 1$ sufficiently large so that $0 < \alpha \le \varepsilon_0/2$. Further conditions on C will be given below. If $m(P)$ denotes the slope of $\mathcal{C}_1$ in $P \in \mathcal{C}_1$ then

$$\sup_{P\in\mathcal{C}_1} |m(P)| \le \max\{|\tan(\alpha \mp \beta)|, |\tan(\alpha \pm \beta)|\} \ll 1. \tag{4.3}$$

Let $P_3 \in \mathcal{C}_1$ be the midpoint of $\mathcal{C}_1$. For $A^{1/d}(a+\tau^{1/\mu-\delta}) \le x \le A^{1/d}(a+\tau^{\lambda})$ there is some θ between ζ and x with

$$|f_A'(x) \pm \tan\beta| = |f_A'(x) - f_A'(\zeta)| = |f_A''(\theta)(x-\zeta)| \ll \tau^{\lambda}.$$

For the slope $m_1 := \tan(\alpha/2 \mp \beta)$ of (P_1P_3) and $0 < \tau \le \tau_0(C)$ it follows that

$$m_1 - f_A'(x) = \pm(\cos\theta_2)^{-2}(\pm\alpha/2) + O(\tau^{\lambda}) \ge \alpha/2 + O(\tau^{\lambda}) > 0$$

with some θ_2 between β and $\beta \mp \alpha/2$. Therefore the graph of f_A between P_1 and P_2 lies below (P_1P_3). The same holds for (P_3P_2) and consequently the graph of f_A lies below $\mathcal{C}_1$. Let $F_A : [A^{1/d}(a+\tau^{1/\mu-\delta}), A^{1/d}(a+\tau^{\lambda})] \to \mathbb{R}$ be the function whose graph is $\mathcal{C}_1$. Then we have $|F_A'| \ll 1$ by (4.3). For $A^{1/d}(a+\tau^{1/\mu-\delta}) \le x \le A^{1/d}(a+\tau^{\lambda})$ we obtain $r^{-1} = |F_A''(x)|(1+F_A'(x)^2)^{-3/2}$ and consequently $|F_A''(x)| \asymp r^{-1} \asymp C^{-1}A^{-1/d}\tau^{-\lambda}$. From van der Corput's theorem (Krätzel [3], Theorem 2.1) it follows that, for $h \in \mathbb{N}$,

$$\begin{aligned} S_1(h) & \\ &:= \sum_{A^{1/d}(a+\tau^{1/\mu-\delta}) < n \le A^{1/d}(a+\tau^{\lambda})} e(hF_A(n)) \ll h^{1/2}A^{(1-\lambda)/(2d)}C^{1/2}. \end{aligned} \tag{4.4}$$

The main task is the estimation of $S_2(h) := S_1(h) - S(h)$. Let $\mathcal{C}_2$ be the part of the graph of f_A between P_1 and P_2 and

$$B := \{(x,y) \in \mathbb{R}^2 \mid A^{1/d}(a+\tau^{1/\mu-\delta}) \le x \le A^{1/d}(a+\tau^{\lambda}),\ f_A(x) \le y \le F_A(x)\}.$$

For $\vec{k} := (k,h) \in \mathbb{Z} \times \mathbb{N}$ define $I(k,h) := \int_B e(kx+hy)\,dx\,dy$ and the vector field $\vec{\nu}(\vec{x}) = \vec{\nu}_{\vec{k}}(\vec{x}) := e(\vec{x}\vec{k})\vec{k}(2\pi i\|\vec{k}\|^2)^{-1}$ where $\|\vec{k}\|$ is the Euclidean norm of $\vec{k}$. From Poisson's sum formula (Krätzel [3], p. 23, equation (1.11)),

$$\begin{aligned} S_2(h) &= \sum_{k\in\mathbb{Z}} \int_{A^{1/d}(a+\tau^{1/\mu-\delta})}^{A^{1/d}(a+\tau^{\lambda})} (e(hF_A(x)) - e(hf_A(x)))e(kx)\,dx + O(1) \\ &= \sum_{k\in\mathbb{Z}} 2\pi i h I(k,h) + O(1). \end{aligned} \tag{4.5}$$

From the divergence theorem it follows that, for $\vec{k} \in \mathbb{Z} \times \mathbb{N}$,

$$I(\vec{k}) = \int_B \operatorname{div} \vec{\nu}(\vec{x})\, d\vec{x} = -\int_{\mathcal{C}_1} \vec{\nu} \cdot \vec{n}^*\, d\sigma + \int_{\mathcal{C}_2} \vec{\nu} \cdot \vec{n}^*\, d\sigma \tag{4.6}$$

where $\vec{n}^*$ is the outer normal unit vector on ∂B. The second integral is estimated first. Let L be the arc length of $\tau\mathcal{C}_2$ and $\vec{u} : [0, L] \to \mathbb{R}^2$ the natural parametrization of $\tau\mathcal{C}_2$. Then $\vec{t}(s) = \vec{u}'(s)$ is the tangent unit vector to $\tau\mathcal{C}_2$ in $\vec{u}(s)$ and $\vec{u}''(s) = \kappa(s)\vec{n}(s)$ where $\kappa(s)$ is the curvature of $\tau\mathcal{C}_2$ in $\vec{u}(s)$ and $\vec{n}(s)$ is the normal unit vector to $\tau\mathcal{C}_2$ in $\vec{u}(s)$. Let $g(s) := \vec{k} \cdot \vec{u}(s)\|\vec{k}\|^{-1}$. Then

$$\int_{\mathcal{C}_2} \vec{\nu} \cdot \vec{n}^*\, d\sigma = -A^{1/d}(2\pi i\|\vec{k}\|^2)^{-1} \int_0^L e(g(s)A^{1/d}\|\vec{k}\|)\vec{k} \cdot \vec{n}(s)\, ds. \tag{4.7}$$

For $0 \le s \le L$ we have

$$\kappa(s) \ll 1 \quad \text{and} \quad \|\vec{n}'(s)\| = \|-\kappa(s)\vec{t}(s)\| \ll 1. \tag{4.8}$$

Furthermore $L \ll \tau^\lambda$. Now $|g'|$ is estimated from below.

Case 1: $|g'(0)| \ge 1/2$. For $0 \le s \le L$ and $0 < \tau \le \tau_0$ it follows from (4.8) and the mean value theorem that $|g'(s) - g'(0)| = |g''(\zeta)s| \le L\kappa(s) \le 1/4$ with $0 \le \zeta \le s$. Therefore $|g'(s)| \ge 1/4$.

Case 2: $|g'(0)| < 1/2$. Then $|\vec{k}\cdot\vec{n}(0)| \cdot \|\vec{k}\|^{-1} = \sqrt{1 - g'(0)^2} \ge \sqrt{3/4}$ and the mean value theorem and (4.8) give $|\vec{k} \cdot \vec{n}(s) - \vec{k} \cdot \vec{n}(0)| \cdot \|\vec{k}\|^{-1} \le \sqrt{3}/4$ for $0 \le s \le L$ and $0 \le \tau \le \tau_0$. Consequently,

$$|\vec{k} \cdot \vec{n}(s)| \cdot \|\vec{k}\|^{-1} \ge \sqrt{3}/4. \tag{4.9}$$

Taylor's formula gives for $a + \tau^{1/\mu-\delta} \le x \le a + \tau^\lambda$

$$\begin{aligned} f_{xx}(\tau, x) &= \frac{\widetilde{f}^{(\mu+2)}(a)}{\mu!}(x-a)^\mu + O(\tau + |x-a|^{\mu+1}) \\ &= \frac{\widetilde{f}^{(\mu+2)}(a)}{\mu!}(x-a)^\mu(1 + O(\tau^{\delta\mu} + \tau^\lambda)) \asymp |x-a|^\mu \end{aligned}$$

and consequently $|\kappa(s)| \asymp |f_{xx}(\tau, u_1(s))| \asymp |u_1(s) - a|^\mu$. Furthermore

$$|u_1(s) - a| \asymp \int_a^{u_1(s)} (1 + f_x(\tau, x)^2)^{1/2}\, dx = \varrho(\tau) + s$$

with $\varrho(\tau) := \int_a^{a+\tau^{1/\mu-\delta}} (1 + f_x(\tau, x)^2)^{1/2}\, dx \ge 0$. With (4.9) it follows that

$$|g''(s)| = |\vec{k} \cdot \vec{n}(s)\kappa(s)| \cdot \|\vec{k}\|^{-1} \asymp \kappa(s) \asymp (\varrho(\tau) + s)^\mu. \tag{4.10}$$

In particular g' is strictly monotonic and consequently there is exactly one $0 \le s_0 \le L$ with $|g'(s_0)| = \min_{0 \le s \le L} |g'(s)|$. Choose $0 \le \gamma = \gamma_{\vec{k}} \le \pi/2$ so that $\pi/2 - \gamma$ is the angle between $\vec{t}(0)$ and $\pm\vec{k}$.

Case 2.1: $|g'(s_0)| \geq |g'(0)|/2$. It follows from $g'(0) = \pm\cos(\pi/2-\gamma)$ that $|g'(s')| \geq |g'(s_0)| \gg \gamma$ for $0 \leq s' \leq L$.

Case 2.2: $|g'(s_0)| < |g'(0)|/2$. It follows from (4.10) that for $0 \leq s' \leq L$,

$$|g'(s')| \geq |g'(s') - g'(s_0)| = \Big|\int_{s_0}^{s'} g''(s)\,ds\Big| \gg \Big|\int_{s_0}^{s'} (\varrho(\tau)+s)^\mu\,ds\Big|$$
$$\gg |(\varrho(\tau)+s')^{\mu+1} - (\varrho(\tau)+s_0)^{\mu+1}| \geq |s'-s_0|(\varrho(\tau)+s_0)^\mu,$$
$$\gamma \ll |g'(0)| \leq 2|g'(0)-g'(s_0)| \ll \Big|\int_0^{s_0} g''(s)\,ds\Big| \ll (\varrho(\tau)+s_0)^{\mu+1}$$

and consequently $|g'(s')| \gg |s'-s_0|\gamma^{\mu/(\mu+1)}$.

Summarizing, in Case 1 we have $|g'(s)| \geq 1/4$ and in Case 2 we have $|g'(s)| \gg \min\{|s-s_0|\gamma^{\mu/(\mu+1)}, \gamma\}$ for $0 \leq s \leq L$.

Let $\gamma > 0$ and $\delta := (A^{1/d}\|\vec{k}\|\gamma^{\mu/(\mu+1)})^{-1/2}$. From (4.7) and (4.8) it follows that in both cases (with arbitrary s_0 in the first case), after partial integration,

$$\begin{aligned}\int_{\mathcal{C}_2} \vec{\nu}\cdot\vec{n}^*\,d\sigma \\ &= -A^{1/d}(2\pi i\|\vec{k}\|^2)^{-1}\Big(\int_{|s-s_0|<\delta} + \int_{|s-s_0|>\delta}\Big) \\ &\ll A^{1/d}\|\vec{k}\|^{-1}\delta \\ &\quad + \|\vec{k}\|^{-2}\Big(\sup_{|s-s_0|\geq\delta}|g'(s)|^{-1} + \int_{|s-s_0|\geq\delta}(|g'(s)|^{-1} + |g''(s)|g'(s)^{-2})\,ds\Big).\end{aligned}$$

In Case 1 the terms in parentheses are $\ll 1$. In Case 2, $g''>0$ or $g''<0$ by (4.10) and the terms in parentheses are $\ll \delta^{-1}\gamma^{-\mu/(\mu+1)} + \gamma^{-1}$. So in both cases

$$\int_{\mathcal{C}_2} \vec{\nu}\cdot\vec{n}^*\,d\sigma \ll A^{1/(2d)}\|\vec{k}\|^{-3/2}\gamma^{-\mu/(2(\mu+1))} + \|\vec{k}\|^{-2}\gamma^{-1}. \tag{4.11}$$

The same arguments hold for the first integral in (4.6) in a simplified form and therefore only the differences are indicated. The corresponding objects are written with a tilde $\widetilde{\ }$. We have $\widetilde{L} \ll \tau^\lambda$ with some $\ll$-constant independent of C. Furthermore $\widetilde{\kappa}(s) = A^{1/d}r^{-1} \asymp C^{-1}\tau^{-\lambda}$ and $\|\widetilde{\vec{n}}'(s)\| = \|-\widetilde{\kappa}(s)\widetilde{\vec{t}}(s)\| \ll C^{-1}\tau^{-\lambda}$ for $0 \leq s \leq \widetilde{L}$.

In the first case $|\widetilde{g}'(0)| \geq 1/2$ use $|\widetilde{g}'(s) - \widetilde{g}'(0)| \ll C^{-1}$ for $0 \leq s \leq \widetilde{L}$. If we choose C sufficiently large the upper bound becomes $\leq 1/4$. In the second case $|\widetilde{g}'(0)| < 1/2$ use $|\vec{k}\cdot\widetilde{\vec{n}}(s) - \vec{k}\cdot\widetilde{\vec{n}}(0)|\cdot\|\vec{k}\|^{-1} \ll C^{-1}$ for $0 \leq s \leq \widetilde{L}$. For C sufficiently large we have $|\vec{k}\cdot\widetilde{\vec{n}}(s)|\cdot\|\vec{k}\|^{-1} \gg 1$. Then $|\widetilde{g}''(s)| \gg C^{-1}\tau^{-\lambda} \gg 1$

for $0 < \tau \le \tau_0(C)$. This was the last condition on C. Now, $\widetilde{g}'$ is strictly monotonic. Let $|\widetilde{g}'|$ take its infimum at s_0. Then $|\widetilde{g}'(s')| \ge |\widetilde{g}'(s') - \widetilde{g}'(s_0)| = |\int_{s_0}^{s'} \widetilde{g}''(s)\,ds| \gg |s' - s_0|$ for $0 \le s' \le \widetilde{L}$. Similar arguments with $\delta := (A^{1/d}\|\vec{k}\|)^{-1/2}$ and $|\widetilde{g}''(s)| \ll C^{-1}\tau^{-\lambda}$ for $0 \le s \le \widetilde{L}$ give

$$\int\limits_{\mathcal{C}_1} \vec{\nu} \cdot \vec{n}^*\, d\sigma \ll A^{1/(2d)}\|\vec{k}\|^{-3/2}.$$

From (4.11) and (4.6) it follows that

$$I(\vec{k}) \ll A^{1/(2d)}\|\vec{k}\|^{-3/2}\gamma_{\vec{k}}^{-\mu/(2(\mu+1))} + \|\vec{k}\|^{-2}\gamma_{\vec{k}}^{-1} \tag{4.12}$$

for all $\vec{k} = (k,h) \in \mathbb{Z} \times \mathbb{N}$ with $0 < \gamma_{\vec{k}} \le \pi/2$ and $\pi/2 - \gamma_{\vec{k}}$ the angle between $\vec{t}(0)$ and $\pm\vec{k}$.

Next $\gamma_{\vec{k}}$ is estimated from below. Set $(\tau_1, \tau_2) := \vec{t}(0)$ and for $h \in \mathbb{N}$ define $k(h) \in \mathbb{Z}$ by $-1/2 < k(h) + h\tau_2/\tau_1 \le 1/2$. From Taylor's theorem it follows that

$$\begin{aligned} \frac{\tau_2}{\tau_1} &= f_x(\tau, a + \tau^{1/\mu-\delta}) \\ &= \widetilde{f}'(a) + \sum_{l=1}^{\mu} \frac{\widetilde{f}^{(l+1)}(a)}{l!}\tau^{(1/\mu-\delta)l} + O(\tau^{(1/\mu-\delta)(\mu+1)} + \tau) \\ &= \widetilde{f}'(a) + O(\tau). \end{aligned} \tag{4.13}$$

In particular $|\tau_2/\tau_1| \asymp 1$ and consequently $k(h) \ll h$ for $h \in \mathbb{N}$. Furthermore $1 \ge \tau_1 = (1 + f_x(\tau, a + \tau^{1/\mu-\delta})^2)^{-1/2} \gg 1$. Applying Roth's theorem to the algebraic irrational $\widetilde{f}'(a)$ gives $|\widetilde{f}'(a) + k(h)/h| \ge K_3(\delta)h^{-(2+\delta)}$ for $h \in \mathbb{N}$ with some constant $K_3(\delta) > 0$. From (4.13) it follows that with some constant $K_4 > 0$ for $\vec{k} = (k(h), h)$,

$$\begin{aligned} \gamma_{\vec{k}} \ge \sin\gamma_{\vec{k}} &= |k(h)\tau_1 + h\tau_2| \cdot \|\vec{k}\|^{-1} \\ &\gg ||k(h) + h\widetilde{f}'(a)| - |h\widetilde{f}'(a) - h\tau_2/\tau_1||h^{-1} \\ &\ge (K_3 h^{-(1+\delta)} - K_4 h\tau)h^{-1}. \end{aligned}$$

For $1 \le h \le A^{(1-\delta/2)/(2d)}$ we have $h\tau/h^{-(1+\delta)} \ll \tau^{\delta^2/4}$ and consequently $\gamma_{\vec{k}} \gg h^{-(2+\delta)}$ for sufficiently large A. (4.12) gives, for $1 \le h \le A^{(1-\delta/2)/(2d)}$,

$$I(k(h), h) \ll A^{1/(2d)}h^{(1+\delta/2)\mu/(\mu+1)-3/2} + h^{\delta}. \tag{4.14}$$

For $\vec{k} = (k,h) \in \mathbb{Z} \times \mathbb{N}$, $k \ne k(h)$ the choice of $k(h)$ gives

$$\begin{aligned} \gamma_{\vec{k}} &\ge |k\tau_1 + h\tau_2| \cdot \|\vec{k}\|^{-1} \ge \|\vec{k}\|^{-1}\tau_1(|k - k(h)| - |k(h) + h\tau_2/\tau_1|) \\ &\ge \|\vec{k}\|^{-1}\tau_1(|k - k(h)| - |k - k(h)|/2) \gg \|\vec{k}\|^{-1}|k - k(h)|. \end{aligned}$$

For $h \in \mathbb{N}$ it follows from (4.12) that

$$\sum_{k \neq k(h)} |I(k,h)| \ll A^{1/(2d)} h^{-1/2} + 1.$$

From (4.5) and (4.14) it follows that $|S_2(h)| \ll A^{1/(2d)} h^{1/2}$ for $1 \le h \le A^{(1-\delta/2)/(2d)}$. The trivial estimate $S(h) \ll A^{(1-\lambda)/d}$ gives, with (4.2) and (4.4) for $H \ge 1$,

$$\begin{aligned} S &\ll A^{(1-\lambda)/d} H^{-1} + \sum_{h \le A^{(1-\delta/2)/(2d)}} \min\left\{\frac{H^2}{h^3}, \frac{1}{h}\right\} A^{1/(2d)} h^{1/2} \\ &\quad + \sum_{h \ge A^{(1-\delta/2)/(2d)}} \frac{H^2}{h^3} A^{(1-\lambda)/d} \\ &\ll A^{(1-\lambda)/d} H^{-1} + A^{1/(2d)} H^{1/2} + H^2 A^{(1/d)(\delta/2-\lambda)}. \end{aligned}$$

The optimal choice $H = A^{2(1/2-\lambda)/(3d)}$ gives $S \ll A^{(2-\lambda)/(3d)}$. Lemma 4.4 and the choice of λ give

$$\begin{aligned} &\sum_{A^{1/d} a(A^{-1/d}) < n \le A^{1/d} b'} \psi(f_A(n)) \\ &\qquad \ll A^{1/d} \tau^{1/\mu - \delta} + A^{(2-\lambda)/(3d)} \\ &\qquad\quad + A^{46/(73d)} (\log A)^{315/146} + A^{(\lambda\mu+1)/(2d)} \\ &\qquad \ll A^{(1/d)(1-1/\mu+\delta)} + A^{46/(73d)} (\log A)^{315/146}. \ \blacksquare \end{aligned}$$

5. The case of rational slope. The following lemma is used in the asymptotic evaluation of the lattice integral.

LEMMA 5.1. *Let* $0 < \omega < 1$. *For* $v \ge 0$, $0 \ne T \in \mathbb{R}$, *define*

$$H(v,T) := \int_v^\infty u^{\omega-1} e^{-iTu}\, du.$$

Then for $v > 0$, $T \ne 0$,

$$|H(v,T)| \ll_\omega |T|^{-1} v^{\omega-1}, \qquad |H(v,T)| \ll_\omega |T|^{-\omega} + v^\omega \tag{5.1}$$

and for $T \ne 0$,

$$H(0,T) = \Gamma(\omega) e^{-(\operatorname{sign} T) i\omega\pi/2} |T|^{-\omega}. \tag{5.2}$$

Proof. The first and the third statement can be found in [4], p. 155. The second statement is a slight generalization of the essential part of the proof of Lemma 5 in [4]. Let $T > 0$, $R > v > 0$, let $\mathcal{K}_1$ resp. $\mathcal{K}_3$ be the straight paths from iv to iR resp. from v to R, and $\mathcal{K}_2$ resp. $\mathcal{K}_4$ the circular arcs with centre 0 from iv to v resp. from R to iR. Cauchy's theorem gives

$$\int_{\mathcal{K}_1} z^{\omega-1}e^{-Tz}dz = \int_{\mathcal{K}_2} + \int_{\mathcal{K}_3} + \int_{\mathcal{K}_4}. \tag{5.3}$$

Decomposing $\mathcal{K}_4$ into those z with $\Re z \geq R^{(1-\omega)/2}$ and their complement gives $\int_{\mathcal{K}_4} \ll R^{\omega}\exp(-TR^{(1-\omega)/2}) + R^{(\omega-1)/2}$. Letting $R \to \infty$ in (5.3) gives

$$\begin{aligned} e^{i\pi\omega/2}H(v,T) &= \int_{\mathcal{K}_2} z^{\omega-1}e^{-Tz}\,dz + \int_v^\infty z^{\omega-1}e^{-Tz}\,dz \\ &\ll v^{\omega} + \int_0^\infty t^{\omega-1}e^{-Tt}\,dt \ll_\omega v^\omega + T^{-\omega}. \end{aligned}$$

The case $T < 0$ can be reduced to this case by complex conjugation. ■

The following proposition is the analogue of Proposition 4.5 in the case of rational slope at the point of vanishing curvature. The proof follows the general line of [8].

PROPOSITION 5.2. *Assume* $0 \leq a < b$ *and* $\widetilde{f} > 0$ *on* $[a,b]$. *Let* $\widetilde{f}'(a) = -p/q$, $p \in \mathbb{N}_0$, $q \in \mathbb{N}$, $(p,q) = 1$, *and* $\mu \in \mathbb{N}$ *with* $\widetilde{f}^{(k)}(a) = 0$ *for* $2 \leq k \leq \mu+1$ *and* $\widetilde{f}^{(\mu+2)}(a) \neq 0$. *Let* $\widetilde{f}', \widetilde{f}'', \widetilde{f}'''$ *be zerofree on* $(a,b]$. *There is some* $\lambda > 0$ *with the property*: *For* $A \geq \tau_0^{-d}$, $a + \tau^\lambda \leq b' \leq b$ *we have*

$$\begin{aligned} N &:= \#\{(x,y) \in \mathbb{Z}^2 \mid A^{1/d}a(A^{-1/d}) < x \leq A^{1/d}b',\ 0 < y \leq f_A(x)\} \\ &= \int_{A^{1/d}a(A^{-1/d})}^{A^{1/d}b'} f_A(x)\,dx - \psi(A^{1/d}b')f_A(A^{1/d}b') \\ &\quad + \psi(A^{1/d}a(A^{-1/d}))f_A(A^{1/d}a(A^{-1/d})) + \frac{1}{2}A^{1/d}(a(A^{-1/d}) - b') \\ &\quad + CA^{(1/d)(1-1/(\mu+2))} \\ &\quad \times \sum_{h\geq 1} \frac{1}{h^{1+1/(\mu+2)}} \sin\left(2\pi h(A^{1/d}(pa+q\widetilde{f}(a))+qf_\tau(0,a)) + \frac{\pi(-1)^{\alpha+1}}{2(\mu+2)}\right) \\ &\quad + O(A^{(1/d)(1-1/(\mu+2)-1/(\mu+2)^2)}\log A + A^{46/(73d)}(\log A)^{315/146}). \end{aligned}$$

The constants are defined by

$$\alpha := \begin{cases} 0, & \widetilde{f}^{(\mu+2)}(a) < 0, \\ 1, & \widetilde{f}^{(\mu+2)}(a) > 0, \end{cases}$$

$$\begin{aligned} C &:= q^{-1-1/(\mu+2)}((\mu+1)!)^{1/(\mu+2)}(\mu+2)^{-(\mu+1)/(\mu+2)}\Gamma\left(\frac{1}{\mu+2}\right) \\ &\quad \times 2^{-1/(\mu+2)}\pi^{-1-1/(\mu+2)}|\widetilde{f}^{(\mu+2)}(a)|^{-1/(\mu+2)}. \end{aligned}$$

Proof. Choose $\varrho := (\mu+3)(\mu+2)^{-2}$ and $0<\lambda<1/(\mu+1)$. In what follows several further conditions will be imposed on λ. Define $\kappa := \min\{\lambda, 1-\varrho(\mu+1)\} > 0$. Define

$$G_\tau(x) := px + qf(\tau,x) \qquad \text{for } a+\tau^\varrho \le x \le a+\tau^\lambda.$$

Taylor's formula gives, for these x,

$$f_x(\tau,x) = -\frac{p}{q} + \frac{\widetilde{f}^{(\mu+2)}(a)}{(\mu+1)!}(x-a)^{\mu+1} + O(|x-a|^{\mu+2}+\tau) \tag{5.4}$$

$$= -\frac{p}{q} + \frac{\widetilde{f}^{(\mu+2)}(a)}{(\mu+1)!}(x-a)^{\mu+1}(1+O(\tau^\kappa)). \tag{5.5}$$

In the following only the case $\widetilde{f}^{(\mu+2)}(a) < 0$ is handled. The opposite case is completely analogous. Then $G'_\tau(x) = p + qf_x(\tau,x) < p + q(-p/q) = 0$ by (5.5). Define $F_\tau := G_\tau^{-1} : [\beta_\tau,\gamma_\tau] \to [a+\tau^\varrho, a+\tau^\lambda]$ with $\beta_\tau = G_\tau(a+\tau^\lambda)$, $\gamma_\tau = G_\tau(a+\tau^\varrho)$. For $t\in\mathbb{Z}$ define

$$N(t) := \#\{x\in\mathbb{Z} \mid px \equiv t\,(q),\ A^{1/d}(a+\tau^\varrho) < x \le A^{1/d}(a+\tau^\lambda),\ px < t \le A^{1/d}G_\tau(A^{-1/d}x)\}.$$

Then

$$\#\{(x,y)\in\mathbb{Z}^2 \mid A^{1/d}(a+\tau^\varrho) < x \le A^{1/d}(a+\tau^\lambda),\ 0<y\le f_A(x)\} = \sum_{t\in\mathbb{Z}} N(t). \tag{5.6}$$

For $t > A^{1/d}\gamma_\tau$ it follows that $N(t)=0$. For $A^{1/d}\beta_\tau \ge t$,

$$N(t) = \#\{(x,y)\in\mathbb{Z}^2 \mid px+qy=t,\ A^{1/d}(a+\tau^\varrho) < x \le A^{1/d}(a+\tau^\lambda),\ y>0\}$$

and consequently

$$\sum_{t\le A^{1/d}\beta_\tau} N(t) = \#\{(x,y)\in\mathbb{Z}^2 \mid A^{1/d}(a+\tau^\varrho) < x \le A^{1/d}(a+\tau^\lambda),\ 0<y\le (1/q)(A^{1/d}\beta_\tau - px)\}.$$

Here $(A^{1/d}\beta_\tau - px)/q \ge A^{1/d}f(\tau, a+\tau^\lambda) > 0$ for sufficiently large A. Partial summation and $\int_u^v \psi(x)\,dx \ll 1$ for $u<v$ give

$$\begin{aligned}\sum_{t\le A^{1/d}\beta_\tau} N(t) &= \frac{1}{q}(A^{1/d}\beta_\tau - \psi(A^{1/d}\beta_\tau))A^{1/d}(\tau^\lambda-\tau^\varrho) - \frac{p}{2q}A^{2/d}(a+\tau^\lambda)^2\\ &\quad + \frac{p}{2q}A^{2/d}(a+\tau^\varrho)^2 - \psi(A^{1/d}(a+\tau^\lambda))A^{1/d}f(\tau,a+\tau^\lambda) + O(1)\\ &\quad + \psi(A^{1/d}(a+\tau^\varrho))\frac{1}{q}A^{1/d}(\beta_\tau - p(a+\tau^\varrho)) - \frac{1}{2}A^{1/d}(\tau^\lambda-\tau^\varrho).\end{aligned} \tag{5.7}$$

For $A^{1/d}\beta_\tau < t \le A^{1/d}\gamma_\tau$ we have

$$N(t) = \#\{x \in \mathbb{Z} \mid px \equiv t\,(q),\ A^{1/d}(a+\tau^\varrho) < x \le A^{1/d}F_\tau(A^{-1/d}t)\}$$

because F_τ is strictly decreasing and $t - px \ge A^{1/d}qf(\tau, a+\tau^\lambda) > 0$ for sufficiently large A. For $t_0 \in \{0,\dots,q-1\}$ there is exactly one $x_0 = x_0(t_0) \in \{0,\dots,q-1\}$ with $t_0 \equiv px_0\,(q)$. Decomposing the range of t into the remainder classes modulo q gives

$$\sum_{A^{1/d}\beta_\tau < t \le A^{1/d}\gamma_\tau} N(t) = H + S_1 + S_2$$

with

$$H = \frac{1}{q}A^{1/d}\sum_{t_0=0}^{q-1}\ \sum_{A^{1/d}\beta_\tau < t \le A^{1/d}\gamma_\tau,\, t\equiv t_0\,(q)} (F_\tau(A^{-1/d}t) - (a+\tau^\varrho)),$$

$$S_1 = -\sum_{t_0=0}^{q-1}\ \sum_{A^{1/d}\beta_\tau < t \le A^{1/d}\gamma_\tau,\, t\equiv t_0\,(q)} \psi\biggl(\frac{1}{q}(A^{1/d}F_\tau(A^{-1/d}t) - x_0(t_0))\biggr), \tag{5.8}$$

$$S_2 = \frac{1}{q}A^{1/d}(\gamma_\tau - \beta_\tau)\psi(A^{1/d}(a+\tau^\varrho)) + O(1). \tag{5.9}$$

Partial summation and substitution $t = A^{1/d}G_\tau(x)$ give

$$\begin{aligned} H = A^{2/d}\int_{a+\tau^\varrho}^{a+\tau^\lambda} f(\tau,x)\,dx + \frac{1}{q}A^{2/d}\beta_\tau(\tau^\varrho - \tau^\lambda) - \frac{p}{2q}A^{2/d}(a+\tau^\varrho)^2 \\ + \frac{p}{2q}A^{2/d}(a+\tau^\lambda)^2 + \frac{1}{q}A^{1/d}\psi(A^{1/d}\beta_\tau)(\tau^\lambda - \tau^\varrho) \\ + \frac{1}{q}\int_{A^{1/d}\beta_\tau}^{A^{1/d}\gamma_\tau} F_\tau'(A^{-1/d}t)\psi(t)\,dt. \end{aligned} \tag{5.10}$$

A trivial estimation of the lattice remainder and $f_A'(x) \ll 1$ give

$$\begin{aligned} &\#\{(x,y) \in \mathbb{Z}^2 \mid A^{1/d}a(A^{-1/d}) < x \le A^{1/d}(a+\tau^\varrho),\ 0 < y \le f_A(x)\} \\ &= \int_{A^{1/d}a(A^{-1/d})}^{A^{1/d}(a+\tau^\varrho)} f_A(x)\,dx - \psi(A^{1/d}(a+\tau^\varrho))f_A(A^{1/d}(a+\tau^\varrho)) \\ &\quad + \psi(A^{1/d}a(A^{-1/d}))f_A(A^{1/d}a(A^{-1/d})) + O(A^{1/d}\tau^\varrho). \end{aligned} \tag{5.11}$$

Choose $0 < \lambda \le 20/(83\mu + 103) < 19/(73\mu)$. Partial summation and Lemma 4.4 give

$$\#\{(x,y) \in \mathbb{Z}^2 \mid A^{1/d}(a+\tau^\lambda) < x \le A^{1/d}b',\ 0 < y \le f_A(x)\}$$

$$= \int\limits_{A^{1/d}(a+\tau^\lambda)}^{A^{1/d}b'} f_A(x)\,dx - \psi(A^{1/d}b')f_A(A^{1/d}b')$$

$$+ \psi(A^{1/d}(a+\tau^\lambda))f_A(A^{1/d}(a+\tau^\lambda)) + \int\limits_{A^{1/d}(a+\tau^\lambda)}^{A^{1/d}b'} f_A'(x)\psi(x)\,dx$$

$$- \tfrac{1}{2}A^{1/d}(b'-a-\tau^\lambda) + O(A^{46/(73d)}(\log A)^{315/146}).$$

The function $\psi_1(x) := \int_0^x \psi(t)\,dt$, $x \in \mathbb{R}$, is continuous, piecewise continuously differentiable and bounded. From $f_A' \ll 1$, $f_A'' \ll A^{-1/d}$ it follows by partial integration that the second integral above is $\ll 1$. Together with (5.6) to (5.11) it follows that

$$N = \int\limits_{A^{1/d}a(\tau)}^{A^{1/d}b'} f_A(x)\,dx + \frac{1}{q}I + S_1 - \frac{1}{2}A^{1/d}(b'-a(\tau)) + \psi(A^{1/d}a(\tau))f_A(A^{1/d}a(\tau)) - \psi(A^{1/d}b')f_A(A^{1/d}b') + O(A^{1/d}\tau^\varrho) + O(A^{46/(73d)}(\log A)^{315/146}) \tag{5.12}$$

with

$$I := \int\limits_{A^{1/d}\beta_\tau}^{A^{1/d}\gamma_\tau} F_\tau'(A^{-1/d}t)\psi(t)\,dt.$$

The asymptotic development of this integral is done with tools from [4]. The asymptotic behaviour of F_τ'' and F_τ''' cannot be determined as in that paper because the influence of τ and x on the value of $f(\tau,x)$ cannot be separated by inverting functions.

Define $\delta_\tau := pa + q\widetilde{f}(a) + qf_\tau(0,a)\tau$. For $a+\tau^\varrho \le x \le a+\tau^\lambda$, Taylor's formula gives

$$f(\tau,x)$$

$$= \frac{1}{q}\delta_\tau - \frac{p}{q}x + \frac{\widetilde{f}^{(\mu+2)}(a)}{(\mu+2)!}(x-a)^{\mu+2}(1+O(|x-a|+\tau|x-a|^{-\mu-1})) \tag{5.13}$$

$$= \frac{1}{q}\delta_\tau - \frac{p}{q}x + \frac{\widetilde{f}^{(\mu+2)}(a)}{(\mu+2)!}(x-a)^{\mu+2}(1+O(\tau^\kappa)). \tag{5.14}$$

We have $pa + q\widetilde{f}(a) > 0$ and consequently $\delta_\tau > 0$ and $\delta_\tau \asymp 1$ for sufficiently large A. Now (5.13) gives

$$\delta_\tau - \gamma_\tau = -\frac{q\widetilde{f}^{(\mu+2)}(a)}{(\mu+2)!}\tau^{\varrho(\mu+2)}(1+O(\tau^{1-\varrho(\mu+1)})) \tag{5.15}$$

and in particular $\gamma_\tau < \delta_\tau$. From (5.14) it follows analogously that

$$\delta_\tau - \beta_\tau \asymp \tau^{\lambda(\mu+2)}. \tag{5.16}$$

For $\beta_\tau \le t \le \gamma_\tau$ it follows from (5.14) that

$$t = G_\tau(F_\tau(t)) = \delta_\tau + q(F_\tau(t) - a)^{\mu+2} \frac{\widetilde{f}^{(\mu+2)}(a)}{(\mu+2)!}(1 + O(\tau^\kappa)) \tag{5.17}$$

and consequently

$$(\delta_\tau - t)^{1/(\mu+2)} \asymp F_\tau(t) - a. \tag{5.18}$$

Define $\Phi_A(t) := F'_\tau(t)(\delta_\tau - t)^{(\mu+1)/(\mu+2)}$ on $[\beta_\tau, \gamma_\tau]$. From (5.5) and (5.17) it follows that

$$\begin{aligned} \Phi_A(t) &= (\delta_\tau - t)^{(\mu+1)/(\mu+2)} G'_\tau(F_\tau(t))^{-1} \\ &= \left| \frac{q\widetilde{f}^{(\mu+2)}(a)}{(\mu+2)!} \right|^{(\mu+1)/(\mu+2)} \left(\frac{q\widetilde{f}^{(\mu+2)}(a)}{(\mu+1)!} \right)^{-1} + O(\tau^\kappa) \end{aligned} \tag{5.19}$$

and in particular $|\Phi_A(t)| \asymp 1$. Furthermore

$$\Phi'_A(t) = -\Phi_A(t) G'_\tau(F_\tau(t))^{-2} \left(G''_\tau(F_\tau(t)) + \frac{\mu+1}{\mu+2}(\delta_\tau - t)^{-1} G'_\tau(F_\tau(t))^2 \right).$$

For the expression in parentheses Taylor's formula and (5.4) give

$$\begin{aligned} &\frac{q}{\mu!}\widetilde{f}^{(\mu+2)}(a)(F_\tau(t) - a)^\mu + O(|F_\tau(t) - a|^{\mu+1}) \\ &\quad + (\delta_\tau - t)^{-1} q \frac{\widetilde{f}^{(\mu+2)}(a)}{\mu!}(F_\tau(t) - a)^\mu \\ &\quad \times \left(\frac{q\widetilde{f}^{(\mu+2)}(a)}{(\mu+2)!}(F_\tau(t) - a)^{\mu+2} + O(\tau|F_\tau(t) - a| + |F_\tau(t) - a|^{\mu+3}) \right). \end{aligned}$$

Using (5.13) gives the more exact asymptotics

$$t = \delta_\tau + q(F_\tau(t) - a)^{\mu+2} \left(\frac{\widetilde{f}^{(\mu+2)}(a)}{(\mu+2)!} + O(|F_\tau(t) - a| + \tau|F_\tau(t) - a|^{-\mu-1}) \right).$$

With (5.18) this gives, for the expression in parentheses,

$$\begin{aligned} &\frac{q\widetilde{f}^{(\mu+2)}(a)}{\mu!}(F_\tau(t) - a)^\mu + O(|F_\tau(t) - a|^{\mu+1}) \\ &\qquad + (\delta_\tau - t)^{-1} \frac{q\widetilde{f}^{(\mu+2)}(a)}{\mu!}(F_\tau(t) - a)^\mu \\ &\qquad \times (t - \delta_\tau + O(|F_\tau(t) - a|^{\mu+3} + \tau|F_\tau(t) - a|)) \\ &\ll |F_\tau(t) - a|^{\mu+1} + \tau|F_\tau(t) - a|^{-1}. \end{aligned}$$

From this together with (5.19) and (5.18) it follows that

$$|\Phi'_A(t)| \ll (\delta_\tau - t)^{-(\mu+1)/(\mu+2)} + \tau(\delta_\tau - t)^{-(2\mu+3)/(\mu+2)}. \tag{5.20}$$

Substituting the Fourier development $\psi(t) = (-1/(2\pi i))\sum_{h\neq 0} h^{-1}e(ht)$, which is valid in $\mathbb{L}^2[0,1]$, into I gives $I = (-1/(2\pi i))\sum_{h\neq 0} h^{-1}I_h$ with

$$I_h = -A^{1/d}\delta_\tau^{1/(\mu+2)}e^{iT}\int_{\beta_\tau/\delta_\tau}^{\gamma_\tau/\delta_\tau} \Phi_A(\delta_\tau t)\frac{\partial H}{\partial v}(1-t,T)\,dt,$$

$$T := 2\pi h A^{1/d}\delta_\tau, \qquad \omega := \frac{1}{\mu+2}.$$

Partial integration together with $\gamma_\tau < \delta_\tau$ and $\delta_\tau \asymp 1$ give

$$\begin{aligned} I_h &= A^{1/d}\delta_\tau^{1/(\mu+2)}e^{iT}\Phi_A(\gamma_\tau)H(1-\gamma_\tau/\delta_\tau,T) \\ &\quad + O(A^{1/d}|\Phi_A(\beta_\tau)H(1-\beta_\tau/\delta_\tau,T)|) \\ &\quad + O\Big(A^{1/d}\int_{\beta_\tau/\delta_\tau}^{\gamma_\tau/\delta_\tau} |\Phi'_A(\delta_\tau t)H(1-t,T)|\,dt\Big). \end{aligned}$$

The first half of (5.1), $|\Phi_A(t)| \asymp 1$ and (5.16) show that the first error term is $\ll A^{1/d}|T|^{-1}\tau^{-\lambda(\mu+1)}$. (5.20), (5.15) and (5.1) show that the second error term is

$$\begin{aligned} &\ll A^{1/d}\int_{\beta_\tau/\delta_\tau}^{\gamma_\tau/\delta_\tau} |1-t|^{-(\mu+1)/(\mu+2)}|T|^{-1}|1-t|^{\omega-1}\,dt \\ &\quad + A^{1/d}\int_{\beta_\tau/\delta_\tau}^{\gamma_\tau/\delta_\tau} \tau|1-t|^{-(2\mu+3)/(\mu+2)}(|T|^{-\omega} + |1-t|^{\omega})\,dt \\ &\ll |h|^{-1}\tau^{-\mu\varrho} + |T|^{-\omega}\tau^{-(\mu+1)\varrho} + \tau^{-\mu\varrho}. \end{aligned}$$

(5.15) gives

$$|H(0,T) - H(1-\gamma_\tau/\delta_\tau,T)| \ll \int_0^{1-\gamma_\tau/\delta_\tau} u^{\omega-1}\,du \ll \tau^{\varrho}.$$

(5.4) yields

$$F'_\tau(\gamma_\tau) = G'_\tau(a+\tau^\varrho)^{-1} = \left(\frac{q\widetilde{f}^{(\mu+2)}(a)}{(\mu+1)!}\right)^{-1}\tau^{-\varrho(\mu+1)}(1 + O(\tau^{1-\varrho(\mu+1)})).$$

This together with (5.15) gives

$$\Phi_A(\gamma_\tau) = \left(\frac{q\widetilde{f}^{(\mu+2)}(a)}{(\mu+1)!}\right)^{-1}\left|\frac{q\widetilde{f}^{(\mu+2)}(a)}{(\mu+2)!}\right|^{(\mu+1)/(\mu+2)}(1 + O(\tau^{1-\varrho(\mu+1)})).$$

Putting everything together and using (5.2) and the choice of ϱ we get

$$(5.21)\quad I_h = -A^{(1/d)(1-1/(\mu+2))}|h|^{-1/(\mu+2)}e\left(hA^{1/d}\delta_\tau - \frac{\operatorname{sign} h}{4(\mu+2)}\right)Cq\pi + O(A^{(1-\varrho)/d})$$

if $0 < \lambda \le (1-\varrho)/(\mu+1)$.

A better estimation is needed for large T. Uniformly in $\beta_\tau/\delta_\tau \le t_0 \le \gamma_\tau/\delta_\tau$ it follows from partial integration and (5.20) that

$$\begin{aligned} I_h = {} & A^{1/d}\delta_\tau^{1/(\mu+2)}\left[\frac{e(hA^{1/d}\delta_\tau t)}{2\pi i h A^{1/d}\delta_\tau}\Phi_A(\delta_\tau t)(1-t)^{-(\mu+1)/(\mu+2)}\right]_{\beta_\tau/\delta_\tau}^{t_0} \\ & - A^{1/d}\delta_\tau^{1/(\mu+2)}\int_{\beta_\tau/\delta_\tau}^{t_0}\frac{e(hA^{1/d}\delta_\tau t)}{2\pi i h A^{1/d}\delta_\tau}\bigg(\delta_\tau \Phi_A'(\delta_\tau t)(1-t)^{-(\mu+1)/(\mu+2)} \\ & + \Phi_A(\delta_\tau t)\frac{\mu+1}{\mu+2}(1-t)^{-(\mu+1)/(\mu+2)-1}\bigg)\,dt \\ & + A^{1/d}\delta_\tau^{1/(\mu+2)}\int_{t_0}^{\gamma_\tau/\delta_\tau} e(hA^{1/d}\delta_\tau t)\Phi_A(\delta_\tau t)(1-t)^{-(\mu+1)/(\mu+2)}\,dt \\ \ll {} & |h|^{-1}(1-t_0)^{-(\mu+1)/(\mu+2)} \\ & + |h|^{-1}\int_{\beta_\tau/\delta_\tau}^{t_0}(1-t)^{-(\mu+1)/(\mu+2)-1}\,dt + A^{1/d}\int_{t_0}^{\gamma_\tau/\delta_\tau}(1-t)^{-(\mu+1)/(\mu+2)}\,dt. \end{aligned}$$

The first term in the $\ll$-estimate only appears if $t_0 > \beta_\tau/\delta_\tau$.

In the case $1-|h|^{-1}A^{-1/d} \le \beta_\tau/\delta_\tau$ take $t_0 := \beta_\tau/\delta_\tau$. Then from $1-\beta_\tau/\delta_\tau \le |h|^{-1}A^{-1/d}$ it follows that $I_h \ll A^{(1/d)(1-1/(\mu+2))}|h|^{-1/(\mu+2)}$.

In the case $1-|h|^{-1}A^{1/d} \ge \gamma_\tau/\delta_\tau$ take $t_0 := \gamma_\tau/\delta_\tau$. Then from $1-\gamma_\tau/\delta_\tau \ge |h|^{-1}A^{-1/d}$ the same estimate for I_h follows.

In the case $\beta_\tau/\delta_\tau < 1-|h|^{-1}A^{-1/d} < \gamma_\tau/\delta_\tau$ take $t_0 := 1-|h|^{-1}A^{-1/d}$ and again the same estimate for I_h follows.

Together with (5.21) this gives

$$\begin{aligned} (5.22)\quad I = {} & -\frac{1}{2\pi i}\sum_{0<|h|\le A}\frac{1}{h}I_h - \frac{1}{2\pi i}\sum_{|h|>A}\frac{1}{h}I_h \\ = {} & \frac{qC}{2i}A^{(1/d)(1-1/(\mu+2))} \\ & \times \sum_{0<|h|\le A}\frac{\operatorname{sign} h}{|h|^{1+1/(\mu+2)}}\exp\left(i\operatorname{sign} h\left(2\pi|h|A^{1/d}\delta_\tau - \frac{\pi}{2(\mu+2)}\right)\right) \\ & + O(A^{(1/d)(1-\varrho)}\log A). \end{aligned}$$

Extending the sum over all integers $\neq 0$ gives the additional error term

$$(5.23)\quad \ll A^{(1/d)(1-1/(\mu+2))} \sum_{|h|>A} \frac{1}{|h|^{1+1/(\mu+2)}} \ll A^{(1/d)(1-1/(\mu+2))-1/(\mu+2)}$$
$$\ll A^{(1/d)(1-\varrho)} \log A.$$

Next the lattice remainder is estimated by the discrete Hardy–Littlewood method. Taylor's formula gives, for $a+\tau^{\varrho} \le x \le a+\tau^{\lambda}$, $k=2,3$,

$$\frac{\partial^k f(\tau,x)}{\partial x^k} = \frac{\widetilde{f}^{(\mu+2)}(a)}{(\mu+2-k)!}(x-a)^{\mu+2-k}(1+O(\tau^{\kappa})).$$

Together with (5.5) it follows that

$$G_\tau'''(x)G_\tau'(x) - 3G_\tau''(x)^2$$
$$= \frac{q^2\widetilde{f}^{(\mu+2)}(a)^2(-2\mu-3)}{\mu!(\mu+1)!}(x-a)^{2\mu}(1+O(\tau^{\kappa})) \asymp |x-a|^{2\mu}$$

and analogously $|G_\tau'(x)| \asymp |x-a|^{\mu+1}$, $|G_\tau''(x)| \asymp |x-a|^{\mu}$.

(5.18) gives, for $\beta_\tau \le t \le \gamma_\tau$,

$$|F_\tau''(t)| = \left| -\frac{G_\tau''(F_\tau(t))}{G_\tau'(F_\tau(t))^3} \right| \asymp |\delta_\tau - t|^{-2+1/(\mu+2)},$$
$$|F_\tau'''(t)| \asymp |\delta_\tau - t|^{-3+1/(\mu+2)}.$$

Set $M_0 := 2q^{-1}A^{46/(73d)}$, $M_J := q^{-1}A^{1/d}(\gamma_\tau - \beta_\tau - 2\tau^{27/73})$, $J := [\log A]$, $B := (M_J/M_0)^{1/J}$, $M_j := M_0 B^j$ for $0 \le j \le J$. Arguments similar to those which led to (5.15) give $\gamma_\tau - \beta_\tau = K_{10}\tau^{\lambda(\mu+2)}(1+o(1)) > 2\tau^{27/73}$ with some constant $K_{10} > 0$ if $0 < \lambda < (27/73)(\mu+2)^{-1}$. Consequently, $\log B = (1/d)(27/73 - \lambda(\mu+2)) + o(1)$ as $A \to \infty$ and therefore $B = K_{11} + o(1)$ with some constant $K_{11} > 1$. For $t_0 \in \{0,\dots,q-1\}$, $1 \le j \le J$, set

$$h_{t_0,j}(x) := q^{-1}A^{1/d}F_\tau(A^{-1/d}t_0 + A^{-1/d}q([q^{-1}A^{1/d}\gamma_\tau]$$
$$- x - [2M_{j-1} - M_j])) - q^{-1}x_0(t_0)$$

on $[M_j - M_{j-1}, 2(M_j - M_{j-1})]$. From (5.15) it follows that, for $M_j - M_{j-1} \le x \le 2(M_j - M_{j-1})$,

$$|\delta_\tau - A^{-1/d}t_0 - A^{-1/d}q([q^{-1}A^{1/d}\gamma_\tau] - x - [2M_{j-1} - M_j])|$$
$$= |\delta_\tau - \gamma_\tau + A^{-1/d}q(x + 2M_{j-1} - M_j) + O(\tau)| \asymp A^{-1/d}M_{j-1}.$$

Choosing $M := M_j - M_{j-1}$, $T := A^{(1/d)(1-1/(\mu+2))}(M_j - M_{j-1})^{(\mu+3)/(\mu+2)}$, it follows that $|h_{t_0,j}''(x)| \asymp TM^{-3}$, $|h_{t_0,j}'''(x)| \asymp TM^{-4}$ for $x \in [M,2M]$. Furthermore $T^{1/3} \le M \ll T^{1/2}$. We have

$$S_1 = -\sum_{t_0=0}^{q-1}\sum_{j=1}^{J} \sum_{M_j - M_{j-1} < n \le 2(M_j - M_{j-1})} \psi(h_{t_0,j}(n)) + O(A^{46/(73d)} + J).$$

Applying Theorem 18.2.2 of [2] gives $S_1 \ll A^{46/(73d)}(\log A)^{315/146}$. From (5.12), (5.22) and (5.23) the proposition follows. ∎

6. The tails of the boundary curve. The result of this section is used in the case where $\widetilde{x}_\nu'(0) = \widetilde{x}_\nu'''(0) = 0$ and $\widetilde{x}_\nu''(0) \neq 0$. Then it is not possible to apply Propositions 4.5 or 5.2. Instead one goes to the inverse function $\widetilde{y}_\nu = \widetilde{x}_\nu^{-1}$. This function is not C^∞ at the point $\widetilde{\xi}_\nu$ but the order of the singularity of $\widetilde{y}_\nu^{(k)}$ for $k = 2, 3$ is small so that no additional main term arises. It is possible to improve this section considerably so that it can be applied generally in the case $j_0^{(\nu)} \geq 2$. This would give a slight improvement of the error term in the contribution of the tails of the boundary curve. For this the reader is refered to [9].

Proposition 6.1. *Let* $\widetilde{x}_\nu'(0) = 0 \neq \widetilde{x}_\nu''(0)$. *There are constants* $0 < \overline{\xi} < \widetilde{\xi}_\nu$ *and* $K > 0$ *with the property*: *For* $A_\nu \geq K$, $\tau_\nu := A_\nu^{-1/d_\nu}$, $\overline{\xi} \leq \xi \leq \xi_\nu(\tau_\nu)$ *we have*

$$\begin{aligned} R &:= \#\{(x,y) \in \mathbb{Z}^2 \mid A_\nu^{1/d_\nu}\xi < x \leq A_\nu^{1/d_\nu}\xi_\nu(\tau_\nu),\ 0 < y \leq A_\nu^{1/d_\nu} y_\nu(\tau_\nu, \tau_\nu x)\} \\ &= A_\nu^{2/d_\nu} \int_\xi^{\xi_\nu(\tau_\nu)} y_\nu(\tau_\nu, \tau_\nu x)\,dx - \tfrac{1}{2}A_\nu^{1/d_\nu}(\xi_\nu(\tau_\nu) - \xi) \\ &\quad + \psi(A_\nu^{1/d_\nu}\xi)A_\nu^{1/d_\nu} y_\nu(\tau_\nu, \xi) + O(A_\nu^{46/(73d_\nu)}(\log A_\nu)^{315/146}). \end{aligned}$$

Proof. Choose $0 < \overline{\xi} < \widetilde{\xi}_\nu$ with $\widetilde{y}_\nu(\overline{\xi}) < \widetilde{\eta}_\nu/2$. A second condition on $\overline{\xi}$ will be given below. For $0 < \widetilde{x}_\nu(2\widetilde{y}_\nu(\overline{\xi})) \leq x \leq \widetilde{\xi}_\nu$ we have

$$a_{d_\nu 0}^{(\nu)} \widetilde{\xi}_\nu^{d_\nu} = \widetilde{g}_\nu(\widetilde{\xi}_\nu, 0) = 1 = \widetilde{g}_\nu(x, \widetilde{y}_\nu(x)) = \sum_{i+j=d_\nu} a_{ij}^{(\nu)} x^i \widetilde{y}_\nu(x)^j$$

and consequently

$$\begin{aligned} \widetilde{\xi}_\nu - x &\asymp a_{d_\nu 0}^{(\nu)} \widetilde{\xi}_\nu^{d_\nu} - a_{d_\nu 0}^{(\nu)} x^{d_\nu} \\ &= \widetilde{y}_\nu(x)^{j_0^{(\nu)}} \sum_{j=j_0^{(\nu)}}^{d_\nu} a_{d_\nu - j, j}^{(\nu)} x^{d_\nu - j} \widetilde{y}_\nu(x)^{j - j_0^{(\nu)}} \asymp \widetilde{y}_\nu(x)^{j_0^{(\nu)}}. \end{aligned}$$

For $0 \leq y \leq 2\widetilde{y}_\nu(\overline{\xi}) < \widetilde{\eta}_\nu$ it follows that

$$|\widetilde{\xi}_\nu - \widetilde{x}_\nu(y)| \asymp y^{j_0^{(\nu)}}. \tag{6.1}$$

For $0 \leq k \leq 3$, $0 \leq y \leq 2\widetilde{y}_\nu(\overline{\xi})$ Taylor's theorem gives

$$\widetilde{x}_\nu^{(k)}(y) = \sum_{l=0}^{j_0^{(\nu)} - k} \frac{1}{l!} \widetilde{x}_\nu^{(k+l)}(0) y^l + O(y^{j_0^{(\nu)} - k + 1}).$$

With $\widetilde{x}_\nu(0) = \widetilde{\xi}_\nu$ and (6.1) the choice $k = 0$ gives $\widetilde{x}_\nu^{(l)}(0) = 0$ for $l = 1, \ldots, j_0^{(\nu)} - 1$ and $\widetilde{x}_\nu^{(j_0^{(\nu)})}(0) \neq 0$. From the assumptions it follows that $j_0^{(\nu)} = 2$.

Let $K_{20} > 0$ be a constant which will be chosen appropriately below. For $1 \le k \le 3$, $K_{20}\tau_\nu \le y \le 2\widetilde{y}_\nu(\overline{\xi})$ Taylor's theorem gives

$$\frac{\partial^k x_\nu}{\partial y^k}(\tau_\nu, y) = ((1-\delta_{3k})\widetilde{x}_\nu''(0) + O(\tau_\nu y^{k-2} + y))y^{2-k}.$$

The error term is

$$\ll \tau_\nu(1 + (K_{20}\tau_\nu)^{-1}) + \widetilde{y}_\nu(\overline{\xi}) \ll \tau_\nu + K_{20}^{-1} + \widetilde{y}_\nu(\overline{\xi}).$$

Choosing K_{20} sufficiently large and $\overline{\xi}$ sufficiently close to $\widetilde{\xi}_\nu$ gives, for $k = 1, 2$,

$$\left|\frac{\partial^k x_\nu}{\partial y^k}(\tau_\nu, y)\right| \asymp y^{2-k} \tag{6.2}$$

and

$$\begin{aligned}|x_{\nu yyy}(\tau_\nu, y)x_{\nu y}(\tau_\nu, y) - 3x_{\nu yy}(\tau_\nu, y)^2|& \\ = |-3\widetilde{x}_\nu''(0)^2 + O(K_{20}^{-1} + \tau_\nu + \widetilde{y}_\nu(\overline{\xi}))| &\asymp 1.\end{aligned}$$

For $\widetilde{x}_\nu(2\widetilde{y}_\nu(\overline{\xi})) \le x \le \widetilde{\xi}_\nu - K_{21}\tau_\nu$ $(\le \xi_\nu(\tau_\nu))$ with sufficiently large $K_{21} > 0$ if follows that $2\widetilde{y}_\nu(\overline{\xi}) \ge \widetilde{y}_\nu(x) \ge y_\nu(\tau_\nu, x) \ge 0$, and (6.1) yields

$$y_\nu(\tau_\nu, x)^2 \asymp |\widetilde{\xi}_\nu - x_\nu(\tau_\nu, y_\nu(\tau_\nu, x)) + O(\tau_\nu)| \asymp |\widetilde{\xi}_\nu - x|.$$

In particular

$$y_\nu(\tau_\nu, x) \gg |\widetilde{\xi}_\nu - x|^{1/2} \gg (K_{21}\tau_\nu)^{1/2} \ge K_{20}\tau_\nu$$

and from (6.2) it follows that

$$\begin{aligned}|y_{\nu x}(\tau_\nu, x)| &= |x_{\nu y}(\tau_\nu, y_\nu(\tau_\nu, x))^{-1}| \asymp |\widetilde{\xi}_\nu - x|^{-1/2},\\ |y_{\nu xx}(\tau_\nu, x)| &= |-x_{\nu yy}(\tau_\nu, y_\nu(\tau_\nu, x))x_{\nu y}(\tau_\nu, y_\nu(\tau_\nu, x))^{-3}| \asymp |\widetilde{\xi}_\nu - x|^{-3/2},\\ |y_{\nu xxx}(\tau_\nu, x)| &= |-(x_{\nu yyy}(\tau_\nu, y_\nu(\tau_\nu, x))x_{\nu y}(\tau_\nu, y_\nu(\tau_\nu, x))\\ &\qquad - 3x_{\nu yy}(\tau_\nu, y_\nu(\tau_\nu, x))^2)x_{\nu y}(\tau_\nu, y_\nu(\tau_\nu, x))^{-5}| \asymp |\widetilde{\xi}_\nu - x|^{-5/2}.\end{aligned}$$

Partial summation gives, for $\overline{\xi} \le \xi \le \xi_\nu(\tau_\nu)$,

$$\begin{aligned}R = &\int_{A_\nu^{1/d_\nu}\xi}^{A_\nu^{1/d_\nu}\xi_\nu(\tau_\nu)} \left(A_\nu^{1/d_\nu} y_\nu(\tau_\nu, \tau_\nu x) - \tfrac{1}{2}\right) dx\\ &+ \psi(A_\nu^{1/d_\nu}\xi)A_\nu^{1/d_\nu} y_\nu(\tau_\nu, \xi)\\ &+ O(1) + \int_{A_\nu^{1/d_\nu}\xi}^{A_\nu^{1/d_\nu}\xi_\nu(\tau_\nu)} y_{\nu x}(\tau_\nu, \tau_\nu x)\psi(x)\, dx\\ &- \sum_{A_\nu^{1/d_\nu}\xi < x \le A_\nu^{1/d_\nu}\xi_\nu(\tau_\nu)} \psi(A_\nu^{1/d_\nu} y_\nu(\tau_\nu, \tau_\nu x)).\end{aligned} \tag{6.3}$$

As in the proof of Proposition 5.2 it follows by partial integration that the second integral in (6.3) is

$$
\begin{aligned}
&= y_{\nu x}(\tau_\nu,\tau_\nu x)\psi_1(x)\Big|_{A_\nu^{1/d_\nu}\xi}^{A_\nu^{1/d_\nu}(\widetilde{\xi}_\nu - K_{21}\tau_\nu)} \\
&\quad - \int\limits_{A_\nu^{1/d_\nu}\xi}^{A_\nu^{1/d_\nu}(\widetilde{\xi}_\nu - K_{21}\tau_\nu)} \tau_\nu y_{\nu xx}(\tau_\nu,\tau_\nu x)\psi_1(x)\,dx \\
&\quad + \int\limits_{A_\nu^{1/d_\nu}(\widetilde{\xi}_\nu - K_{21}\tau_\nu)}^{A_\nu^{1/d_\nu}\xi_\nu(\tau_\nu)} y_{\nu x}(\tau_\nu,\tau_\nu x)\psi(x)\,dx \\
&\ll (K_{21}\tau_\nu)^{-1/2} + \tau_\nu \int\limits_{A_\nu^{1/d_\nu}\xi}^{A_\nu^{1/d_\nu}(\widetilde{\xi}_\nu - K_{21}\tau_\nu)} |\widetilde{\xi}_\nu - \tau_\nu x|^{-3/2}dx \\
&\quad + \int\limits_{A_\nu^{1/d_\nu}(\widetilde{\xi}_\nu - K_{21}\tau_\nu)}^{A_\nu^{1/d_\nu}\xi_\nu(\tau_\nu)} |\widetilde{\xi}_\nu - \tau_\nu x|^{-1/2}\,dx \\
&\ll \tau_\nu^{-1/2}.
\end{aligned}
$$

Let $J := [\log A]$, $M_0 := A_\nu^{46/(73d_\nu)}$, $M_J := [A_\nu^{1/d_\nu}(\widetilde{\xi}_\nu - \overline{\xi})]$, $B := (M_J M_0^{-1})^{1/J}$, $M_j := M_0 B^j$ for $0 \le j \le J$. Then $B = B_0 + o(1)$ with some constant $B_0 > 1$. For $x \in [M_j - M_{j-1}, 2(M_j - M_{j-1})]$ define

$$f_j(x) := A_\nu^{1/d_\nu} y_\nu(\tau_\nu, \tau_\nu([A_\nu^{1/d_\nu}\widetilde{\xi}_\nu] - [2M_{j-1} - M_j] - x)).$$

Then

$$
\begin{aligned}
|f_j''(x)| &\asymp A_\nu^{-1/d_\nu}|\widetilde{\xi}_\nu - A_\nu^{-1/d_\nu}(A_\nu^{1/d_\nu}\widetilde{\xi}_\nu - 2M_{j-1} + M_j - x + O(1))|^{-3/2} \\
&\asymp A_\nu^{-1/d_\nu}(A_\nu^{-1/d_\nu}M_{j-1})^{-3/2} \asymp T(M_j - M_{j-1})^{-3}, \\
|f_j'''(x)| &\asymp T(M_j - M_{j-1})^{-4}
\end{aligned}
$$

with $T := A_\nu^{1/(2d_\nu)} M_{j-1}^{3/2}$. Furthermore $T^{63/146}(\log T)^{63/292} \le (M_j - M_{j-1}) \ll T^{1/2}$. From Theorem 18.2.2 of [2] it follows that the lattice remainder in (6.3) is

$$
\begin{aligned}
&= \sum_{j=1}^{J} \sum_{M_j - M_{j-1} \le n \le \min\{2(M_j - M_{j-1}), A_\nu^{1/d_\nu}(\widetilde{\xi}_\nu - \xi) - 2M_{j-1} + M_j\}} \psi(f_j(n)) + O(J) \\
&\quad + O(A_\nu^{46/(73d_\nu)}) \\
&\ll A_\nu^{46/(73d_\nu)}(\log A_\nu)^{315/146}. \ \blacksquare
\end{aligned}
$$

7. Combinatorial composition of the results. $R^\dagger$ is handled first. Choose $Z \subseteq \mathbb{R}_0^+$, $|Z| < \infty$, with the properties:

(Z†1) The zeros of $\widetilde{y}_j''\widetilde{y}_j'''$ in $[0, \widetilde{\xi}_j]$ are contained in Z for $j = 1, 2$.

(Z†2) The zeros of $\widetilde{x}_j''\widetilde{x}_j'''$ in $[0, \widetilde{\eta}_j]$ are contained in $\widetilde{y}_j(Z)$ for $j = 1, 2$.

(Z†3) $\xi_0, \ldots, \xi_r, x_0, x_2 \in Z$.

(Z†4) If $\widetilde{y}_j'(0) = 0 \neq \widetilde{y}_j''(0)$ for $j = 1$ or $j = 2$ then the value $\overline{\eta}$ which comes from the application of Proposition 6.1 with x and y interchanged is contained in $\widetilde{y}_j(Z)$.

Choose $0 = \zeta_0 < \ldots < \zeta_n$ with the properties:

(Z†5) $Z \subseteq \{\zeta_0, \ldots, \zeta_n\}$.

(Z†6) For each $\zeta, \zeta' \in Z$ with $\zeta < \zeta'$ there is some $1 \leq \nu \leq n$ with $\zeta < \zeta_\nu < \zeta'$.

The next two lemmas combine the results of Sections 4 to 6.

LEMMA 7.1. *Let $j \in \{1, 2\}$ and $1 \leq \nu \leq n$ with $\zeta_\nu < \widetilde{\xi}_j$. There are constants $\kappa, K > 0$ so that for $\zeta \in (\zeta_{\nu-1} + \tau_j^\kappa, \zeta_\nu - \tau_j^\kappa)$ and $A_j \geq K$,*

$$\begin{aligned}(7.1)\quad R := \#\{(x,y) \in \mathbb{Z}^2 \mid A_j^{1/d_j}\zeta_{\nu-1} < x \leq A_j^{1/d_j}\zeta,\\ 0 < y \leq A_j^{1/d_j} y_j(\tau_j, \tau_j x)\}\\ = A_j^{2/d_j} \int\limits_{\zeta_{\nu-1}}^{\zeta} y_j(\tau_j, x)\,dx - \tfrac{1}{2}A_j^{1/d_j}(\zeta - \zeta_{\nu-1}) + T(A_j) + U(A_j)\\ + \psi(A_j^{1/d_j}\zeta_{\nu-1})A_j^{1/d_j} y_j(\tau_j, \zeta_{\nu-1}) - \psi(A_j^{1/d_j}\zeta)A_j^{1/d_j} y_j(\tau_j, \zeta)\\ + O(A_j^{46/(73d_j)}(\log A_j)^{315/146}).\end{aligned}$$

The representation

$$\begin{aligned}(7.2)\qquad T(A_j) = A_j^{(1/d_j)(1-1/(\mu+2))} H(A_j^{1/d_j})\\ + O(A_j^{(1/d_j)(1-1/(\mu+2)-1/(\mu+2)^2)} \log A_j)\end{aligned}$$

holds with some constant $\mu \in \mathbb{N}$ and some periodic function H which is given by an absolutely convergent Fourier series. $T(A_j)$ can only occur if $\widetilde{y}_j''$ has a zero of order μ at $\zeta_{\nu-1}$ and $\widetilde{y}_j'(\zeta_{\nu-1}) \in \mathbb{Q}$.

Furthermore

$$(7.3)\qquad U(A_j) = O_\delta(A_j^{(1/d_j)(1-1/\mu+\delta)})$$

for each $\delta > 0$ and this function can only occur if $\widetilde{y}_j''$ has a zero of order μ at $\zeta_{\nu-1}$ and $\widetilde{y}_j'(\zeta_{\nu-1}) \notin \mathbb{Q}$.

Proof. From the choice of the ζ_ν it follows that $\widetilde{y}_j''\widetilde{y}_j'''$ has zeros in $[\zeta_{\nu-1}, \zeta_\nu]$ at most at $\zeta_{\nu-1}$ or ζ_ν but not at both points.

Case 1: $\widetilde{y}_j''\widetilde{y}_j'''(\zeta_{\nu-1}) \neq 0$ *and* $\widetilde{y}_j''\widetilde{y}_j'''(\zeta_\nu) \neq 0$. Partial summation gives

$$R = \int_{A_j^{1/d_j}\zeta_{\nu-1}}^{A_j^{1/d_j}\zeta} \big(A_j^{1/d_j} y_j(\tau_j, \tau_j x) - \tfrac{1}{2}\big)\, dx \tag{7.4}$$
$$+ \int_{A_j^{1/d_j}\zeta_{\nu-1}}^{A_j^{1/d_j}\zeta} y_{jx}(\tau_j, \tau_j x)\psi(x)\, dx$$
$$+ \psi(A_j^{1/d_j}\zeta_{\nu-1})\big(A_j^{1/d_j} y_j(\tau_j, \zeta_{\nu-1}) - \tfrac{1}{2}\big)$$
$$- \psi(A_j^{1/d_j}\zeta)\big(A_j^{1/d_j} y_j(\tau_j, \zeta) - \tfrac{1}{2}\big)$$
$$- \sum_{A_j^{1/d_j}\zeta_{\nu-1} < x \le A_j^{1/d_j}\zeta} \psi(A_j^{1/d_j} y_j(\tau_j, \tau_j x)).$$

The lattice remainder is $O(A_j^{46/(73d_j)}(\log A_j)^{315/146})$ as follows from Lemma 4.3. Partial integration gives the bound $O(1)$ for the second integral.

Case 2: $\widetilde{y}_j''$ *has at* ζ_ν *a zero of order* $\mu \in \mathbb{N}$. Then $\zeta_\nu, 0 \in Z$ and by ($\mathrm{Z}^\dagger 6$) it follows $\zeta_{\nu-1} \notin Z$, i.e. $\nu \ge 2$. Therefore $\widetilde{y}_j'(x) \neq 0$ for $x \in [\zeta_{\nu-1}, \zeta_\nu]$. Choose $0 < \kappa < \lambda_0 := 20/(83\mu+103)$. By Lemma 4.4 the lattice remainder in (7.4) is

$$= \sum_{A_j^{1/d_j}(-\zeta_\nu + \tau_j^{\lambda_0}) < x \le -A_j^{1/d_j}\zeta_{\nu-1}} \psi(A_j^{1/d_j} y_j(\tau_j, -\tau_j x))$$
$$- \sum_{A_j^{1/d_j}(-\zeta_\nu + \tau_j^{\lambda_0}) < x \le -A_j^{1/d_j}\zeta} \psi(A_j^{1/d_j} y_j(\tau_j, -\tau_j x)) + O(1)$$
$$\ll A_j^{46/(73d_j)}(\log A_j)^{315/146}.$$

Case 3: $\widetilde{y}_j''$ *has at* $\zeta_{\nu-1}$ *a zero of order* $\mu \in \mathbb{N}$.

Case 3.1: $\widetilde{y}_j'(\zeta_{\nu-1}) \in \mathbb{Q}$. Apply Proposition 5.2. Choose $\kappa > 0$ smaller than the value λ which is given by this proposition.

Case 3.2: $\widetilde{y}_j'(\zeta_{\nu-1}) \notin \mathbb{Q}$. Then $\widetilde{y}_j'(\zeta_{\nu-1})$ is algebraic over $\mathbb{Q}$ by Lemma 4.2. Apply Proposition 4.5 to the lattice remainder in (7.4) and choose $0 < \kappa < 20/(83\mu + 103)$.

Case 4: *There is some* $\nu' \in \{\nu - 1, \nu\}$ *with* $\widetilde{y}_j'''(\zeta_{\nu'}) = 0$ *and* $\widetilde{y}_j''(\zeta_{\nu'}) \neq 0$. We have

$$R = \#\{(x, y) \in \mathbb{Z}^2 \mid A_j^{1/d_j} y_j(\tau_j, \zeta) < y \le A_j^{1/d_j} y_j(\tau_j, \zeta_{\nu-1}), \tag{7.5}$$
$$0 < x \le A_j^{1/d_j} x_j(\tau_j, \tau_j y)\}$$
$$+ [A_j^{1/d_j}\zeta][A_j^{1/d_j} y_j(\tau_j, \zeta)] - [A_j^{1/d_j}\zeta_{\nu-1}][A_j^{1/d_j} y_j(\tau_j, \zeta_{\nu-1})].$$

$\widetilde{x}_j''\widetilde{x}_j'''$ has no zeros in $(\widetilde{y}_j(\zeta_\nu),\widetilde{y}_j(\zeta_{\nu-1}))$. On $[\widetilde{y}_j(\zeta_\nu),\widetilde{y}_j(\zeta_{\nu-1}))$ we have

$$\begin{gathered}\widetilde{x}_j'(y)=\widetilde{y}_j'(\widetilde{x}_j(y))^{-1},\qquad \widetilde{x}_j''(y)=-\widetilde{y}_j'(\widetilde{x}_j(y))^{-3}\widetilde{y}_j''(\widetilde{x}_j(y)),\\ \widetilde{x}_j'''(y)=-\widetilde{y}_j'(\widetilde{x}_j(y))^{-5}(\widetilde{y}_j'''(\widetilde{x}_j(y))\widetilde{y}_j'(\widetilde{x}_j(y))-3\widetilde{y}_j''(\widetilde{x}_j(y))^2).\end{gathered}\tag{7.6}$$

Case 4.1: $\nu\ge 2$. Then $0<\zeta_{\nu-1}<\zeta_\nu<\widetilde{\xi}_j$ and (7.6) is valid also in $\widetilde{y}_j(\zeta_{\nu-1})$. From (7.6) and the assumption of Case 4 it follows that $\widetilde{x}_j''\widetilde{x}_j'''(\widetilde{y}_j(\zeta_{\nu'}))\ne 0$. If $\widetilde{x}_j''\widetilde{x}_j'''$ had a zero in $[\widetilde{y}_j(\zeta_\nu),\widetilde{y}_j(\zeta_{\nu-1})]$ then this zero would be of the form $\widetilde{y}_j(\zeta)$, $\zeta\in Z$. Then $\zeta_{\nu-1}\le\zeta\le\zeta_\nu$, $\zeta\ne\zeta_{\nu'}$, $\zeta,\zeta_{\nu'}\in Z$. Then there would be some $\zeta_{\nu''}$ between ζ and $\zeta_{\nu'}$ and consequently between $\zeta_{\nu-1}$ and ζ_ν, which is a contradiction. Therefore $\widetilde{x}_j''\widetilde{x}_j'''$ has no zeros in $[\widetilde{y}_j(\zeta_\nu),\widetilde{y}_j(\zeta_{\nu-1})]$. Applying partial summation and Lemma 4.3 to (7.5) gives

$$\begin{aligned}R=A_j^{2/d_j}\int\limits_{\zeta_{\nu-1}}^{\zeta} y_j(\tau_j,x)\,dx-\psi(A_j^{1/d_j}\zeta)A_j^{1/d_j}y_j(\tau_j,\zeta)&\\ +\psi(A_j^{1/d_j}\zeta_{\nu-1})A_j^{1/d_j}y_j(\tau_j,\zeta_{\nu-1})-\tfrac12 A_j^{1/d_j}(\zeta-\zeta_{\nu-1})&\\ +O(A_j^{46/(73d_j)}(\log A_j)^{315/146}).&\end{aligned}$$

Case 4.2: $\nu=1$, $\widetilde{y}_j'(0)\ne 0$. Then $\zeta_{\nu-1}=\zeta_0=0$ and

$$0\ne-\widetilde{y}_j'(0)\widetilde{g}_{jy}(0,\widetilde{y}_j(0))=\widetilde{g}_{jx}(0,\widetilde{y}_j(0))=a_{1,d_1-1}^{(j)}\widetilde{\eta}_j^{d_j-1}$$

and consequently $i_0^{(j)}=1$ and $\widetilde{x}_j\in C^\infty[0,\widetilde{\eta}_j]$. The proof is exactly the same as in Case 4.1.

Case 4.3: $\nu=1$, $\widetilde{y}_j'(0)=0$. Now the argument of Cases 4.1 and 4.2 is no longer valid because $\widetilde{x}_j$ is not C^∞ at $\widetilde{\eta}_j$. We have $\widetilde{y}_j''(0)\ne 0$ because otherwise $\nu'=1$ and $\widetilde{y}_j''\widetilde{y}_j'''$ would have zeros at $\zeta_{\nu-1}$ and ζ_ν, which is impossible. Apply Proposition 6.1 for x and y interchanged. If $0<\overline{\eta}<\widetilde{\eta}_j$ is the value which corresponds to $\overline{\xi}$ in Proposition 6.1 then $\overline{\eta}\in\widetilde{y}_j(Z)$ by ($\mathrm{Z}^\dagger 4$). Furthermore there is some $1\le\nu''\le n$ with $\zeta_0<\zeta_{\nu''}<\widetilde{x}_j(\overline{\eta})$. Consequently, $\zeta_1<\widetilde{x}_j(\overline{\eta})$ and $\widetilde{y}_j(\zeta_1)>\overline{\eta}$. Therefore for sufficiently large A_j we have $\overline{\eta}\le y_j(\tau_j,\zeta)=:\eta\le\eta_j(\tau_j)$. Proposition 6.1 gives

$$\begin{aligned}R=\int\limits_{A_j^{1/d_j}\eta}^{A_j^{1/d_j}\eta_j(\tau_j)} A_j^{1/d_j}x_j(\tau_j,\tau_j y)\,dy+\psi(A_j^{1/d_j}\eta)A_j^{1/d_j}x_j(\tau_j,\eta)&\\ -\tfrac12 A_j^{1/d_j}(\eta_j(\tau_j)-\eta)+O(A_j^{46/(73d_j)}(\log A_j)^{315/146})&\\ +[A_j^{1/d_j}\zeta][A_j^{1/d_j}y_j(\tau_j,\zeta)].&\end{aligned}$$

Substituting $y = A_j^{1/d_j} y_j(\tau_j, x)$ and partial integration give

$$R = A_j^{2/d_j} \int\limits_0^{\zeta} y_j(\tau_j, x)\,dx - \tfrac{1}{2} A_j^{1/d_j} \zeta - \psi(A_j^{1/d_j}\zeta) A_j^{1/d_j} y_j(\tau_j, \zeta)$$
$$+ \psi(A_j^{1/d_j} 0) A_j^{1/d_j} y_j(\tau_j, 0) + O(A_j^{46/(73 d_j)} (\log A_j)^{315/146}). \blacksquare$$

LEMMA 7.2. *Let $j \in \{1,2\}$ and $1 \le \nu \le n$ with $\zeta_\nu < \widetilde{\xi}_j$. There is some constant $K > 0$ so that for $A_j \ge K$, $\zeta \in (\zeta_{\nu-1}, \zeta_\nu]$ with $|\zeta - \zeta_\nu| \ll \tau_j$ the asymptotics* (7.1) *hold. $T(A_j)$ is of the form* (7.2) *and can only appear if $\widetilde{y}_j''$ has a zero of order μ at $\zeta_{\nu-1}$ or ζ_ν and $\widetilde{y}_j'$ is rational at this point. $U(A_j)$ is of the form* (7.3) *and can only appear if $\widetilde{y}_j''$ has a zero of order μ at $\zeta_{\nu-1}$ or ζ_ν and $\widetilde{y}_j'$ is irrational at this point.*

Proof. It is quite similar to that of Lemma 7.1.

CASE 1: *$\widetilde{y}_j''$ has a zero of order μ at $\zeta_{\nu-1}$*. Apply the reasoning of Case 3 of Lemma 7.1.

CASE 2: *$\widetilde{y}_j''$ has a zero of order μ at ζ_ν*. From ($\mathrm{Z}^\dagger$1) it follows that $\zeta_\nu \in Z$. From $\zeta_0 = 0 \in Z$ it follows that $\nu \ge 2$ because there is some $\zeta_{\nu'}$ between ζ_0 and ζ_ν. Therefore $0 < \zeta_{\nu-1} < \zeta_\nu < \widetilde{\xi}_j$. It follows from (7.6) and $\widetilde{x}_j'(\widetilde{y}_j(\zeta_\nu)) \ne 0$ that

$$|\widetilde{x}_j''(y)| \asymp |\widetilde{y}_j''(\widetilde{x}_j(y))| \asymp |\widetilde{x}_j(y) - \zeta_\nu|^\mu \asymp |y - \widetilde{y}_j(\zeta_\nu)|^\mu$$

as $y \to \widetilde{y}_j(\zeta_\nu)$. Therefore $\widetilde{x}_j''$ has a zero of order μ at $\widetilde{y}_j(\zeta_\nu)$. Furthermore $\widetilde{x}_j'(\widetilde{y}_j(\zeta_\nu)) \in \mathbb{Q}$ if and only if $\widetilde{y}_j'(\zeta_\nu) \in \mathbb{Q}$. From the construction of the ζ_ν it follows that $\widetilde{x}_j''\widetilde{x}_j'''$ has no zero in $(\widetilde{y}_j(\zeta_\nu), \widetilde{y}_j(\zeta_{\nu-1})]$. From the assumptions it follows that

$$y_j(\tau_j, \zeta) = \widetilde{y}_j(\zeta) + O(\tau_j) = \widetilde{y}_j(\zeta_\nu) + O(\tau_j), \quad y_j(\tau_j, \zeta_{\nu-1}) = \widetilde{y}_j(\zeta_{\nu-1}) + O(\tau_j).$$

Applying Proposition 5.2 or Proposition 4.5 to (7.5) gives the desired result.

CASE 3: $\widetilde{y}_j''(\zeta_{\nu-1}) \ne 0 \ne \widetilde{y}_j''(\zeta_\nu)$.

CASE 3.1: $\widetilde{y}_j'''(\zeta_{\nu-1}) \ne 0 \ne \widetilde{y}_j'''(\zeta_\nu)$. Use the reasoning of Case 1 of Lemma 7.1.

CASE 3.2: *There is some $\nu' \in \{\nu - 1, \nu\}$ with $\widetilde{y}_j'''(\zeta_{\nu'}) = 0$*. Use the reasoning of Case 4 of Lemma 7.1. $\blacksquare$

The following lemma is used in the construction of the decomposition of $[C_1, C_2]$. For $1 \le \varrho \le r$ define $[a_\varrho', b_\varrho'] := k_\varrho^{-1}((-\infty, x_0])$ if the right hand side is nonempty.

LEMMA 7.3. *For each $\theta \in \mathbb{R}$, $\kappa > 0$, $1 \le \varrho \le r$, there is a decomposition $a_\varrho' = C_0^* < \ldots < C_m^* = b_\varrho'$ and constants $\varepsilon, K > 0$ with the property: If*

$A_1, A_2 \geq K$, $C := A_2 A_1^{-d_2/d_1} \in [a'_\varrho, b'_\varrho]$, $|C - C^*_\mu| \geq A_1^{-\varepsilon}$ *for each* $0 \leq \mu \leq m$, *then*

$$|k_\varrho(C) - \theta| \geq \tau_1^\kappa, \qquad |C^{-1/d_2} k_\varrho(C) - \theta| \geq \tau_2^\kappa.$$

P r o o f. The function $\widetilde{k}_\varrho : [a'_\varrho, b'_\varrho] \to \mathbb{C}$, $C \mapsto C^{-1/d_2} k_\varrho(C)$, is injective and continuous: If $C \in [a'_\varrho, b'_\varrho]$, $x := C^{-1/d_2} k_\varrho(C)$, then

$$\widetilde{g}_2(x, C^{-1/d_2} \widetilde{y}_1(C^{1/d_2} x)) = C^{-1} h(C^{1/d_2} x) = 1$$

and consequently $0 \leq x \leq \widetilde{\xi}_2$, $\widetilde{y}_1(C^{1/d_2} x) = C^{1/d_2} \widetilde{y}_2(x)$. The left hand side of the last equation is strictly decreasing in C whereas the right hand side is strictly increasing. Therefore the value x is assumed at no other argument C.

If θ is not in the range of values of $\widetilde{k}_\varrho$ then

$$|C^{-1/d_2} k_\varrho(C) - \theta| \gg 1 \quad \text{ for } C \in [a'_\varrho, b'_\varrho].$$

In the opposite case $\theta = \widetilde{k}_\varrho(C^*)$ for some $C^* \in [a'_\varrho, b'_\varrho]$. Then $k_\varrho(C^*) \in [0, x_0] \cap [\xi_{\varrho-1}, \xi_\varrho]$. The function h is holomorphic in some neighbourhood of $[0, x_0]$.

If $h'(k_\varrho(C^*)) \neq 0$ then h has a holomorphic inverse in some neighbourhood of $k_\varrho(C^*)$ and therefore k_ϱ and $\widetilde{k}_\varrho$ are holomorphic in some neighbourhood of C^*. The function $\widetilde{k}_\varrho$ is nonconstant. Consequently, there are $p \in \mathbb{N}$ and $\delta > 0$ with $|\widetilde{k}_\varrho(C) - \theta| \asymp |C - C^*|^p$ for $C \in [a'_\varrho, b'_\varrho]$, $|C - C^*| \leq \delta$. Choosing $0 < \varepsilon < \kappa/(d_1 p)$ gives $|\widetilde{k}_\varrho(C) - \theta| \gg A_1^{-p\varepsilon} \geq \tau_2^\kappa$ for $C \in [a'_\varrho, b'_\varrho]$ in the case $A_1^{-\varepsilon} \leq |C - C^*| \leq \delta$. In the case $|C - C^*| \geq \delta$ we have

$$|\widetilde{k}_\varrho(C) - \theta| \geq \min_{C' \in [a'_\varrho, b'_\varrho],\, |C' - C^*| \geq \delta} |\widetilde{k}_\varrho(C') - \theta| = \text{const.} > 0.$$

If $h'(k_\varrho(C^*)) = 0$ it follows from Lemma 2.1 that there are $p \in \mathbb{N} \setminus \{1\}$ and some neighbourhood U of $k_\varrho(C^*)$ with

$$|h(x) - h(k_\varrho(C^*))| \asymp |x - k_\varrho(C^*)|^p \quad \text{ for } x \in U.$$

Consequently, there are constants $\delta, K_i > 0$ with

$$\begin{aligned}
|\widetilde{k}_\varrho(C) - \theta| &= |(C^*)^{-1/d_2} k_\varrho(C) - (C^*)^{-1/d_2} k_\varrho(C^*)| \\
&\qquad + O(|C^{-1/d_2} - (C^*)^{-1/d_2}| \cdot |k_\varrho(C)|) \\
&\geq K_3 |k_\varrho(C) - k_\varrho(C^*)| - K_4 |C - C^*| \\
&\geq K_5 |h(k_\varrho(C)) - h(k_\varrho(C^*))|^{1/p} - K_4 |C - C^*| \\
&= K_5 |C - C^*|^{1/p} - K_4 |C - C^*| \geq K_6 |C - C^*|^{1/p}
\end{aligned}$$

for $C \in [a'_\varrho, b'_\varrho]$, $|C - C^*| \leq \delta$. The remainder of the argument is as above.

The function k_ϱ is injective on $[a'_\varrho, b'_\varrho]$. If $\theta \notin k_\varrho([a'_\varrho, b'_\varrho])$ then $|k_\varrho(C) - \theta| \gg 1$ for $C \in [a'_\varrho, b'_\varrho]$. In the opposite case there is some $C^* \in [a'_\varrho, b'_\varrho]$ with

$\theta = k_\varrho(C^*) \in [0, x_0]$. From $h \in C^\infty[0, x_0]$ it follows that

$$|C - C^*| = |h(k_\varrho(C)) - h(k_\varrho(C^*))| \ll |k_\varrho(C) - k_\varrho(C^*)|$$

for $C \in [a'_\varrho, b'_\varrho]$. The remainder of the argument is as above. ∎

Now use Lemma 7.3 and the injectivity of k_ϱ and $\widetilde{k}_\varrho$ to construct a decomposition $C_1 = C_0^* < \ldots < C_m^* = C_2$ and some $\varepsilon > 0$ with the properties:

(C†1) If $a'_\varrho \in [C_1, C_2]$ for some $1 \le \varrho \le r$ then $a'_\varrho \in \{C_0^*, \ldots, C_m^*\}$. The same holds for $a_\varrho, b'_\varrho, b_\varrho$.

(C†2) For $1 \le \varrho \le r$, $1 \le \mu \le m$, $0 \le \nu \le n$, with $[C_{\mu-1}^*, C_\mu^*] \subseteq [a'_\varrho, b'_\varrho]$: For $A_1, A_2 \ge K$, $C := A_2 A_1^{-d_2/d_1} \in (C_{\mu-1}^* + A_1^{-\varepsilon}, C_\mu^* - A_1^{-\varepsilon})$ we have

$$|k_\varrho(C) - \zeta_\nu| \ge \tau_1^\kappa, \qquad |C^{-1/d_2} k_\varrho(C) - \zeta_\nu| \ge \tau_2^\kappa.$$

(C†3) For $1 \le \mu \le m$, $1 \le \nu' \le n$, $0 \le \nu \le n$: For $A_1, A_2 \ge K$, $C := A_2 A_1^{-d_2/d_1} \in (C_{\mu-1}^* + A_1^{-\varepsilon}, C_\mu^* - A_1^{-\varepsilon})$ we have $|C^{-1/d_2}\zeta_{\nu'} - \zeta_\nu| \ge \tau_2^\kappa$.

(C†4) For $1 \le \varrho \le r$, $1 \le \mu \le m$, $1 \le \nu' \le n$, $0 \le \nu \le n$ we have

$$\zeta_\nu \notin k_\varrho((C_{\mu-1}^*, C_\mu^*) \cap [a'_\varrho, b'_\varrho]), \qquad \zeta_\nu \notin \widetilde{k}_\varrho((C_{\mu-1}^*, C_\mu^*) \cap [a'_\varrho, b'_\varrho]),$$
$$\zeta_\nu \ne C^{-1/d_2}\zeta_{\nu'} \quad \text{for } C \in (C_{\mu-1}^*, C_\mu^*).$$

With this decomposition the following lemma holds which contains one half of Theorem 1.1.

Lemma 7.4. *Let $1 \le \mu \le m$, $1 \le \varrho \le r$ with $\xi_{\varrho-1} < x_0$. Then for $A_1, A_2 \ge K$, $C := A_2 A_1^{-d_2/d_1} \in (C_{\mu-1}^* + A_1^{-\varepsilon}, C_\mu^* - A_1^{-\varepsilon})$ we have*

$$\begin{aligned}(7.7)\quad R_\varrho^\dagger(A_1, A_2) = & \int\limits_{A_1^{1/d_1}\xi_{\varrho-1}}^{A_1^{1/d_1}\min\{\xi_\varrho, x_0\}} f_{A_1,A_2}(x)\,dx \\ & - \tfrac{1}{2}A_1^{1/d_1}(\min\{\xi_\varrho, x_0\} - \xi_{\varrho-1}) + T_1^{\dagger(\mu,\varrho)}(A_1) \\ & + T_2^{\dagger(\mu,\varrho)}(A_2) + U_1^{\dagger(\mu,\varrho)}(A_1) + U_2^{\dagger(\mu,\varrho)}(A_2) \\ & - \psi(A_1^{1/d_1}\min\{\xi_\varrho, x_0\}) f_{A_1,A_2}(A_1^{1/d_1}\min\{\xi_\varrho, x_0\}) \\ & + \psi(A_1^{1/d_1}\xi_{\varrho-1}) f_{A_1,A_2}(A_1^{1/d_1}\xi_{\varrho-1}) \\ & + O(A_1^{46/(73d_1)}(\log A_1)^{315/146}).\end{aligned}$$

The functions $T_j^{\dagger(\mu,\varrho)}$ and $U_j^{\dagger(\mu,\varrho)}$ are of the form described in Theorem 1.1.

Proof. Several cases have to be distinguished in order to apply Lemma 3.1.

Case 1: $[C^*_{\mu-1}, C^*_\mu] \subseteq [a'_\varrho, b'_\varrho]$. From ($Z^\dagger 3$), ($Z^\dagger 5$) and ($C^\dagger 4$) it follows that x_0 is not contained in the interval $k_\varrho((C^*_{\mu-1}, C^*_\mu))$.

Case 1.1: $k_\varrho((C^*_{\mu-1}, C^*_\mu)) \subseteq (-\infty, x_0)$. Then $\xi_{\varrho-1} \le k_\varrho(C) < x_0$. Lemma 3.1 gives the decomposition in the following two cases.

Case 1.1.1: If $h\restriction[\xi_{\varrho-1}, \xi_\varrho]$ is increasing then

$$\begin{aligned} R^\dagger_\varrho = \#\{(x,y) \in \mathbb{Z}^2 \mid A_1^{1/d_1}\xi_{\varrho-1} < x \le A_1^{1/d_1} k_\varrho(C),& \\ 0 < y \le A_1^{1/d_1} y_1(\tau_1, \tau_1 x)\}& \\ + \#\{(x,y) \in \mathbb{Z}^2 \mid A_1^{1/d_1} k_\varrho(C) < x \le A_1^{1/d_1} \min\{\xi_\varrho, x_0\},& \\ 0 < y \le A_2^{1/d_2} y_2(\tau_2, \tau_2 x)\}& \\ + O(A_1^{46/(73 d_1)}).& \end{aligned}$$

Case 1.1.2: If $h\restriction[\xi_{\varrho-1}, \xi_\varrho]$ is decreasing then

$$\begin{aligned} R^\dagger_\varrho = \#\{(x,y) \in \mathbb{Z}^2 \mid A_1^{1/d_1}\xi_{\varrho-1} < x \le A_1^{1/d_1} k_\varrho(C),& \\ 0 < y \le A_2^{1/d_2} y_2(\tau_2, \tau_2 x)\}& \\ + \#\{(x,y) \in \mathbb{Z}^2 \mid A_1^{1/d_1} k_\varrho(C) < x \le A_1^{1/d_1} \min\{\xi_\varrho, x_0\},& \\ 0 < y \le A_1^{1/d_1} y_1(\tau_1, \tau_1 x)\}& \\ + O(A_1^{46/(73 d_1)}).& \end{aligned}$$

Case 1.2: $k_\varrho((C^*_{\mu-1}, C^*_\mu)) \subseteq (x_0, \infty)$. Then $\xi_{\varrho-1} < x_0 < k_\varrho(C) \le \xi_\varrho$. Lemma 3.1 gives the decomposition in the following two cases.

Case 1.2.1: If $h\restriction[\xi_{\varrho-1}, \xi_\varrho]$ is increasing then

$$\begin{aligned} R^\dagger_\varrho = \#\{(x,y) \in \mathbb{Z}^2 \mid A_1^{1/d_1}\xi_{\varrho-1} < x \le A_1^{1/d_1} x_0,\ 0 < y \le A_1^{1/d_1} y_1(\tau_1, \tau_1 x)\}& \\ + O(A_1^{46/(73 d_1)}).& \end{aligned}$$

Case 1.2.2: If $h\restriction[\xi_{\varrho-1}, \xi_\varrho]$ is decreasing then

$$\begin{aligned} R^\dagger_\varrho = \#\{(x,y) \in \mathbb{Z}^2 \mid A_1^{1/d_1}\xi_{\varrho-1} < x \le A_1^{1/d_1} x_0,\ 0 < y \le A_2^{1/d_2} y_2(\tau_2, \tau_2 x)\}& \\ + O(A_1^{46/(73 d_1)}).& \end{aligned}$$

Case 2: $[C^*_{\mu-1}, C^*_\mu] \subseteq [a_\varrho, a'_\varrho]$ *or* $[C^*_{\mu-1}, C^*_\mu] \subseteq [b'_\varrho, b_\varrho]$. Then $C \in [a_\varrho, b_\varrho]$ and $k_\varrho(C) > x_0$. For $R^\dagger_\varrho$ the same holds as in Case 1.2.

Case 3: $C^*_\mu \le a_\varrho$. Then by Lemma 3.1

$$\begin{aligned} R^\dagger_\varrho = \#\{(x,y) \in \mathbb{Z}^2 \mid A_1^{1/d_1}\xi_{\varrho-1} < x \le A_1^{1/d_1} \min\{\xi_\varrho, x_0\},& \\ 0 < y \le A_2^{1/d_2} y_2(\tau_2, \tau_2 x)\}.& \end{aligned}$$

CASE 4: $C^*_{\mu-1} \geq b_\varrho$. Then by Lemma 3.1

$$R^\dagger_\varrho = \#\{(x,y) \in \mathbb{Z}^2 \mid A_1^{1/d_1}\xi_{\varrho-1} < x \leq A_1^{1/d_1}\min\{\xi_\varrho, x_0\}, \\ 0 < y \leq A_1^{1/d_1} y_1(\tau_1, \tau_1 x)\}.$$

In the following the proof of (7.7) is given only in Cases 1.1.1 and 1.1.2. The other cases are similar but somewhat easier.

In Case 1.1.1 use ($C^\dagger$4), (2.7) and ($Z^\dagger$3) and choose $1 \leq \nu_j \leq n$ with the properties

$$k_\varrho((C^*_{\mu-1}, C^*_\mu)) \subseteq (\zeta_{\nu_0-1}, \zeta_{\nu_0}) \subseteq (\xi_{\varrho-1}, \min\{\xi_\varrho, x_0\}), \quad \xi_{\varrho-1} = \zeta_{\nu_1-1},$$
$$\min\{\xi_\varrho, x_0\} = \zeta_{\nu_2}, \quad \nu_1 \leq \nu_0 \leq \nu_2, \quad \widetilde{k}_\varrho((C^*_{\mu-1}, C^*_\mu)) \subseteq (\zeta_{\nu_3-1}, \zeta_{\nu_3}),$$
$$C^{-1/d_2}\zeta_{\nu_2} \in (\zeta_{\nu_4-1}, \zeta_{\nu_4}) \subseteq [0, x_2] \quad \text{for } C \in (C^*_{\mu-1}, C^*_\mu), \quad \nu_3 \leq \nu_4.$$

Then

$$\begin{aligned} R^\dagger_\varrho = \sum_{\nu=\nu_1}^{\nu_0-1} & \#\{(x,y) \in \mathbb{Z}^2 \mid A_1^{1/d_1}\zeta_{\nu-1} < x \leq A_1^{1/d_1}\zeta_\nu, \\ & \qquad 0 < y \leq A_1^{1/d_1} y_1(\tau_1, \tau_1 x)\} \\ & + \#\{(x,y) \in \mathbb{Z}^2 \mid A_1^{1/d_1}\zeta_{\nu_0-1} < x \leq A_1^{1/d_1} k_\varrho(C), \\ & \qquad 0 < y \leq A_1^{1/d_1} y_1(\tau_1, \tau_1 x)\} \\ & - \#\{(x,y) \in \mathbb{Z}^2 \mid A_2^{1/d_2}\zeta_{\nu_3-1} < x \leq A_2^{1/d_2} C^{-1/d_2} k_\varrho(C), \\ & \qquad 0 < y \leq A_2^{1/d_2} y_2(\tau_2, \tau_2 x)\} \\ & + \sum_{\nu=\nu_3}^{\nu_4-1} \#\{(x,y) \in \mathbb{Z}^2 \mid A_2^{1/d_2}\zeta_{\nu-1} < x \leq A_2^{1/d_2}\zeta_\nu, \\ & \qquad 0 < y \leq A_2^{1/d_2} y_2(\tau_2, \tau_2 x)\} \\ & + \#\{(x,y) \in \mathbb{Z}^2 \mid A_2^{1/d_2}\zeta_{\nu_4-1} < x \leq A_2^{1/d_2} C^{-1/d_2}\zeta_{\nu_2}, \\ & \qquad 0 < y \leq A_2^{1/d_2} y_2(\tau_2, \tau_2 x)\} \\ & + O(A_1^{46/(73 d_1)}). \end{aligned}$$

It follows from ($C^\dagger$2) and ($C^\dagger$3) that the endpoints of the respective intervals of x when renormalized with the corresponding A_j^{-1/d_j} have a distance $\geq \tau_j^\kappa$ from all ζ_ν with $0 \leq \nu \leq n$, or are equal to one of these ζ_ν. Furthermore $\zeta_{\nu_0} < \widetilde{\xi}_1$, $\zeta_{\nu_4} < \widetilde{\xi}_2$. Application of Lemmas 7.1 and 7.2 to each of the summands gives

$$\begin{aligned} \text{(7.8)} \qquad R^\dagger_\varrho = A_1^{2/d_1} \int_{\zeta_{\nu_1-1}}^{k_\varrho(C)} y_1(\tau_1, x)\, dx + A_2^{2/d_2} \int_{C^{-1/d_2} k_\varrho(C)}^{C^{-1/d_2}\zeta_{\nu_2}} y_2(\tau_2, x)\, dx \\ + \tfrac{1}{2} A_1^{1/d_1}\zeta_{\nu_1-1} - \tfrac{1}{2} A_2^{1/d_2} C^{-1/d_2}\zeta_{\nu_2} \\ + T_1^{\dagger(\mu,\varrho)}(A_1) + T_2^{\dagger(\mu,\varrho)}(A_2) \end{aligned}$$

$$
\begin{aligned}
&+\psi(A_1^{1/d_1}\zeta_{\nu_1-1})A_1^{1/d_1}y_1(\tau_1,\zeta_{\nu_1-1})\\
&-\psi(A_1^{1/d_1}k_\varrho(C))A_1^{1/d_1}y_1(\tau_1,k_\varrho(C))\\
&+\psi(A_2^{1/d_2}C^{-1/d_2}k_\varrho(C))A_2^{1/d_2}y_2(\tau_2,C^{-1/d_2}k_\varrho(C))\\
&+U_1^{\dagger(\mu,\varrho)}(A_1)+U_2^{\dagger(\mu,\varrho)}(A_2)\\
&-\psi(A_2^{1/d_2}C^{-1/d_2}\zeta_{\nu_2})A_2^{1/d_2}y_2(\tau_2,C^{-1/d_2}\zeta_{\nu_2})\\
&+O(A_1^{46/(73d_1)}(\log A_1)^{315/146}).
\end{aligned}
$$

Lemma 2.2, (3.3) and (3.5) give for the first integral

$$
\int\limits_{\xi_{\varrho-1}}^{k_\varrho(C)-\tau_1^{50/73}} \tau_1 f_{A_1,A_2}(A_1^{1/d_1}x)\,dx
+\int\limits_{k_\varrho(C)-\tau_1^{50/73}}^{k_\varrho(C)} (\tau_1 f_{A_1,A_2}(A_1^{1/d_1}x)+O(\tau_1^{50/73}))\,dx,
$$

for the second integral

$$
C^{-1/d_2}\int\limits_{k_\varrho(C)}^{k_\varrho(C)+\tau_1^{50/73}} (\tau_2 f_{A_1,A_2}(A_1^{1/d_1}x)+O(\tau_2^{50/73}))\,dx
+C^{-1/d_2}\int\limits_{k_\varrho(C)+\tau_1^{50/73}}^{\zeta_{\nu_2}} \tau_2 f_{A_1,A_2}(A_1^{1/d_1}x)\,dx
$$

and

$$
\begin{aligned}
y_1(\tau_1,k_\varrho(C))&=\tau_1 f_{A_1,A_2}(A_1^{1/d_1}k_\varrho(C))+O(\tau_1^{50/73}),\\
y_2(\tau_2,C^{-1/d_2}k_\varrho(C))&=\tau_2 f_{A_1,A_2}(A_1^{1/d_1}k_\varrho(C))+O(\tau_2^{50/73}).
\end{aligned}
$$

Substituting into (7.8) gives (7.7).

In Case 1.1.2 choose $1\le\nu_j\le n$ with the properties

$$
\begin{gathered}
k_\varrho((C_{\mu-1}^*,C_\mu^*))\subseteq(\zeta_{\nu_0-1},\zeta_{\nu_0})\subseteq(\xi_{\varrho-1},\min\{\xi_\varrho,x_0\}),\\
\xi_{\varrho-1}=\zeta_{\nu_1-1},\quad \min\{\xi_\varrho,x_0\}=\zeta_{\nu_2},\quad \nu_1\le\nu_0\le\nu_2,\quad \zeta_{\nu_2}<\widetilde{\xi}_1,\\
\widetilde{k}_\varrho((C_{\mu-1}^*,C_\mu^*))\subseteq(\zeta_{\nu_3-1},\zeta_{\nu_3})\subseteq[0,x_2],\quad \zeta_{\nu_3}<\widetilde{\xi}_2,\\
C^{-1/d_2}\zeta_{\nu_1-1}\in(\zeta_{\nu_4-1},\zeta_{\nu_4})\text{ or }\equiv\zeta_{\nu_4-1}\ (\text{if }\varrho=1)\\
\text{for } C\in(C_{\mu-1}^*,C_\mu^*),\ \nu_4\le\nu_3.
\end{gathered}
$$

Then

$$
\begin{aligned}
R_\varrho^\dagger=-\#\{(x,y)\in\mathbb{Z}^2\mid A_2^{1/d_2}\zeta_{\nu_4-1}<x\le A_2^{1/d_2}C^{-1/d_2}\zeta_{\nu_1-1},\\
0<y\le A_2^{1/d_2}y_2(\tau_2,\tau_2x)\}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\nu=\nu_4}^{\nu_3-1} \#\{(x,y)\in\mathbb{Z}^2 \mid A_2^{1/d_2}\zeta_{\nu-1} < x \le A_2^{1/d_2}\zeta_\nu,\\
&\qquad\qquad\qquad\qquad\qquad\qquad 0 < y \le A_2^{1/d_2} y_2(\tau_2,\tau_2 x)\}\\
&+\#\{(x,y)\in\mathbb{Z}^2 \mid A_2^{1/d_2}\zeta_{\nu_3-1} < x \le A_2^{1/d_2} C^{-1/d_2} k_\varrho(C),\\
&\qquad\qquad\qquad\qquad\qquad\qquad 0 < y \le A_2^{1/d_2} y_2(\tau_2,\tau_2 x)\}\\
&-\#\{(x,y)\in\mathbb{Z}^2 \mid A_1^{1/d_1}\zeta_{\nu_0-1} < x \le A_1^{1/d_1} k_\varrho(C),\\
&\qquad\qquad\qquad\qquad\qquad\qquad 0 < y \le A_1^{1/d_1} y_1(\tau_1,\tau_1 x)\}\\
&+\sum_{\nu=\nu_0}^{\nu_2} \#\{(x,y)\in\mathbb{Z}^2 \mid A_1^{1/d_1}\zeta_{\nu-1} < x \le A_1^{1/d_1}\zeta_\nu,\\
&\qquad\qquad\qquad\qquad\qquad\qquad 0 < y \le A_1^{1/d_1} y_1(\tau_1,\tau_1 x)\}\\
&+O(A_1^{46/(73d_1)}).
\end{aligned}
$$

The remainder of the proof is as above. ∎

Summing over $1 \le \varrho \le r$ gives

Corollary 7.5. *Let $1 \le \mu \le m$. Then for $A_1, A_2 \ge K$, $C := A_2 A_1^{-d_2/d_1} \in (C^*_{\mu-1} + A_1^{-\varepsilon}, C^*_\mu - A_1^{-\varepsilon})$ we have*

$$
\begin{aligned}
R^\dagger(A_1,A_2) = {} & \int_0^{A_1^{1/d_1}x_0} f_{A_1,A_2}(x)\,dx - \tfrac12 A_1^{1/d_1}x_0 + T_1^{\dagger(\mu)}(A_1) + T_2^{\dagger(\mu)}(A_2)\\
& + U_1^{\dagger(\mu)}(A_1) + U_2^{\dagger(\mu)}(A_2) - \tfrac12 f_{A_1,A_2}(0)\\
& - \psi(A_1^{1/d_1}x_0) f_{A_1,A_2}(A_1^{1/d_1}x_0)\\
& + O(A_1^{46/(73d_1)}(\log A_1)^{315/146}).
\end{aligned}
$$

Now $R^\#$ is handled similarly to $R^\dagger$ but with the roles of x and y interchanged. The analogue of $h : [0,\widetilde{\xi}_1] \to \mathbb{R}^+$ is $l : [0,\widetilde{\eta}_1] \to \mathbb{R}^+$ with $l(y) := \widetilde{g}_2(\widetilde{x}_1(y), y)$. Let $0 = \eta_0 < \ldots < \eta_t = \widetilde{\eta}_1$ be the decomposition which is analogous to $0 = \xi_0 < \ldots < \xi_r = \widetilde{\xi}_1$ and $m_\tau := (l\restriction[\eta_{\tau-1},\eta_\tau])^{-1} : [c_\tau, d_\tau] \to [\eta_{\tau-1},\eta_\tau]$ for $1 \le \tau \le t$.

The situations for $R^\dagger$ and $R^\#$ are not completely symmetric because $y_0 = y_{0,A_1,A_2}$ depends on A_1 and A_2 whereas x_0 is constant. Therefore some additional lemmas are needed.

Lemma 7.6. *For $\overline{y}, \kappa > 0$ there is a decomposition $C_1 = C_0^* < \ldots < C_m^* = C_2$ and constants $\varepsilon, K > 0$ so that for $1 \le \mu \le m$ the following holds*:

(1) *Uniformly for all $A_1, A_2 \ge K$ with $C := A_2 A_1^{-d_2/d_1} \in (C^*_{\mu-1} + A_1^{-\varepsilon}, C^*_\mu - A_1^{-\varepsilon})$, one of the two cases $y_{0,A_1,A_2} \le \overline{y}$ or $y_{0,A_1,A_2} > \overline{y}$ is valid.*

(2) *Uniformly for all* $A_1, A_2 \geq K$ *with* $C := A_2 A_1^{-d_2/d_1} \in (C^*_{\mu-1} + A_1^{-\varepsilon}, C^*_\mu - A_1^{-\varepsilon})$ *one of the two estimates* $|y_{0,A_1,A_2} - \overline{y}| \geq \tau_1^\kappa$ *or* $|y_{0,A_1,A_2} - \overline{y}| \ll \tau_1$ *is valid.*

P r o o f. Choose $\widetilde{g}_2(x_0, \overline{y})$ as point of decomposition if it lies in $[C_1, C_2]$.

CASE 1: $C^*_\mu \leq \widetilde{g}_2(x_0, \overline{y})$. Then $C \leq \widetilde{g}_2(x_0, \overline{y}) - A_1^{-\varepsilon}$ and consequently

$$\begin{aligned}\widetilde{g}_2(C^{-1/d_2}x_0, C^{-1/d_2}\overline{y}) &\geq 1 + C^{-1}A_1^{-\varepsilon} \\ &\geq \widetilde{g}_2(C^{-1/d_2}x_0, \widetilde{y}_2(C^{-1/d_2}x_0)) + C_2^{-1}A_1^{-\varepsilon}.\end{aligned}$$

Therefore $C^{-1/d_2}\overline{y} > \widetilde{y}_2(C^{-1/d_2}x_0)$ and

$$\begin{aligned}&|C^{-1/d_2}\overline{y} - \widetilde{y}_2(C^{-1/d_2}x_0)| \\ &\quad \gg |\widetilde{g}_2(C^{-1/d_2}x_0, C^{-1/d_2}\overline{y}) - \widetilde{g}_2(C^{-1/d_2}x_0, \widetilde{y}_2(C^{-1/d_2}x_0))| \gg A_1^{-\varepsilon}.\end{aligned}$$

It follows that $\overline{y} \geq C^{1/d_2}\widetilde{y}_2(C^{-1/d_2}x_0) + K_{30}A_1^{-\varepsilon}$ with some constant $K_{30} > 0$. From the definition of y_{0,A_1,A_2} it follows that

$$y_{0,A_1,A_2} \leq C^{1/d_2} y_2(\tau_2, C^{-1/d_2}x_0) \leq C^{1/d_2}\widetilde{y}_2(C^{-1/d_2}x_0) \leq \overline{y} - K_{30}A_1^{-\varepsilon}.$$

Choosing $0 < \varepsilon < \kappa/d_1$ it follows that in (1) and (2) always the first cases are valid.

CASE 2: $C^*_{\mu-1} \geq \widetilde{g}_2(x_0, \overline{y})$. The same reasoning as above gives $C^{1/d_2}\widetilde{y}_2(C^{-1/d_2}x_0) \geq \overline{y} + K_{30}A_1^{-\varepsilon}$ with some constant $K_{30} > 0$.

CASE 2.1: $\widetilde{y}_1(x_0) > \overline{y}$. Then with some constant $K_{31} > 0$,

$$\begin{aligned}y_{0,A_1,A_2} &= \min\{y_1(\tau_1, x_0), C^{1/d_2} y_2(\tau_2, C^{-1/d_2}x_0)\} \\ &\geq \min\{\widetilde{y}_1(x_0) - K_{31}\tau_1, C^{1/d_2}\widetilde{y}_2(C^{-1/d_2}x_0) - K_{31}\tau_1\} \\ &\geq \overline{y} + K_{30}A_1^{-\varepsilon}/2\end{aligned}$$

and so if $0 < \varepsilon < \min\{1/d_1, \kappa/d_1\}$ is chosen then in (1) always the second case is valid and in (2) always the first case.

CASE 2.2: $\widetilde{y}_1(x_0) = \overline{y}$. Then with some constant $K_{32} > 0$ we have

$$\begin{aligned}C^{1/d_2} y_2(\tau_2, C^{-1/d_2}x_0) &\geq C^{1/d_2}\widetilde{y}_2(C^{-1/d_2}x_0) - K_{32}\tau_1 \\ &\geq \widetilde{y}_1(x_0) - K_{32}\tau_1 + K_{30}A_1^{-\varepsilon} \geq y_1(\tau_1, x_0)\end{aligned}$$

and so $y_0 = y_1(\tau_1, x_0)$. Consequently, $y_0 \leq \widetilde{y}_1(x_0) = \overline{y}$ and $|y_{0,A_1,A_2} - \overline{y}| = |y_1(\tau_1, x_0) - \widetilde{y}_1(x_0)| \ll \tau_1$. So in (1) always the first case and in (2) always the second case is valid.

CASE 2.3: $\widetilde{y}_1(x_0) < \overline{y}$. Then $y_{0,A_1,A_2} \leq y_1(\tau_1, x_0) \leq \widetilde{y}_1(x_0) < \overline{y}$ and $|y_{0,A_1,A_2} - \overline{y}| \geq \overline{y} - \widetilde{y}_1(x_0) = \text{const.} > 0$. So in (1) and (2) always the first cases are valid. ∎

LEMMA 7.7. *For $\overline{y}, \kappa > 0$ there is a decomposition $C_1 = C_0^* < \ldots < C_m^* = C_2$ and $K, \varepsilon > 0$ so that for $1 \le \mu \le m$ and uniformly for $A_1, A_2 \ge K$ with $C := A_2 A_1^{-d_2/d_1} \in (C_{\mu-1}^* + A_1^{-\varepsilon}, C_\mu^* - A_1^{-\varepsilon})$ one of the two cases $C^{-1/d_2} y_{0,A_1,A_2} \ge \overline{y} + \tau_1^\kappa$ or $C^{-1/d_2} y_{0,A_1,A_2} \le \overline{y} - \tau_1^\kappa$ is valid.*

Proof. CASE 1: $\overline{y} \ge \widetilde{\eta}_2$. Then

$$\begin{aligned}\overline{y} - C^{-1/d_2} y_{0,A_1,A_2} &\ge \widetilde{\eta}_2 - y_2(\tau_2, C^{-1/d_2} x_0) \ge \widetilde{\eta}_2 - \widetilde{y}_2(C^{-1/d_2} x_0) \\ &\ge \widetilde{\eta}_2 - \widetilde{y}_2(C_2^{-1/d_2} x_0) = \text{const.} > 0\end{aligned}$$

and therefore always the second case is valid.

CASE 2: $\overline{y} < \widetilde{\eta}_2$. Choose $(x_0/\widetilde{x}_2(\overline{y}))^{d_2}$ and $(\widetilde{y}_1(x_0)/\overline{y})^{d_2}$ as division points if they lie in $[C_1, C_2]$.

CASE 2.1: $C_\mu^* \le (x_0/\widetilde{x}_2(\overline{y}))^{d_2}$. Then $C \le (x_0/\widetilde{x}_2(\overline{y}))^{d_2} - A_1^{-\varepsilon}$ and by Taylor's theorem $C^{-1/d_2} \ge \widetilde{x}_2(\overline{y})/x_0 + K_{40} A_1^{-\varepsilon}$ with some constant $K_{40} > 0$. Applying Taylor's theorem again gives

$$\widetilde{y}_2(C^{-1/d_2} x_0) \le \widetilde{y}_2(\widetilde{x}_2(\overline{y}) + x_0 K_{40} A_1^{-\varepsilon}) \le \overline{y} - K_{41} A_1^{-\varepsilon}$$

with some constant $K_{41} > 0$ and so

$$\begin{aligned}\overline{y} - C^{-1/d_2} y_{0,A_1,A_2} &\ge \overline{y} - y_2(\tau_2, C^{-1/d_2} x_0) \ge \overline{y} - \widetilde{y}_2(C^{-1/d_2} x_0) \\ &\ge K_{41} A_1^{-\varepsilon} \ge \tau_1^\kappa\end{aligned}$$

if $0 < \varepsilon < \kappa/d_1$. Therefore always the second case is valid.

CASE 2.2: $C_{\mu-1}^* \ge (x_0/\widetilde{x}_2(\overline{y}))^{d_2}$.

CASE 2.2.1: $C_{\mu-1}^* \ge (\widetilde{y}_1(x_0)/\overline{y})^{d_2}$. Then $C \ge (\widetilde{y}_1(x_0)/\overline{y})^{d_2} + A_1^{-\varepsilon}$ and by Taylor's theorem $C^{1/d_2} \ge \widetilde{y}_1(x_0)/\overline{y} + K_{40} A_1^{-\varepsilon}$. Consequently,

$$\begin{aligned}\overline{y} - C^{-1/d_2} y_{0,A_1,A_2} &\ge \overline{y} - C^{-1/d_2} y_1(\tau_1, x_0) \ge \overline{y} - C^{-1/d_2} \widetilde{y}_1(x_0) \\ &\ge \overline{y} C^{-1/d_2} K_{40} A_1^{-\varepsilon} \ge \tau_1^\kappa\end{aligned}$$

if $0 < \varepsilon < \kappa/d_1$. Therefore always the second case is valid.

CASE 2.2.2: $C_\mu^* \le (\widetilde{y}_1(x_0)/\overline{y})^{d_2}$. Then $(x_0/\widetilde{x}_2(\overline{y}))^{d_2} + A_1^{-\varepsilon} \le C \le (\widetilde{y}_1(x_0)/\overline{y})^{d_2} - A_1^{-\varepsilon}$ and by Taylor's theorem

$$C^{-1/d_2} \le \frac{\widetilde{x}_2(\overline{y})}{x_0} - K_{40} A_1^{-\varepsilon}, \qquad C^{1/d_2} \le \frac{\widetilde{y}_1(x_0)}{\overline{y}} - K_{40} A_1^{-\varepsilon}.$$

Applying Taylor's theorem again gives

$$\widetilde{y}_2(C^{-1/d_2} x_0) \ge \widetilde{y}_2(\widetilde{x}_2(\overline{y}) - K_{40} x_0 A_1^{-\varepsilon}) \ge \overline{y} + K_{41} A_1^{-\varepsilon}.$$

With further constants $K_{42}, K_{43} > 0$ it follows that

$$\begin{aligned}C^{-1/d_2} y_{0,A_1,A_2} &= \min\{C^{-1/d_2} y_1(\tau_1, x_0), y_2(\tau_2, C^{-1/d_2} x_0)\} \\ &\ge \min\{C^{-1/d_2}(\widetilde{y}_1(x_0) - K_{42}\tau_1), \widetilde{y}_2(C^{-1/d_2} x_0) - K_{42}\tau_2\}\end{aligned}$$

$$\begin{aligned}&\geq \min\{\overline{y}+\overline{y}C^{-1/d_2}K_{40}A_1^{-\varepsilon}-C^{-1/d_2}K_{42}\tau_1,\\ &\qquad\qquad\qquad\qquad \overline{y}+K_{41}A_1^{-\varepsilon}-K_{42}\tau_2\}\\ &\geq \overline{y}+K_{43}A_1^{-\varepsilon}\geq \overline{y}+\tau_1^{\kappa}\end{aligned}$$

if $0<\varepsilon<\min\{1/d_1,\kappa/d_1\}$ and so always the first case is valid. ∎

LEMMA 7.8. *There is a decomposition* $C_1=C_0^*<\ldots<C_m^*=C_2$ *and constants* $\varepsilon, K>0$ *with the property*: *For* $1\leq\mu\leq m$, $1\leq\tau\leq t$ *we have*

$$(C_{\mu-1}^*,C_\mu^*)\cap[c_\tau,d_\tau]=\emptyset \quad \textit{or} \quad [C_{\mu-1}^*,C_\mu^*]\subseteq[c_\tau,d_\tau].$$

In the latter case uniformly for $A_1,A_2\geq K$ *with* $C:=A_2A_1^{-d_2/d_1}\in(C_{\mu-1}^*+A_1^{-\varepsilon},C_\mu^*-A_1^{-\varepsilon})$, *one of the two cases* $m_\tau(C)\leq y_{0,A_1,A_2}$ *or* $m_\tau(C)\geq y_{0,A_1,A_2}$ *is valid.*

Proof. As decomposition points the following points are chosen if they lie in $[C_1,C_2]$:

(1) $h(x_0)$ and c_τ,d_τ for $1\leq\tau\leq t$.

(2) The decomposition points arising from Lemma 7.6 applied to $\overline{y}=\eta_\tau$, $1\leq\tau\leq t$.

Let $1\leq\mu\leq m$, $1\leq\tau\leq t$. Then $(C_{\mu-1}^*,C_\mu^*)\cap[c_\tau,d_\tau]=\emptyset$ or $[C_{\mu-1}^*,C_\mu^*]\subseteq[c_\tau,d_\tau]$. Assume the latter case. Then uniformly for $A_1,A_2\geq K$ with $C:=A_2A_1^{-d_2/d_1}\in(C_{\mu-1}^*+A_1^{-\varepsilon},C_\mu^*-A_1^{-\varepsilon})$ one of the three cases $y_{0,A_1,A_2}\leq\eta_{\tau-1}$, $y_{0,A_1,A_2}\in(\eta_{\tau-1},\eta_\tau]$ or $y_{0,A_1,A_2}>\eta_\tau$ is valid. In the first case $y_{0,A_1,A_2}\leq m_\tau(C)$. In the last case $y_{0,A_1,A_2}>m_\tau(C)$. For the remainder of the proof assume that the second case is valid. From the definition of m_τ it follows that

$$m_\tau(C)\leq y_{0,A_1,A_2}\Leftrightarrow\begin{cases}C\leq l(y_{0,A_1,A_2}), & l\restriction[\eta_{\tau-1},\eta_\tau]\text{ increasing},\\ C\geq l(y_{0,A_1,A_2}), & l\restriction[\eta_{\tau-1},\eta_\tau]\text{ decreasing}.\end{cases}\tag{7.9}$$

From (2.10) it follows that

$$|\widetilde{y}_1(x_0)-C^{1/d_2}\widetilde{y}_2(C^{-1/d_2}x_0)|\gg|h(x_0)-C|\geq A_1^{-\varepsilon}.\tag{7.10}$$

CASE 1: $C_{\mu-1}^*\geq h(x_0)$. Then $\widetilde{y}_1(x_0)<C^{1/d_2}\widetilde{y}_2(C^{-1/d_2}x_0)$ by (2.16), and with (7.10) this gives $\widetilde{y}_1(x_0)\leq C^{1/d_2}\widetilde{y}_2(C^{-1/d_2}x_0)-K_{50}A_1^{-\varepsilon}$ with some constant $K_{50}>0$. If $0<\varepsilon<1/d_1$ this gives

$$\begin{aligned}y_1(\tau_1,x_0)&=\widetilde{y}_1(x_0)+O(\tau_1)\\ &\leq C^{1/d_2}y_2(\tau_2,C^{-1/d_2}x_0)=C^{1/d_2}\widetilde{y}_2(C^{-1/d_2}x_0)+O(\tau_1)\end{aligned}$$

and consequently $y_0=y_1(\tau_1,x_0)=\widetilde{y}_1(x_0)+O(\tau_1)$. Therefore

$$\begin{aligned}l(y_0) &= \widetilde{g}_2(\widetilde{x}_1(y_0), y_0) = \widetilde{g}_2(x_0 + O(\tau_1), \widetilde{y}_1(x_0) + O(\tau_1)) \\ &= \widetilde{g}_2(x_0, \widetilde{y}_1(x_0)) + O(\tau_1) \\ &= h(x_0) + O(\tau_1) \le C^*_{\mu-1} + O(\tau_1) \\ &\le C - A_1^{-\varepsilon} + O(\tau_1) < C.\end{aligned}$$

CASE 2: $C^*_\mu \le h(x_0)$. From (2.16) and (7.10) it follows that

$$\widetilde{y}_1(x_0) \ge C^{1/d_2}\widetilde{y}_2(C^{-1/d_2}x_0) + K_{50}A_1^{-\varepsilon}. \tag{7.11}$$

Just as above,

$$y_{0,A_1,A_2} = C^{1/d_2}y_2(\tau_2, C^{-1/d_2}x_0) = C^{1/d_2}\widetilde{y}_2(C^{-1/d_2}x_0) + O(\tau_1).$$

Taylor's theorem gives $\widetilde{x}_1(\widetilde{y}_1(x_0) - K_{50}A_1^{-\varepsilon}) \ge x_0 + K_{51}A_1^{-\varepsilon}$ with some constant $K_{51} > 0$. From (7.11) it follows that

$$\begin{aligned}l(y_0) &= \widetilde{g}_2(\widetilde{x}_1(y_0), y_0) \\ &= \widetilde{g}_2(\widetilde{x}_1(C^{1/d_2}\widetilde{y}_2(C^{-1/d_2}x_0)) + O(\tau_1), C^{1/d_2}\widetilde{y}_2(C^{-1/d_2}x_0) + O(\tau_1)) \\ &\ge \widetilde{g}_2(\widetilde{x}_1(\widetilde{y}_1(x_0) - K_{50}A_1^{-\varepsilon}), C^{1/d_2}\widetilde{y}_2(C^{-1/d_2}x_0)) + O(\tau_1) \\ &\ge \widetilde{g}_2(x_0 + K_{51}A_1^{-\varepsilon}, C^{1/d_2}\widetilde{y}_2(C^{-1/d_2}x_0)) + O(\tau_1) \\ &\ge \widetilde{g}_2(x_0, C^{1/d_2}\widetilde{y}_2(C^{-1/d_2}x_0)) + a^{(2)}_{d_2 0}(x_0 + K_{51}A_1^{-\varepsilon})^{d_2} - a^{(2)}_{d_2 0}x_0^{d_2} \\ &\quad + O(\tau_1) \\ &\ge C + a^{(2)}_{d_2 0}d_2 x_0^{d_2-1}K_{51}A_1^{-\varepsilon} + O(\tau_1) > C.\end{aligned}$$

(7.9) shows in Cases 1 and 2 that uniformly throughout the given range of (A_1, A_2) one of the two cases $m_\tau(C) \le y_0$ or $m_\tau(C) \ge y_0$ is valid. ∎

Choose $Z \subseteq \mathbb{R}_0^+$, $|Z| < \infty$ with the properties:

($Z^\#1$) The zeros of $\widetilde{x}_j''\widetilde{x}_j'''$ in $[0, \widetilde{\eta}_j]$ are contained in Z for $j = 1, 2$.

($Z^\#2$) The zeros of $\widetilde{y}_j''\widetilde{y}_j'''$ in $[0, \widetilde{\xi}_j]$ are contained in $\widetilde{x}_j(Z)$ for $j = 1, 2$.

($Z^\#3$) $\eta_0, \ldots, \eta_t, y_1, y_2, h(x_0) \in Z$.

($Z^\#4$) If $\widetilde{x}_j'(0) = 0 \ne \widetilde{x}_j''(0)$ for $j = 1$ or $j = 2$, then the value of $\overline{\xi}$ which comes from the application of Proposition 6.1 is contained in $\widetilde{x}_j(Z)$.

Choose $0 = \zeta_0 < \ldots < \zeta_n$ with the properties:

($Z^\#5$) $Z \subseteq \{\zeta_0, \ldots, \zeta_n\}$.

($Z^\#6$) For each $\zeta, \zeta' \in Z$ with $\zeta < \zeta'$ there is some $1 \le \nu \le n$ with $\zeta < \zeta_\nu < \zeta'$.

For $1 \le \tau \le t$ define $[c'_\tau, d'_\tau] := m_\tau^{-1}((-\infty, y_1])$ and $\widetilde{m}_\tau(C) := C^{-1/d_2}m_\tau(C)$ on $[c'_\tau, d'_\tau]$. Now choose a decomposition $C_1 = C^*_0 < \ldots < C^*_m = C_2$ and constant $\varepsilon > 0$ with the properties:

(C$^{\#}$1) If $c'_\tau \in [C_1, C_2]$ for some $1 \le \tau \le t$ then $c'_\tau \in \{C_0^*, \dots, C_m^*\}$. The same holds for c_τ, d'_τ and d_τ.

(C$^{\#}$2) For $1 \le \tau \le t$, $1 \le \mu \le m$, $0 \le \nu \le n$ with $[C^*_{\mu-1}, C^*_\mu] \subseteq [c'_\tau, d'_\tau]$: For $A_1, A_2 \ge K$ with $C := A_2 A_1^{-d_2/d_1} \in (C^*_{\mu-1} + A_1^{-\varepsilon}, C^*_\mu - A_1^{-\varepsilon})$ we have

$$|m_\tau(C) - \zeta_\nu| \ge \tau_1^\kappa, \qquad |\widetilde{m}_\tau(C) - \zeta_\nu| \ge \tau_2^\kappa.$$

(C$^{\#}$3) For $1 \le \mu \le m$, $1 \le \nu' \le n$, $0 \le \nu \le n$: For $A_1, A_2 \ge K$ with $C := A_2 A_1^{-d_2/d_1} \in (C^*_{\mu-1} + A_1^{-\varepsilon}, C^*_\mu - A_1^{-\varepsilon})$ we have

$$|C^{-1/d_2}\zeta_{\nu'} - \zeta_\nu| \ge \tau_2^\kappa.$$

(C$^{\#}$4) For $1 \le \tau \le t$, $1 \le \mu \le m$, $1 \le \nu' \le n$, $0 \le \nu \le n$ we have

$$\zeta_\nu \notin m_\tau((C^*_{\mu-1}, C^*_\mu) \cap [c'_\tau, d'_\tau]), \qquad \zeta_\nu \notin \widetilde{m}_\tau((C^*_{\mu-1}, C^*_\mu) \cap [c'_\tau, d'_\tau]),$$
$$\zeta_\nu \ne C^{-1/d_2}\zeta_{\nu'} \quad \text{for } C \in (C^*_{\mu-1}, C^*_\mu).$$

(C$^{\#}$5) All the decomposition points which arise from the application of Lemmas 7.6 and 7.7 to $\overline{y} = \zeta_1, \dots, \zeta_n$ and from Lemma 7.8 are contained in $\{C_0^*, \dots, C_m^*\}$.

The following lemma is the analogue of Lemma 7.4.

Lemma 7.9. *Let $1 \le \mu \le m$, $1 \le \tau \le t$. Then uniformly for $A_1, A_2 \ge K$ with $C := A_2 A_1^{-d_2/d_1} \in (C^*_{\mu-1} + A_1^{-\varepsilon}, C^*_\mu - A_1^{-\varepsilon})$ one of the two cases is valid*:

(1) $y_{0,A_1,A_2} \le \eta_{\tau-1}$ *and* $R^{\#}_\tau(A_1, A_2) = 0$.
(2) $y_{0,A_1,A_2} \ge \eta_{\tau-1}$ *and*

$$\begin{aligned} R^{\#}_\tau(A_1, A_2) = & \int\limits_{A_1^{1/d_1}\eta_{\tau-1}}^{A_1^{1/d_1}\min\{y_0,\eta_\tau\}} f^{-1}_{A_1,A_2}(y)\,dy - \tfrac{1}{2}A_1^{1/d_1}(\min\{\eta_\tau, y_0\} - \eta_{\tau-1}) \\ & + T_1^{\#(\mu,\varrho)}(A_1) + T_2^{\#(\mu,\varrho)}(A_2) + U_1^{\#(\mu,\varrho)}(A_1) + U_2^{\#(\mu,\varrho)}(A_2) \\ & + \psi(A_1^{1/d_1}\eta_{\tau-1}) f^{-1}_{A_1,A_2}(A_1^{1/d_1}\eta_{\tau-1}) \\ & - \psi(A_1^{1/d_1}\min\{\eta_\tau, y_0\}) f^{-1}_{A_1,A_2}(A_1^{1/d_1}\min\{\eta_\tau, y_0\}) \\ & + O(A_1^{46/(73d_1)}(\log A_1)^{315/146}). \end{aligned}$$

Proof. Again several cases have to be distinguished. From Lemma 7.6 for $\overline{y} = \eta_{\tau-1}$ (in case $\tau \ge 2$) it follows that always $y_0 \le \eta_{\tau-1}$ or $y_0 > \eta_{\tau-1}$. Assume the latter case.

Case 1: $[C^*_{\mu-1}, C^*_\mu] \subseteq [c'_\tau, d'_\tau]$. From Lemma 7.6 for $\overline{y} = \eta_\tau$ and Lemma 7.8 it follows that uniformly in the given range of (A_1, A_2) one of the two cases $y_0 \le \eta_\tau$ or $y_0 > \eta_\tau$ and one of the two cases $m_\tau(C) \le y_0$ or $m_\tau(C) \ge y_0$ is valid.

CASE 1.1: $m_\tau(C) \le y_0 \le \eta_\tau$. Applying Lemma 3.1 with x, y interchanged gives the decomposition in the following two cases.

CASE 1.1.1: If $l\restriction[c_\tau, d_\tau]$ increases then

$$\begin{aligned} R_\tau^{\#} = \#\{(x,y) \in \mathbb{Z}^2 \mid A_1^{1/d_1}\eta_{\tau-1} < y \le A_1^{1/d_1} m_\tau(C),& \\ 0 < x \le A_1^{1/d_1} x_1(\tau_1, \tau_1 y)\}& \\ + \#\{(x,y) \in \mathbb{Z}^2 \mid A_1^{1/d_1} m_\tau(C) < y \le A_1^{1/d_1} y_0,& \\ 0 < x \le A_2^{1/d_2} x_2(\tau_2, \tau_2 y)\}& \\ + O(A_1^{46/(73d_1)}).& \end{aligned}$$

CASE 1.1.2: If $l\restriction[c_\tau, d_\tau]$ decreases then

$$\begin{aligned} R_\tau^{\#} = \#\{(x,y) \in \mathbb{Z}^2 \mid A_1^{1/d_1}\eta_{\tau-1} < y \le A_1^{1/d_1} m_\tau(C),& \\ 0 < x \le A_2^{1/d_2} x_2(\tau_2, \tau_2 y)\}& \\ + \#\{(x,y) \in \mathbb{Z}^2 \mid A_1^{1/d_1} m_\tau(C) < y \le A_1^{1/d_1} y_0,& \\ 0 < x \le A_1^{1/d_1} x_1(\tau_1, \tau_1 y)\}& \\ + O(A_1^{46/(73d_1)}).& \end{aligned}$$

CASE 1.2: $y_0 \le m_\tau(C) \le \eta_\tau$.

CASE 1.2.1: If $l\restriction[c_\tau, d_\tau]$ increases then

$$\begin{aligned} R_\tau^{\#} = \#\{(x,y) \in \mathbb{Z}^2 \mid A_1^{1/d_1}\eta_{\tau-1} < y \le A_1^{1/d_1} y_0,\ 0 < x \le A_1^{1/d_1} x_1(\tau_1, \tau_1 y)\}& \\ + O(A_1^{46/(73d_1)}).& \end{aligned}$$

CASE 1.2.2: If $l\restriction[c_\tau, d_\tau]$ decreases then

$$\begin{aligned} R_\tau^{\#} = \#\{(x,y) \in \mathbb{Z}^2 \mid A_1^{1/d_1}\eta_{\tau-1} < y \le A_1^{1/d_1} y_0,\ 0 < x \le A_2^{1/d_2} x_2(\tau_2, \tau_2 y)\}& \\ + O(A_1^{46/(73d_1)}).& \end{aligned}$$

CASE 1.3: $y_0 > \eta_\tau$, $y_0 \ge m_\tau(C)$.

CASE 1.3.1: If $l\restriction[c_\tau, d_\tau]$ increases then

$$\begin{aligned} R_\tau^{\#} = \#\{(x,y) \in \mathbb{Z}^2 \mid A_1^{1/d_1}\eta_{\tau-1} < y \le A_1^{1/d_1} m_\tau(C),& \\ 0 < x \le A_1^{1/d_1} x_1(\tau_1, \tau_1 y)\}& \\ + \#\{(x,y) \in \mathbb{Z}^2 \mid A_1^{1/d_1} m_\tau(C) < y \le A_1^{1/d_1} \eta_\tau,& \\ 0 < x \le A_2^{1/d_2} x_2(\tau_2, \tau_2 y)\}& \\ + O(A_1^{46/(73d_1)}).& \end{aligned}$$

CASE 1.3.2: If $l\restriction[c_\tau, d_\tau]$ decreases then

$$\begin{aligned} R_\tau^{\#} = \#\{(x,y) \in \mathbb{Z}^2 \mid A_1^{1/d_1}\eta_{\tau-1} < y \le A_1^{1/d_1} m_\tau(C),& \\ 0 < x \le A_2^{1/d_2} x_2(\tau_2, \tau_2 y)\}& \end{aligned}$$

$$+\#\{(x,y)\in\mathbb{Z}^2 \mid A_1^{1/d_1}m_\tau(C)<y\le A_1^{1/d_1}\eta_\tau,\\ 0<x\le A_1^{1/d_1}x_1(\tau_1,\tau_1 y)\}\\ +O(A_1^{46/(73d_1)}).$$

The fourth combination $m_\tau(C)\ge y_0>\eta_\tau$ is not possible because always $m_\tau(C)\le\eta_\tau$.

CASE 2: $[C^*_{\mu-1},C^*_\mu]\subseteq[c_\tau,c'_\tau]$ *or* $\subseteq[d'_\tau,d_\tau]$. Then $y_0\le y_1<m_\tau(C)\le\eta_\tau$.

CASE 2.1: If $l\restriction[c_\tau,d_\tau]$ increases then

$$R^\#_\tau=\#\{(x,y)\in\mathbb{Z}^2 \mid A_1^{1/d_1}\eta_{\tau-1}<y\le A_1^{1/d_1}y_0,\ 0<x\le A_1^{1/d_1}x_1(\tau_1,\tau_1 y)\}\\ +O(A_1^{46/(73d_1)}).$$

CASE 2.2: If $l\restriction[c_\tau,d_\tau]$ decreases then

$$R^\#_\tau=\#\{(x,y)\in\mathbb{Z}^2 \mid A_1^{1/d_1}\eta_{\tau-1}<y\le A_1^{1/d_1}y_0,\ 0<x\le A_2^{1/d_2}x_2(\tau_2,\tau_2 y)\}\\ +O(A_1^{46/(73d_1)}).$$

CASE 3: $C^*_\mu\le c_\tau$. From Lemma 7.6 for $\overline{y}=\eta_\tau$ it follows that always $y_0\le\eta_\tau$ or $y_0>\eta_\tau$.

CASE 3.1: If $y_0\le\eta_\tau$ then

$$R^\#_\tau=\#\{(x,y)\in\mathbb{Z}^2 \mid A_1^{1/d_1}\eta_{\tau-1}<y\le A_1^{1/d_1}y_0,\ 0<x\le A_2^{1/d_2}x_2(\tau_2,\tau_2 y)\}.$$

CASE 3.2: If $y_0>\eta_\tau$ then

$$R^\#_\tau=\#\{(x,y)\in\mathbb{Z}^2 \mid A_1^{1/d_1}\eta_{\tau-1}<y\le A_1^{1/d_1}\eta_\tau,\ 0<x\le A_2^{1/d_2}x_2(\tau_2,\tau_2 y)\}.$$

CASE 4: $C^*_{\mu-1}\ge d_\tau$.

CASE 4.1: If $y_0\le\eta_\tau$ then

$$R^\#_\tau=\#\{(x,y)\in\mathbb{Z}^2 \mid A_1^{1/d_1}\eta_{\tau-1}<y\le A_1^{1/d_1}y_0,\ 0<x\le A_1^{1/d_1}x_1(\tau_1,\tau_1 y)\}.$$

CASE 4.2: If $y_0>\eta_\tau$ then

$$R^\#_\tau=\#\{(x,y)\in\mathbb{Z}^2 \mid A_1^{1/d_1}\eta_{\tau-1}<y\le A_1^{1/d_1}\eta_\tau,\ 0<x\le A_1^{1/d_1}x_1(\tau_1,\tau_1 y)\}.$$

As in the proof of Lemma 7.4 only Cases 1.1.1 and 1.1.2 are pursued further. The other cases are similar but somewhat easier.

In Case 1.1.1 use ($C^\#4$), ($C^\#5$), (2.8) and ($Z^\#3$) and choose $1\le\nu_j\le n$ with the properties

$$m_\tau((C^*_{\mu-1},C^*_\mu))\subseteq(\zeta_{\nu_0-1},\zeta_{\nu_0})\subseteq(\eta_{\tau-1},\min\{\eta_\tau,y_1\}),\qquad \zeta_{\nu_1-1}=\eta_{\tau-1},\\ \zeta_{\nu_2}=\min\{\eta_\tau,y_1\},\qquad \nu_1\le\nu_0\le\nu_2,\qquad \zeta_{\nu_2}<\widetilde{\eta}_1,\\ \widetilde{m}_\tau((C^*_{\mu-1},C^*_\mu))\subseteq(\zeta_{\nu_3-1},\zeta_{\nu_3}),\\ C^{-1/d_2}y_{0,A_1,A_2}\in(\zeta_{\nu_4-1},\zeta_{\nu_4})\quad \text{for } A_1,A_2\ge K$$

with

$$C := A_2 A_1^{-d_2/d_1} \in (C^*_{\mu-1} + A_1^{-\varepsilon}, C^*_\mu - A_1^{-\varepsilon}), \quad \nu_3 \le \nu_4, \quad \zeta_{\nu_4} < \widetilde{\eta}_2.$$

Then

$$\begin{aligned}
R^{\#}_\tau = \sum_{\nu=\nu_1}^{\nu_0-1} \#\{(x,y)\in\mathbb{Z}^2 \mid A_1^{1/d_1}\zeta_{\nu-1} < y \le A_1^{1/d_1}\zeta_\nu,& \\
0 < x \le A_1^{1/d_1} x_1(\tau_1,\tau_1 y)\}& \\
+ \#\{(x,y)\in\mathbb{Z}^2 \mid A_1^{1/d_1}\zeta_{\nu_0-1} < y \le A_1^{1/d_1} m_\tau(C),& \\
0 < x \le A_1^{1/d_1} x_1(\tau_1,\tau_1 y)\}& \\
- \#\{(x,y)\in\mathbb{Z}^2 \mid A_2^{1/d_2}\zeta_{\nu_3-1} < y \le A_2^{1/d_2} C^{-1/d_2} m_\tau(C),& \\
0 < x \le A_2^{1/d_2} x_2(\tau_2,\tau_2 y)\}& \\
+ \sum_{\nu=\nu_3}^{\nu_4-1} \#\{(x,y)\in\mathbb{Z}^2 \mid A_2^{1/d_2}\zeta_{\nu-1} < y \le A_2^{1/d_2}\zeta_\nu,& \\
0 < x \le A_2^{1/d_2} x_2(\tau_2,\tau_2 y)\}& \\
+ \#\{(x,y)\in\mathbb{Z}^2 \mid A_2^{1/d_2}\zeta_{\nu_4-1} < y \le A_2^{1/d_2} C^{-1/d_2} y_0,& \\
0 < x \le A_2^{1/d_2} x_2(\tau_2,\tau_2 y)\}& \\
+ O(A_1^{46/(73d_1)}).&
\end{aligned}$$

In Case 1.1.2 choose $1 \le \nu_j \le n$ with the properties

$$\begin{gathered}
m_\tau((C^*_{\mu-1}, C^*_\mu)) \subseteq (\zeta_{\nu_0-1}, \zeta_{\nu_0}) \subseteq (\eta_{\tau-1}, \min\{\eta_\tau, y_1\}), \quad \zeta_{\nu_1-1} = \eta_{\tau-1}, \\
\zeta_{\nu_2} = \min\{\eta_\tau, y_1\}, \quad \nu_1 \le \nu_0 \le \nu_2, \quad \zeta_{\nu_2} < \widetilde{\eta}_1, \\
\widetilde{m}_\tau((C^*_{\mu-1}, C^*_\mu)) \subseteq (\zeta_{\nu_3-1}, \zeta_{\nu_3}) \subseteq [0, y_2], \\
\zeta_{\nu_3} < \widetilde{\eta}_2, \quad C^{-1/d_2}\widetilde{\eta}_{\tau-1} \in (\zeta_{\nu_4-1}, \zeta_{\nu_4}) \text{ or } \equiv \zeta_{\nu_4-1} \text{ (if } \tau = 1), \quad \nu_4 \le \nu_3, \\
y_{0,A_1,A_2} \in (\zeta_{\nu_5-1}, \zeta_{\nu_5}] \quad \text{for } A_1, A_2 \ge K
\end{gathered}$$

with

$$C := A_2 A_1^{-d_2/d_1} \in (C^*_{\mu-1} + A_1^{-\varepsilon}, C^*_\mu - A_1^{-\varepsilon}), \quad \nu_0 \le \nu_5.$$

Then

$$\begin{aligned}
R^{\#}_\tau = -\#\{(x,y)\in\mathbb{Z}^2 \mid A_2^{1/d_2}\zeta_{\nu_4-1} < y \le A_2^{1/d_2} C^{-1/d_2}\eta_{\tau-1},& \\
0 < x \le A_2^{1/d_2} x_2(\tau_2,\tau_2 y)\}& \\
+ \sum_{\nu=\nu_4}^{\nu_3-1} \#\{(x,y)\in\mathbb{Z}^2 \mid A_2^{1/d_2}\zeta_{\nu-1} < y \le A_2^{1/d_2}\zeta_\nu,& \\
0 < x \le A_2^{1/d_2} x_2(\tau_2,\tau_2 y)\}& \\
+ \#\{(x,y)\in\mathbb{Z}^2 \mid A_2^{1/d_2}\zeta_{\nu_3-1} < y \le A_2^{1/d_2} C^{-1/d_2} m_\tau(C),& \\
0 < x \le A_2^{1/d_2} x_2(\tau_2,\tau_2 y)\}&
\end{aligned}$$

$$
\begin{aligned}
&- \#\{(x,y) \in \mathbb{Z}^2 \mid A_1^{1/d_1}\zeta_{\nu_0-1} < y \le A_1^{1/d_1} m_\tau(C), \\
&\qquad\qquad\qquad\qquad\qquad\qquad 0 < x \le A_1^{1/d_1} x_1(\tau_1, \tau_1 y)\} \\
&+ \sum_{\nu=\nu_0}^{\nu_5-1} \#\{(x,y) \in \mathbb{Z}^2 \mid A_1^{1/d_1}\zeta_{\nu-1} < y \le A_1^{1/d_1}\zeta_\nu, \\
&\qquad\qquad\qquad\qquad\qquad\qquad 0 < x \le A_1^{1/d_1} x_1(\tau_1, \tau_1 y)\} \\
&+ \#\{(x,y) \in \mathbb{Z}^2 \mid A_1^{1/d_1}\zeta_{\nu_5-1} < y \le A_1^{1/d_1} y_0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad 0 < x \le A_1^{1/d_1} x_1(\tau_1, \tau_1 y)\} \\
&+ O(A_1^{46/(73d_1)}).
\end{aligned}
$$

It follows from (C$^\#$2) and (C$^\#$3) that in each but the last set the endpoints of the respective intervals of y when renormalized with the corresponding A_j^{-1/d_j} have a distance $\ge \tau_j^\kappa$ from all ζ_ν with $0 \le \nu \le n$ or are equal to one of these ζ_ν. From (C$^\#$5) and Lemma 7.6 applied to $\overline{y} = \zeta_{\nu_5-1}$ (if $\nu_5 > 1$) and $\overline{y} = \zeta_{\nu_5}$ it follows that for the last set the same holds or that $|y_0 - \zeta_{\nu_5}| \ll \tau_1$ or $|y_0 - \zeta_{\nu_5-1}| \ll \tau_1$ uniformly in (A_1, A_2). In the last case apply the trivial estimation to the last set.

The remainder of the proof is as in the proof of Lemma 7.4. ∎

COROLLARY 7.10. *Let $1 \le \mu \le m$. Then for $A_1, A_2 \ge K$ with $C := A_2 A_1^{-d_2/d_1} \in (C_{\mu-1}^* + A_1^{-\varepsilon}, C_\mu^* - A_1^{-\varepsilon})$ we have*

$$
\begin{aligned}
R^\#(A_1, A_2) = &\int_0^{A_1^{1/d_1} y_0} f_{A_1,A_2}^{-1}(y)\,dy - \tfrac{1}{2} A_1^{1/d_1} y_0 + T_1^{\#(\mu)}(A_1) + T_2^{\#(\mu)}(A_2) \\
&+ U_1^{\#(\mu)}(A_1) + U_2^{\#(\mu)}(A_2) - \tfrac{1}{2} f_{A_1,A_2}^{-1}(0) \\
&- \psi(A_1^{1/d_1} y_0) f_{A_1,A_2}^{-1}(A_1^{1/d_1} y_0) \\
&+ O(A_1^{46/(73d_1)} (\log A_1)^{315/146}).
\end{aligned}
$$

Corollaries 7.5 and 7.10 and (3.1) together with

$$
\begin{aligned}
f_{A_1,A_2}(0) &= \min\{A_1^{1/d_1}\widetilde{\eta}_1, A_2^{1/d_2}\widetilde{\eta}_2\} + O(1), \\
f_{A_1,A_2}^{-1}(0) &= \varrho_{A_1,A_2} = \min\{A_1^{1/d_1}\widetilde{\xi}_1, A_2^{1/d_2}\widetilde{\xi}_2\} + O(1)
\end{aligned}
$$

give the asymptotics of Theorem 1.1. The estimation $p_{\mu,\nu}, q_{\mu,\nu} \le d_\nu - 2$ follows from

LEMMA 7.11. *For each $x_0 \in [0, \widetilde{\xi}_\nu)$ there is some $2 \le k \le d_\nu$ with $\widetilde{y}_\nu^{(k)}(x_0) \ne 0$. If $j_0^{(\nu)} = 1$ then the statement is valid also for $x_0 = \widetilde{\xi}_\nu$.*

Proof. Assume $\widetilde{y}^{(k)}(x_0) = 0$ for $2 \le k \le d_\nu$. Then Taylor's theorem implies

$$
\widetilde{y}_\nu(x) = \widetilde{y}_\nu(x_0) + \widetilde{y}_\nu'(x_0)(x - x_0) + O(|x - x_0|^{d_\nu+1})
$$

in some neighbourhood of x_0 in $[0,\widetilde{\xi}_\nu]$. Consequently,

$$1=\sum_{j=0}^{d_\nu} a^{(\nu)}_{d_\nu-j,j}x^{d_\nu-j}(\widetilde{y}_\nu(x_0)+\widetilde{y}'_\nu(x_0)(x-x_0)+O(|x-x_0|^{d_\nu+1}))^j$$

and

$$1-\sum_{j=0}^{d_\nu} a^{(\nu)}_{d_\nu-j,j}x^{d_\nu-j}(\widetilde{y}_\nu(x_0)+\widetilde{y}'_\nu(x_0)(x-x_0))^j=O(|x-x_0|^{d_\nu+1})$$

for x close to x_0. The left hand polynomial of order $\le d_\nu$ vanishes therefore at x_0 with order $\ge d_\nu+1$ and is consequently the zero polynomial. Therefore $\widetilde{g}_\nu(x,\widetilde{y}_\nu(x_0)+\widetilde{y}'_\nu(x_0)(x-x_0))=1$ on $\mathbb{R}$, which contradicts Lemma 4.1. ■

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Received on 24.4.1998
and in revised form 31.5.1999 (3370)