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A p-adic Nevanlinna–Diophantine correspondence

by

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Dedicated to Professor Pit-Mann Wong
on the occasion of his sixtieth birthday

1. Introduction. Beginning with the work of Osgood, Vojta, and Lang, it has been observed that many statements in Nevanlinna theory closely resemble statements in Diophantine approximation. Qualitatively, in the simplest case, holomorphic curves in a variety X should correspond to infinite sets of integral points on X. A detailed dictionary between Nevanlinna theory and Diophantine approximation has been developed by Vojta [13]. This correspondence has been influential, inspiring conjectures and results in both subjects.

Relatively recently, a p-adic analogue of Nevanlinna theory and value distribution theory has been developed, and many analogous results proven. Similar to the correspondence between classical Nevanlinna theory and Diophantine approximation, we discuss here a correspondence between p-adic Nevanlinna theory and certain Diophantine statements over the integers $\mathbb{Z}$ or the rational numbers $\mathbb{Q}$. Roughly speaking, at least for certain classes of varieties, a nonconstant p-adic analytic map into a variety X should correspond to an infinite set of $\mathbb{Z}$-integral points on X. We discuss this in the next section, making some observations towards a precise formulation. While we lack such a precise formulation in general, this correspondence already appears to be useful in suggesting both results and proofs of statements concerning p-adic analytic maps and integral points on varieties. We illustrate this in the last section, giving several examples of parallel p-adic and arithmetic results. Aside from their illustrative purpose, some of these results may be of independent interest.

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2. A correspondence. We begin with a motivating example. A basic fact that separates the theory of p-adic entire functions from its complex counterpart is the following:

THEOREM 2.1. *A p-adic entire function without zeros (over $\mathbb{C}_p$) is constant.*

Here $\mathbb{C}_p$ denotes the completion of the algebraic closure of the field of p-adic numbers $\mathbb{Q}_p$. In the classical analogy (Vojta's dictionary) between value distribution theory and arithmetic, an entire function without zeros is analogous to an infinite set of units in some ring of integers. Thus, to obtain a Diophantine analogue of p-adic entire functions, it is natural that we consider only rings of integers which contain finitely many units, i.e., $\mathbb{Z}$ or the (classical) ring of integers of an imaginary quadratic field. In a more geometric way, we may rephrase Theorem 2.1 as:

THEOREM 2.2A. *There are no nonconstant analytic maps from $\mathbb{C}_p$ to $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$.*

To give an arithmetic counterpart to this theorem, we introduce a bit of notation. Let X be an affine variety over a number field k. Let S be a finite set of places of k containing the archimedean places and let $\mathcal{O}_{k,S}$ denote the ring of S-integers of k. We define a set $R \subset X(k)$ to be a set of *$\mathcal{O}_{k,S}$-integral points* on X if there exists an affine embedding $\phi : X \hookrightarrow \mathbb{A}^n$ such that $\phi(R) \subset X \cap \mathbb{A}^n(\mathcal{O}_{k,S})$. With this terminology, the arithmetic counterpart to Theorem 2.2A is the following theorem:

THEOREM 2.2B. *There exists an infinite set of $\mathcal{O}_{k,S}$-integral points on $\mathbb{G}_m$ if and only if $\mathcal{O}_{k,S}$ is neither $\mathbb{Z}$ nor the ring of integers of an imaginary quadratic field.*

Thus, there is a dichotomy between $\mathcal{O}_{k,S}$-integral points with $|S| = 1$ and $\mathcal{O}_{k,S}$-integral points with $|S| > 1$. While the general case of integral points ($|S| > 1$) corresponds to classical holomorphic curves, we would like to make the case that nonconstant p-adic analytic maps to a variety behave similarly to infinite sets of $\mathbb{Z}$-integral points on the variety (or integral points over the ring of integers of an imaginary quadratic field). Actually, to obtain a more precise analogy, one must impose some natural restrictions on the varieties considered, as we now discuss.

Firstly, note that there is also a notion of integral points for arbitrary varieties. For instance, for a projective variety X over $\mathbb{Q}$, the $\mathbb{Z}$-integral points on X are just the rational points $X(\mathbb{Q})$. In this case, however, $X(\mathbb{Z}) = X(\mathbb{Z}_S) = X(\mathbb{Q})$, where S is any finite set of places of $\mathbb{Q}$ containing the archimedean place. Since we do not expect any correspondence with p-adic analytic maps to hold for $\mathbb{Z}_S$-integral points ($|S| > 1$), it is not surprising that the above correspondence fails for projective varieties. Indeed, there exist elliptic curves over $\mathbb{Q}$ with infinitely many rational points, while every analytic map from $\mathbb{C}_p$ to an elliptic curve is constant. Thus, it is reasonable to restrict our correspondence to affine varieties (or varieties close to affine varieties).

Another difficulty is illustrated by a curve such as $C : x^2 - 2y^2 = 1$. Over $\overline{\mathbb{Q}}$, we have $C \cong \mathbb{G}_m$. However, as is well-known, C, which is defined by a so-called Pell equation, does admit infinitely many $\mathbb{Z}$-integral points, while by Theorem 2.2A, it does not admit a nonconstant analytic map from $\mathbb{C}_p$. In an attempt to understand this phenomenon, we note an equivalent definition for $\mathcal{O}_{k,S}$-integral points on an affine variety X over k:

DEFINITION 2.3. Let X be an affine variety over k. A set $R \subset X(k)$ is a set of $\mathcal{O}_{k,S}$-*integral points* if and only if for every regular function ϕ in the ring of regular functions $\mathcal{O}(X)$ on X (over k), there exists a constant $\alpha \in k^*$ such that $\alpha\phi(R) \subset \mathcal{O}_{k,S}$.

Now let $\tilde{C}$ be the projective closure of C defined in the projective plane by $x^2 - 2y^2 = z^2$. Then C has two points at infinity, given in homogeneous coordinates by $P_\pm = (\pm\sqrt{2}, 1, 0)$. Every regular function on C over $\mathbb{Q}$ has a pole at both P_+ and P_- on $\tilde{C}$. Over $\mathbb{Q}(\sqrt{2})$, however, there are regular functions on C with a pole only at, say, P_+ on $\tilde{C}$. Thus, in view of Definition 2.3, one might view the problem here as being that C does not have enough regular functions over $\mathbb{Q}$ to have a good notion of $\mathbb{Z}$-integral points, at least for our purposes.

Given $k = \mathbb{Q}$ or an imaginary quadratic field and a nonsingular affine variety X over k, a hypothesis that will appear (implicitly) throughout Section 3 that is related to avoiding the above phenomenon is the following:

($*$) There exists a nonsingular projective closure $\tilde{X}$ of X such that every (geometric) irreducible component of $\tilde{X} \setminus X$ is defined over k.

If an affine variety X over k satisfies ($*$), it is easy to see that $\mathcal{O}(X) \otimes \overline{k} = \mathcal{O}(X_{\overline{k}})$, where $X_{\overline{k}} = X \times_k \overline{k}$ (in fact, the converse also holds). It is in this sense that the curve C did not have "enough" regular functions over $\mathbb{Q}$ (i.e., $\mathcal{O}(C) \otimes \overline{\mathbb{Q}} \neq \mathcal{O}(C_{\overline{\mathbb{Q}}})$).

Define X to be *arithmetically* $\mathcal{O}_{k,S}$-*hyperbolic* if any set of $\mathcal{O}_{k,S}$-integral points on X is finite. We define X to be $\mathbb{C}_p$-*hyperbolic* if every analytic map from $\mathbb{C}_p$ to X is constant. For curves, the condition ($*$) yields a sufficient hypothesis under which our correspondence holds:

THEOREM 2.4. *Let $k = \mathbb{Q}$ or an imaginary quadratic field. If X is an affine curve over k satisfying* ($*$) *then X is $\mathbb{C}_p$-hyperbolic if and only if X is arithmetically $\mathcal{O}_k$-hyperbolic.*

This follows from Theorems 2.2A and 2.2B, Siegel's theorem, and the nonarchimedean analogue of Picard's theorem. In higher dimensions, the condition $(*)$ is probably necessary to obtain a nice correspondence, but it is no longer sufficient. Although it is not clear what exactly the right conditions in higher dimensions should be, in Section 3 we give several higher-dimensional results demonstrating the correspondence between p-adic analytic maps and $\mathcal{O}_k$-integral points ($k = \mathbb{Q}$ or an imaginary quadratic field).

Quantitatively, along the lines of the condition $(*)$, it appears that p-adic Nevanlinna theory statements can be made to correspond to Diophantine approximation statements where one restricts to looking at $\mathbb{Q}$-rational numbers approximating $\mathbb{Q}$-rational divisors. We give an example of this in Section 3.1. In Section 3.2, we discuss generalizations of Berkovich's Picard theorem to higher-dimensional varieties, and their arithmetic analogues. In Section 3.3, we recall some results concerning the degeneration of p-adic analytic maps into the complements of hypersurface divisors in nonsingular projective varieties, and establish their arithmetic analogues. In particular, we give necessary and sufficient conditions for $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ to be arithmetically $\mathcal{O}_k$-hyperbolic when $k = \mathbb{Q}$ or an imaginary quadratic field and D_1 and D_2 are nonsingular projective curves in $\mathbb{P}^2$ intersecting transversally. This result has its own interest in the study of ternary form (homogeneous polynomial in three variables) equations. For example, it is fundamental to study when a ternary form $F(X, Y, Z) \in \mathbb{Z}[X, Y, Z]$ has infinitely many solutions in $\mathbb{Z}^3$ satisfying $F(X, Y, Z) = 1$. We refer to [3] for an introduction and more general statements. We should also mention that all the statements for $\mathbb{C}_p$ hold for any algebraically closed field of arbitrary characteristic, complete with respect to a nonarchimedean absolute value.

3. Some examples of the correspondence

3.1. Second Main Theorems. We first give some basic notation and definitions in p-adic Nevanlinna theory. Let $h(z) = \sum_{j=0}^{\infty} a_j z^j$ be an entire function on $\mathbb{C}_p$. For each $r \geq 0$, we define

$$\begin{aligned} |h|_r := \sup_j |a_j| r^j &= \sup\{|h(z)| : z \in \mathbb{C}_p \text{ with } |z| \leq r\} \\ &= \sup\{|h(z)| : z \in \mathbb{C}_p \text{ with } |z| = r\}. \end{aligned}$$

Let $f : \mathbb{C}_p \to \mathbb{P}^N(\mathbb{C}_p)$ be a nonconstant analytic curve in projective space. Let $\tilde{f} = (f_0, \ldots, f_N)$ be a reduced representative of f, where $f_0, \ldots, f_N$ are entire functions on $\mathbb{C}_p$ without common zeros, at least one of which is nonconstant.

The Nevanlinna characteristic function $T_f(r)$ is defined by $T_f(r) = \log \|f\|_r$, where $\|f\|_r = \max\{|f_0|_r, \ldots, |f_N|_r\}$. The above definition of $T_f(r)$ is independent, up to an additive constant, of the choice of the reduced representation of f.

Let D be a hypersurface in $\mathbb{P}^N(\mathbb{C}_p)$ of degree d. Let Q be a homogeneous polynomial in $N+1$ variables with coefficients in $\mathbb{C}_p$ defining D. We consider the entire function $Q \circ f = Q(f_0, \ldots, f_N)$ on $\mathbb{C}_p$. If $Q \circ f \not\equiv 0$, then the *proximity function* with respect to D is defined by

$$m_f(r, D) = m_f(r, Q) = \log \frac{\|f\|_r^d}{|Q \circ f|_r}.$$

Note that up to a constant term, $m_f(r, D)$ is independent of the choice of defining form Q.

Let $X \subset \mathbb{P}^N$ be a projective variety over a field K of dimension n. A collection of hypersurfaces $D_1, \ldots, D_q \subset \mathbb{P}^N$ over K is said to be *in general position with* X if for each $1 \le l \le n+1$ and each choice of indices $i_1 < \cdots < i_l$, each irreducible component of

$$D_{i_1}(\bar{K}) \cap \cdots \cap D_{i_l}(\bar{K}) \cap X(\bar{K})$$

has codimension l in X, so in particular is empty when $l = n+1$. A p-adic Second Main Theorem was proven by Ru in [11] for the case of projective space and by An in [1] for arbitrary projective varieties.

Theorem 3.1A (An, Ru). *Let $X \subset \mathbb{P}^N$ be a projective subvariety of dimension $n \ge 1$ over $\mathbb{C}_p$. Let $D_1, \ldots, D_q$ be hypersurfaces in $\mathbb{P}^N$ in general position with X. Let $f : \mathbb{C}_p \to X$ be a nonconstant analytic map whose image is not completely contained in any of the hypersurfaces $D_1, \ldots, D_q$. Then, for any positive real number r,*

$$\sum_{j=1}^{q} \frac{m_f(r, D_j)}{\deg D_j} \le nT_f(r) + O(1),$$

where $O(1)$ is a constant independent of r.

As noted in [11], in contrast to the case of classical Nevanlinna theory, the p-adic Second Main Theorem follows from the p-adic First Main Theorem. We now prove a Diophantine analogue of the p-adic Second Main Theorem. As in the p-adic case, the theorem follows essentially from the "Diophantine First Main Theorem" (the definition of the height as a sum of local heights and the equivalence, up to an additive constant, of heights associated to linearly equivalent divisors).

We first recall some basic definitions in Diophantine geometry. Let k be a number field. We have a set M_k of absolute values (or places) of k consisting of one place for each prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$, one place for each real embedding $\sigma : k \to \mathbb{R}$, and one place for each pair of conjugate embeddings $\sigma, \overline{\sigma} : k \to \mathbb{C}$. Let k_v denote the completion of k with respect to $v \in M_k$. We normalize our absolute values so that $|p|_v = p^{-[k_v:\mathbb{Q}_p]/[k:\mathbb{Q}]}$ if v corresponds to $\mathfrak{p}$ and $\mathfrak{p} \mid p$, and $|x|_v = |\sigma(x)|^{[k_v:\mathbb{R}]/[k:\mathbb{Q}]}$ if v corresponds to an embedding σ.

For a point $P = (x_0, \dots, x_n) \in \mathbb{P}^n(k)$ we define the *height* of P to be

$$h(P) = \sum_{v \in M_k} \log \max\{|x_0|_v, \dots, |x_n|_v\}.$$

This is the analogue of the Nevanlinna characteristic function. It follows from the product formula that $h(P)$ is independent of the choice of homogeneous coordinates for P. It is also easy to see that the height is independent of the choice of k.

Let D be a hypersurface over k in $\mathbb{P}^N$ of degree d. Let Q be a homogeneous polynomial over k in $N+1$ variables defining D. Let $\mathbf{x} = (x_0, \dots, x_N)$ be a representation of $P \in X(k)$ and let $\|\mathbf{x}\|_v = \max_{0 \le j \le N} |x_j|_v$. Let $\|Q\|_v$ be the maximum absolute value of the coefficients of Q with respect to $v \in M_k$. We define a *local Weil function* for D at v by

$$\lambda_{D,v}(P) = \log \frac{\|\mathbf{x}\|_v^d \cdot \|Q\|_v}{|Q(\mathbf{x})|_v}$$

for $P \notin D$ $(Q(\mathbf{x}) \neq 0)$. This is clearly independent of the choice of the coordinates of P and the choice of Q.

If S is a finite set of places of k, we let $m_S(P, D) = \sum_{v \in S} \lambda_{D,v}(P)$ be a sum of local Weil functions, the Diophantine analogue of the proximity function in Nevanlinna theory. When $S = \{\infty\}$, the unique archimedean place of $\mathbb{Q}$ or an imaginary quadratic field, we will just write $m_\infty(P, D)$.

With this notation, we now give the Diophantine approximation analogue of Theorem 3.1A.

THEOREM 3.1B. *Let $k = \mathbb{Q}$ or an imaginary quadratic field. Let $X \subset \mathbb{P}^N$ be a projective subvariety of dimension $n \ge 1$ over k. Let $D_1, \dots, D_q$ be hypersurfaces in $\mathbb{P}^N$ defined over k and in general position with X. Then there exists an effectively computable real constant C, depending only on $D_1, \dots, D_q$, such that*

$$\sum_{j=1}^{q} \frac{m_\infty(P, D_j)}{\deg D_j} \le nh(P) + C \quad \text{for all } P \in X(k) \setminus \textstyle\bigcup_{j=1}^q D_j.$$

Proof. Let Q_j be a homogeneous polynomial in $N+1$ variables of degree d_j with coefficients in k defining D_j. Let d be the least common multiple of $d_1, \dots, d_q$ and $G_j = Q_j^{d/d_j}$. Then $\deg G_j = d$ for all j. Let $\mathbf{x} = (x_0, \dots, x_N)$ be a representation of $P \in X(k) \setminus \bigcup_{j=1}^q D_j$. Then we have

$$m_\infty(P, D_j) = \log \frac{\|\mathbf{x}\|_\infty^{d_j} \cdot \|Q_j\|_\infty}{|Q_j(\mathbf{x})|_\infty}.$$

For every nonarchimedean place v, it is clear that $\lambda_{D,v}(P) \geq 0$. It follows easily from this and the product formula that

$$m_\infty(P, D_j) \leq d_j h(P) + h(Q_j) \tag{1}$$

where $h(Q_j) = \sum_{v \in M_k} \log \|Q_j\|_v$. The assertion then holds trivially if $q \leq n$. Therefore, we only need to consider when $q \geq n+1$.

For a fixed $P = \mathbf{x} = (x_0, \ldots, x_N) \in X(k) \setminus \bigcup_{j=1}^q D_j$, by rearranging the indices if necessary, we may assume that

$$|G_1(\mathbf{x})|_\infty \leq \cdots \leq |G_q(\mathbf{x})|_\infty. \tag{2}$$

Since $D_1, \ldots, D_q$ are in general position with X,

$$X \cap \{G_1 = 0, \ldots, G_{n+1} = 0\} = \emptyset.$$

Applying Hilbert's Nullstellensatz to the ideal generated by the forms defining X and $\{Q_1, \ldots, Q_{n+1}\}$, we see that for any integer l, $0 \leq l \leq N$, there is an integer $m_l \geq d$ such that

$$x_l^{m_l} = \sum_{j=1}^{n+1} A_{l,j}(\mathbf{x}) G_j(\mathbf{x}) \quad \text{on } X(k),$$

where $A_{l,j}$ are homogeneous polynomials with coefficients in k of degree $m_l - d$. Then

$$|x_l|_\infty^{m_l} \leq c \|\mathbf{x}\|_\infty^{m_l - d} \max_{1 \leq j \leq n+1} |G_j(\mathbf{x})|_\infty,$$

where c is a positive constant that depends only on the coefficients of the $A_{l,j}$. Therefore,

$$\|\mathbf{x}\|_\infty^d \leq c \max_{1 \leq j \leq n+1} |G_j(\mathbf{x})|_\infty.$$

From our assumption (2), we have

$$\log \frac{\|\mathbf{x}\|_\infty^d}{|G_j(\mathbf{x})|_\infty} \leq \log c \quad \text{for } j = n+1, \ldots, q.$$

Thus,

$$\begin{aligned}
d \sum_{j=1}^q \frac{m_\infty(P, D_j)}{d_j} &= \sum_{j=1}^q \log \frac{\|\mathbf{x}\|_\infty^d}{|G_j(\mathbf{x})|_\infty} + \sum_{j=1}^q \frac{d}{d_j} \log \|Q_j\|_\infty \\
&\leq \sum_{j=1}^n \log \frac{\|\mathbf{x}\|_\infty^d}{|G_j(\mathbf{x})|_\infty} + \sum_{j=1}^q \frac{d}{d_j} \log \|Q_j\|_\infty + (q-n) \log c \\
&\leq \frac{d}{d_j} \sum_{j=1}^n m_\infty(P, D_j) + \sum_{j=n+1}^q \frac{d}{d_j} \log \|Q_j\|_\infty + (q-n) \log c \\
&\leq n d h(P) + C,
\end{aligned}$$

where $C = (q-n)\big(\log c + \max_{1\le j\le q} \frac{d}{d_j}\log\|Q_j\|_\infty\big) + n\max_{1\le j\le q}\frac{d}{d_j}h(Q_j)$ by (1). Because the right hand side of the inequality no longer depends on the arrangement of the indices (2), dividing both sides through by d gives the theorem for all $P \in X(k)\setminus\bigcup_{j=1}^q D_j$. We note that it is well-known that the effectiveness follows from any effective version of Hilbert's Nullstellensatz and the method of Masser–Wüstholz [8]. ∎

As corollaries of Theorems 3.1A and 3.1B, we obtain:

Corollary 3.2A. *Let $X \subset \mathbb{P}^N$ be a projective subvariety of dimension $n \ge 1$ over $\mathbb{C}_p$. Let $D_1,\dots,D_q$ be hypersurfaces in $\mathbb{P}^n$ in general position with X. If $q \ge n+1$, then $X\setminus\bigcup_{i=1}^q D_i$ is $\mathbb{C}_p$-hyperbolic.*

Corollary 3.2B. *Let $k = \mathbb{Q}$ or an imaginary quadratic field. Let $X \subset \mathbb{P}^N$ be a projective subvariety of dimension $n \ge 1$ over k. Let $D_1,\dots,D_q$ be hypersurfaces in $\mathbb{P}^n$ defined over k and in general position with X. If $q \ge n+1$, then $X\setminus\bigcup_{i=1}^q D_i$ is arithmetically $\mathcal{O}_k$-hyperbolic.*

A generalization of these corollaries is given in the next section.

3.2. Varieties with many components at infinity. Berkovich's Picard theorem may be viewed as stating that any analytic map from $\mathbb{C}_p$ to a projective curve omitting two points must be constant. A generalization of this result to higher dimensions was recently given by Lin and Wang [7]. We will prove a refinement of [7]. Before stating the theorem, we introduce some more notation and definitions.

Let D be a divisor on a nonsingular projective variety X over a field k. For a nonzero rational function $\phi \in \overline{k}(X)$, we let $\operatorname{div}(\phi)$ denote the divisor associated to ϕ. Then we let $L(D) = \{\phi \in \overline{k}(X) : \operatorname{div}(\phi) + D \ge 0\}$ and $h^0(D) = \dim H^0(X, \mathcal{O}(D)) = \dim L(D)$. If $h^0(nD) = 0$ for all $n > 0$ then we let $\kappa(D) = -\infty$. Otherwise, we define the *Kodaira–Iitaka dimension* of D to be the integer $\kappa(D)$ such that there exist positive constants c_1 and c_2 with

$$c_1 n^{\kappa(D)} \le h^0(nD) \le c_2 n^{\kappa(D)}$$

for all sufficiently divisible $n > 0$. We define a divisor D on X to be *big* if $\kappa(D) = \dim X$.

Theorem 3.3A. *Let X be a nonsingular projective variety over $\mathbb{C}_p$. Let $D_1,\dots,D_m$ be effective divisors on X with empty intersection. Let $D = \sum_{i=1}^m D_i$.*

(a) *If $\kappa(D_i) > 0$ for all i, then the image of an analytic map $f : \mathbb{C}_p \to X\setminus D$ is contained in a proper subvariety of X.*

(b) *If D_i is big for all i, then there exists a proper Zariski-closed subset $Z \subset X$ such that the image of any analytic map $f : \mathbb{C}_p \to X\setminus D$ is contained in Z.*

(c) *If D_i is ample for all i, then there is no nonconstant analytic map from $\mathbb{C}_p$ to $X \setminus D$.*

The arithmetic analogue of Theorem 3.3A is implicit in the proof of a more general result proved in [6] using a higher-dimensional version of “Runge’s method”.

THEOREM 3.3B (Levin). *Let $k = \mathbb{Q}$ or an imaginary quadratic field. Let X be a nonsingular projective variety over k. Let $D_1, \dots, D_m$ be effective divisors on X, defined over k, with empty intersection. Let $D = \sum_{i=1}^{m} D_i$.*

(a) *If $\kappa(D_i) > 0$ for all i, then any set R of $\mathcal{O}_k$-integral points on $X \setminus D$ is contained in a proper Zariski-closed subset of X.*
(b) *If D_i is big for all i, then there exists a proper Zariski-closed subset $Z \subset X$ such that for any set R of $\mathcal{O}_k$-integral points on $X \setminus D$, the set $R \setminus Z$ is finite.*
(c) *If D_i is ample for all i, then all sets R of $\mathcal{O}_k$-integral points on $X \setminus D$ are finite.*

Furthermore, as shown in [6], in each of the above cases, the integral points can be effectively computed.

We will give a proof of Theorem 3.3A following the idea of Levin [6] in the arithmetic situation. First, we need the following lemma.

LEMMA 3.4. *Let X be a nonsingular projective variety over $\mathbb{C}_p$. Let $\phi_1, \dots, \phi_m \in \mathbb{C}_p(X)$ be rational functions on X without a common pole. Then there exists a constant λ such that*

$$\min_{1 \le i \le m} |\phi_i(P)| \le \lambda \quad \text{for all } P \in X(\mathbb{C}_p).$$

Proof. Since $\phi_1, \dots, \phi_m$ have no common pole, we may take a finite affinoid covering $\mathcal{U}$ such that for each affinoid subdomain U in $\mathcal{U}$ there is at least one ϕ_i, $1 \le i \le m$, which is regular on U. Since a regular function on an affinoid subdomain is bounded, we can find a constant λ_U such that $\min_{1 \le i \le m} |\phi_i(P)| \le \lambda_U$ for all $P \in U$. As $\mathcal{U}$ is a finite affinoid covering, the assertion of the lemma holds by taking $\lambda = \max_{U \in \mathcal{U}} \lambda_U$. ∎

Proof of Theorem 3.3A. We first prove (a). Since $\kappa(D_i) > 0$ for all i, there exists a nonconstant rational function $\phi_i \in \mathbb{C}_p(X)$ such that the poles of ϕ_i lie in the support of D_i. Since the intersection of the supports of $D_1, \dots, D_m$ is empty, $\phi_1, \dots, \phi_m$ have no common pole. By Lemma 3.4, there exists a constant λ such that

$$\min_{1 \le i \le m} |\phi_i(P)| \le \lambda \tag{3}$$

for all $P \in X(\mathbb{C}_p)$. Let f be an analytic map from $\mathbb{C}_p$ to $X \setminus D$. Then $\phi_1 \circ f, \dots, \phi_m \circ f$ are analytic functions. If $\phi_i \circ f$ is constant for some $1 \le i \le m$, then this algebraic relation implies that the image of f is contained in a proper algebraic subset of X. Therefore, we may assume all the $\phi_i \circ f$ are nonconstant analytic functions. A classical result on the growth modulus of nonarchimedean analytic functions (cf. [10, Section 6.1.4]) shows that for all $r \geq 0$ except a discrete subset of $[0, \infty)$,

$$\sup_{|z|=r} |\phi_i(f(z))| = |\phi_i(f(w))| \quad \text{for all } |w| = r. \tag{4}$$

Since there are only a finite number of ϕ_i's,

$$\sup_{|z|=r} |\phi_i(f(z))| = |\phi_i(f(w))| \quad \text{for all } |w| = r \text{ and } 1 \leq i \leq m$$

for all $r \geq 0$ except a discrete subset of $[0, \infty)$. For such r, we can easily deduce that

$$\min_{1 \leq i \leq m} \sup_{|z|=r} |\phi_i(f(z))| = \sup_{|z|=r} \min_{1 \leq i \leq m} |\phi_i(f(z))|. \tag{5}$$

By (3), we have

$$\min_{1 \leq i \leq m} |\phi_i(f(z))| \leq \lambda \tag{6}$$

for all $z \in \mathbb{C}_p$. This gives an upper bound for the right hand side of (5). On the other hand, since each of the $\phi_i(f(z))$ is a nonconstant analytic function, the left hand side of (5) tends to infinity as r grows to infinity, giving a contradiction. We conclude that the image of f is contained in a proper algebraic set.

We now prove (b). Since D_i is big for all i, we may choose a sufficiently large integer N such that for each i the map Φ_{ND_i} associated to $L(ND_i)$ is birational onto its image, and an isomorphism onto its image outside of a proper Zariski-closed subset $Z_i \subset X$. Let $Z = \bigcup_{i=1}^m Z_i$. For each i, let $\phi_{i,1}, \ldots, \phi_{i,l(ND_i)}$ be a basis of $L(ND_i)$. We may take

$$\Phi_{ND_i} = (\phi_{i,1}, \ldots, \phi_{i,l(ND_i)}).$$

Let f be a nonconstant analytic map from $\mathbb{C}_p$ to $X \setminus D$. Suppose that the image of f is not a subset of Z. Since f is not constant, there exist at least two distinct points P and Q in the image of f but not in Z. As for each i the map Φ_{ND_i} is one-to-one outside of Z, we have $\Phi_{ND_i}(P) \neq \Phi_{ND_i}(Q)$. Therefore, there exists some $1 \leq j_i \leq l(ND_i)$ such that $\phi_{i,j_i}(P) \neq \phi_{i,j_i}(Q)$. Since P and Q are two distinct points in the image of f, this shows that for each i, $\phi_{i,j_i} \circ f$ is not constant. Since the poles of ϕ_{i,j_i} lie in D_i, $\phi_{1,j_1}, \ldots, \phi_{m,j_m}$ have no common pole. We can repeat the arguments in (a) to reach a contradiction and conclude that the image of f must be contained in Z.

The proof of (c) follows from the proof of (b) since Z_i is empty when D_i is ample. ∎

3.3. Complements of hypersurface divisors. In [2], An, Wang, and Wong studied the degeneration of p-adic analytic maps into the complements of hypersurface divisors in nonsingular projective varieties. They proved:

Theorem 3.5A (An, Wang, Wong). *Let X be a nonsingular projective subvariety of $\mathbb{P}^N$ of dimension n. Let $P_1, \dots, P_q$ be nonconstant homogeneous polynomials in $N+1$ variables. Let $D_i = X \cap \{P_i = 0\}$, $1 \le i \le q$, be divisors of X in general position. Let f be an analytic map from $\mathbb{C}_p$ to $X \setminus \bigcup_{i=1}^q D_i$. Then the image of f is contained in a subvariety of X of codimension* $\min\{n+1, q\} - 1$ *in X. In particular, f is algebraically degenerate if $q \ge 2$, and $X \setminus \bigcup_{i=1}^q D_i$ is $\mathbb{C}_p$-hyperbolic if $q \ge n+1$.*

We now prove an arithmetic analogue.

Theorem 3.5B. *Let $k = \mathbb{Q}$ or an imaginary quadratic field. Let X be a nonsingular projective subvariety of $\mathbb{P}^N$ of dimension n defined over k. Let $P_1, \dots, P_q$ be nonconstant homogeneous polynomials over k in $N+1$ variables. Let $D_i = X \cap \{P_i = 0\}$, $1 \le i \le q$, be divisors of X in general position. Let R be a set of $\mathcal{O}_k$-integral points of $X \setminus \bigcup_{i=1}^q D_i$. Then R is contained in a finite union of subvarieties of X of codimension* $\min\{n+1, q\} - 1$ *in X. In particular, R is algebraically degenerate if $q \ge 2$, and $X \setminus \bigcup_{i=1}^q D_i$ is arithmetically $\mathcal{O}_k$-hyperbolic if $q \ge n+1$.*

Proof. When $q \ge n+1$, the result follows from Corollary 3.2B or Theorem 3.3B(c). Therefore, we only need to consider when $q \le n$. It follows from the definition of integral points that for some constant $\alpha \in k^*$, $\alpha(P_i/P_1)(R) \subset \mathcal{O}_k$, $i = 1, \dots, q$. Similarly, there is a constant $\beta \in k^*$ such that $\beta(P_1/P_i)(R) \subset \mathcal{O}_k$, $i = 1, \dots, q$. It follows that $(P_1/P_i)(R)$ lies in a finite number of cosets of $\mathcal{O}_k^*$ in k^* for $i = 1, \dots, q$. Since $\mathcal{O}_k^*$ is finite, this implies that R lies in a finite union of closed subsets of the form $P_i - c_i P_1 = 0$, $c_i \in k^*$, $i = 1, \dots, q$. The rest of the proof now proceeds as in [2]. ∎

An, Wang, and Wong proved more precise results on complements of hypersurface divisors in projective space.

Theorem 3.6 (An, Wang, Wong). *Let $D_1, \dots, D_n$ be nonsingular hypersurfaces in $\mathbb{P}^n$ intersecting transversally. Then $\mathbb{P}^n \setminus \bigcup_{i=1}^n D_i$ is $\mathbb{C}_p$-hyperbolic if* $\deg D_i \ge 2$ *for each $1 \le i \le n$.*

We obtain both a mild improvement to this theorem, and an arithmetic version. Before stating the two theorems, we make a convenient definition.

Definition 3.7. Let D be a hypersurface of degree d in $\mathbb{P}^n$. A nonsingular point x of D is said to be a *maximal inflexion point* if there exists a line intersecting D at x with multiplicity d.

THEOREM 3.8A. *Let $D_1, \dots, D_n$ be nonsingular hypersurfaces in $\mathbb{P}^n$. Assume that $D_1, \dots, D_n$ intersect transversally and* $\deg D_1 \le \cdots \le \deg D_n$. *Then $\mathbb{P}^n \setminus \bigcup_{i=1}^n D_i$ is $\mathbb{C}_p$-hyperbolic if either of the following conditions holds:*

(i) $\deg D_1 \ge 2$,
(ii) $\deg D_1 = 1$, $\deg D_n \ge 3$, *and D_1 does not simultaneously intersect $D_2, \dots, D_n$ at a point that is a maximal inflexion point of each of $D_2, \dots, D_n$.*

THEOREM 3.8B. *Let $k = \mathbb{Q}$ or an imaginary quadratic field. Let $D_1, \dots, D_n$ be nonsingular hypersurfaces defined over k in $\mathbb{P}^n$. Assume that $D_1, \dots, D_n$ intersect transversally and* $\deg D_1 \le \cdots \le \deg D_n$. *When* $\deg D_1 \le 2$, *if $\mathfrak{p} \in \bigcap_{i=1}^n D_i(\bar{k})$ and $[k(\mathfrak{p}) : k] = 2$, then we assume further that $k = \mathbb{Q}$ and $k(\mathfrak{p})$ is an imaginary quadratic field. Then $\mathbb{P}^n \setminus \bigcup_{i=1}^n D_i$ is arithmetically $\mathcal{O}_k$-hyperbolic if either of the following conditions holds:*

(i) $\deg D_1 \ge 2$,
(ii) $\deg D_1 = 1$, $\deg D_n \ge 3$, *and D_1 does not simultaneously intersect $D_2, \dots, D_n$ at a point that is a maximal inflexion point of each of $D_2, \dots, D_n$.*

We will only give the proof of Theorem 3.8B, as the proof of Theorem 3.8A follows from obvious modifications (and simplifications) of this proof.

Proof. Suppose $D_1, \dots, D_n$ satisfy either condition (i) or condition (ii). Let R be a set of $\mathcal{O}_k$-integral points on $\mathbb{P}^n \setminus \bigcup_{i=1}^n D_i$. By Theorem 3.5B, R lies on a finite union of curves in $\mathbb{P}^n$. We note that if a curve C is minimally defined over a proper finite extension of k, then the $\mathcal{O}_k$-integral points of C will lie in the intersection of C and its conjugate curves over k and hence the number is finite. Now, let C be any (irreducible) projective curve over k in $\mathbb{P}^n$. Then it suffices to show that $C \setminus \bigcup_{i=1}^n D_i$ contains only finitely many $\mathcal{O}_k$-integral points. By Siegel's theorem, this will be true if $\# C \cap \bigcup_{i=1}^n D_i > 2$. So we can assume that $C \cap \bigcup_{i=1}^n D_i$ consists of either a single point $\{\mathfrak{p}\}$ or two distinct points $\{\mathfrak{p}, \mathfrak{q}\}$. We use throughout (implicitly) a higher-dimensional version of Noether's formula for intersection numbers (e.g., [5, Th. 12.4]).

Suppose first that $C \cap \bigcup_{i=1}^n D_i$ consists of a single point $\{\mathfrak{p}\}$, which must be k-rational. Suppose that C is not a line in $\mathbb{P}^n$. Then $m_{\mathfrak{p}}(C) \le \deg(C) - 1$, where $m_{\mathfrak{p}}(C)$ is the multiplicity of C at $\mathfrak{p}$ (otherwise one could find a hyperplane in $\mathbb{P}^n$ intersecting C in $> \deg C$ points, counting multiplicities). Let X be the blow-up of $\mathbb{P}^n$ at $\mathfrak{p}$ with exceptional divisor E ($\cong \mathbb{P}^{n-1}$) and let $\tilde{C}$ be the strict transform of C, and $\tilde{D}_i$ the strict transform of D_i, $i = 1, \dots, n$. Since each divisor D_i is smooth at $\mathfrak{p}$ and $m_{\mathfrak{p}}(C) m_{\mathfrak{p}}(D_i) = m_{\mathfrak{p}}(C) \le \deg(C) - 1 < \deg(C)\deg(D_i)$, it follows that each $\tilde{D}_i$ must intersect $\tilde{C}$ at some point on X lying above $\mathfrak{p}$. Since the D_i intersect transversally, $\bigcap_{i=1}^n \tilde{D}_i \cap E = \emptyset$. Thus, there must be at least two points on $\tilde{C}$ lying above $\mathfrak{p}$ ($\#(\tilde{C} \cap E) \geq 2$). If there are strictly more than two such points, then by looking at a normalization of C, Siegel's theorem implies that $C \setminus \{\mathfrak{p}\}$ contains only finitely many integral points. On the other hand, if there are exactly two points $\tilde{\mathfrak{p}}_1$, $\tilde{\mathfrak{p}}_2$ on $\tilde{C}$ lying above $\mathfrak{p}$, then $\tilde{\mathfrak{p}}_1$ and $\tilde{\mathfrak{p}}_2$ must both be k-rational, as from the above, $\{\tilde{\mathfrak{p}}_1\} = \tilde{C} \cap \tilde{D}_i$ for some i, and $\tilde{C}$ and $\tilde{D}_i$ are both defined over k. Then Theorem 3.3B(c) (applied to a normalization of C) implies that $C \setminus \{\mathfrak{p}\}$ contains only finitely many $\mathcal{O}_k$-integral points. If C is a line and $\deg D_1 \geq 2$, then again, for every i, $\tilde{D}_i$ and $\tilde{C}$ must intersect and we can use the above argument. If C is a line and $\deg D_1 = 1$, then the assumption on maximal inflexion points in (ii) implies that C cannot intersect $\bigcup_{i=1}^n D_i$ in a single point.

Now suppose that $C \cap \bigcup_{i=1}^n D_i$ consists of two points $\{\mathfrak{p}, \mathfrak{q}\}$. If $\mathfrak{p}$ and $\mathfrak{q}$ are k-rational, then $C \setminus \{\mathfrak{p}, \mathfrak{q}\}$ contains only finitely many $\mathcal{O}_k$-integral points by Theorem 3.3B(c). So we can assume that $[k(\mathfrak{p}) : k] = 2$, and $\mathfrak{p}$ and $\mathfrak{q}$ are conjugate, lying in a quadratic extension of k. Note that $\{\mathfrak{p}, \mathfrak{q}\} \subset \bigcap_{i=1}^n D_i(\bar{k})$, since otherwise C would intersect $\bigcup_{i=1}^n D_i$ in a third point. If $\deg D_1 \leq 2$, then by assumption, $k = \mathbb{Q}$ and $k(\mathfrak{p})$ is an imaginary quadratic field. Hence, in this case, $C \setminus \{\mathfrak{p}, \mathfrak{q}\}$ contains only finitely many $\mathcal{O}_k$-integral points. Finally, suppose that $\deg D_1 \geq 3$. Consider the blow-up X of $\mathbb{P}^n$ at $\mathfrak{p}$ as before. If there is more than one point on $\tilde{C}$ lying above $\mathfrak{p}$, then as in previous arguments, $C \setminus \{\mathfrak{p}, \mathfrak{q}\}$ contains only finitely many integral points by Siegel's theorem. So suppose that there is a unique point $\tilde{\mathfrak{p}}$ on $\tilde{C}$ lying above $\mathfrak{p}$. As in our previous argument, transversality implies that some $\tilde{D}_i$ does not pass through $\tilde{\mathfrak{p}}$. Thus, the contribution to the intersection number (C, D_i) from the intersection at $\mathfrak{p}$ is $m_{\mathfrak{p}}(C)$. By using the Galois action of $k(\mathfrak{p})/k$, we deduce similarly that the contribution to (C, D_i) from the intersection at $\mathfrak{q}$ is $m_{\mathfrak{q}}(C)$. So $(C, D_i) = m_{\mathfrak{p}}(C) + m_{\mathfrak{q}}(C) \leq 2 \deg C$. Since $\deg D_i \geq 3$, this yields a contradiction. ■

The $n = 2$ case of Theorem 3.8A was proved by An, Wang, and Wong in [2]. In fact, in this case they also proved the converse.

THEOREM 3.9A (An, Wang, Wong). *Let D_1 and D_2 be nonsingular projective curves in $\mathbb{P}^2$. Assume that D_1 and D_2 intersect transversally and $\deg D_1 \leq \deg D_2$. Then $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ is $\mathbb{C}_p$-hyperbolic if and only if either $\deg D_1, \deg D_2 \geq 2$ or $\deg D_1 = 1$, $\deg D_2 \geq 3$ and D_1 does not intersect D_2 at any maximal inflexion point.*

Similarly, we can give a complete characterization of when $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ is arithmetically $\mathcal{O}_k$-hyperbolic, where $k = \mathbb{Q}$ or an imaginary quadratic field and D_1 and D_2 are curves over k intersecting transversally.

THEOREM 3.9B. *Let $k = \mathbb{Q}$ or an imaginary quadratic field. Let D_1 and D_2 be nonsingular projective curves defined over k in $\mathbb{P}^2$. Assume that D_1 and D_2 intersect transversally and* $\deg D_1 \le \deg D_2$. *Then $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ is arithmetically $\mathcal{O}_k$-hyperbolic if and only if one of the following holds:*

(i) $\deg D_1 \geq 3$.
(ii) $\deg D_1 = 2$, *and if there exists $\mathfrak{p} \in D_1(\bar{k}) \cap D_2(\bar{k})$ with $[k(\mathfrak{p}) : k] = 2$ such that the line through $\mathfrak{p}$ and its conjugate point $\mathfrak{q}$ only intersects D_1 and D_2 at $\mathfrak{p}$ and $\mathfrak{q}$, then we assume further that $k = \mathbb{Q}$ and $k(\mathfrak{p})$ is an imaginary quadratic field.*
(iii) $\deg D_1 = 1$, $\deg D_2 = 2$, *$D_1(\bar{k}) \cap D_2(\bar{k}) = \{\mathfrak{p}, \mathfrak{q}\}$, $k = \mathbb{Q}$, and $k(\mathfrak{p})$ is an imaginary quadratic field.*
(iv) $\deg D_1 = 1$, $\deg D_2 \geq 3$, *D_1 does not intersect D_2 at any k-rational maximal inflexion point, and if there exists $\mathfrak{p} \in D_1(\bar{k}) \cap D_2(\bar{k})$ with $[k(\mathfrak{p}) : k] = 2$ and a conic C intersecting with $D_1 \cup D_2$ only at $\mathfrak{p}$ and its conjugate point, then we assume that either*

   (a) *$k = \mathbb{Q}$ and $k(\mathfrak{p})$ is an imaginary quadratic field, or*
   (b) *C has no k-rational point.*

In the p-adic case, we have the following for generic curves.

COROLLARY 3.10A. *If D_1 and D_2 are two generic curves in $\mathbb{P}^2(\mathbb{C}_p)$ with* $\deg D_1 + \deg D_2 \geq 4$, *then $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ is $\mathbb{C}_p$-hyperbolic.*

For the arithmetic case, we need to introduce Serre's thin sets (cf. [12]) for a similar statement. We will use the following definitions from [4]. Let K be a field and let n be a positive integer. Let $\mathcal{T}$ be a subset of the affine space K^n. The set $\mathcal{T}$ is called a *basic thin set of the first type* if there exists a nonzero polynomial $F(\underline{t}) \in K[\underline{t}]$ (where $\underline{t} = (t_1, \dots, t_n)$) such that $(\underline{\tau}) \in \mathcal{T}$ if and only if $F(\underline{\tau}) = 0$. The set $\mathcal{T}$ is a *basic thin set of the second type* if there exists a K-irreducible polynomial $F(\underline{t}, X) \in K[\underline{t}, X]$ with $\deg_X F \geq 2$ such that $(\underline{\tau}) \in \mathcal{T}$ if and only if the specialized polynomial $F(\underline{\tau}, X)$ has a root in K. The set $\mathcal{T}$ is called *thin* if it is contained in a finite union of basic thin sets. We recall the following basic fact.

LEMMA 3.11. *If $F(\underline{t}, X) \in K[\underline{t}, X]$ is an irreducible polynomial over $K(\underline{t})$, then there exists a thin set $\mathcal{T} \subset K^n$ such that $F(\underline{\tau}, X)$ is irreducible over K if $\underline{\tau} \notin \mathcal{T}$.*

Proof. See [12, Section 9.2, Propositions 1 and 2] or [4, Theorem 2.2.] ∎

DEFINITION 3.12. Let D_1 and D_2 be two curves in $\mathbb{P}^2$ over a field K, and let P_1 and P_2 be their defining polynomials of degrees d_1 and d_2 respectively. We may write $P_1 = \sum_I a_I \underline{U}^I$ where I runs through the set of (i_0, i_1, i_2) with $i_0 + i_1 + i_2 = d_1$ and $\underline{U}^I = X^{i_0} Y^{i_1} Z^{i_2}$, and write $P_2 = \sum_J b_J \underline{W}^J$ where J runs through the set of (j_0, j_1, j_2) with $j_0 + j_1 + j_2 = d_2$ and

$\underline{W}^J = X^{j_0}Y^{j_1}Z^{j_2}$. We can associate to the two curves D_1 and D_2 the point $(\dots, a_I, \dots, b_J, \dots) \in K^M$, where $M = \binom{d_1+2}{2} + \binom{d_2+2}{2}$. We say that a statement is true for two *general curves* D_1 and D_2 over k if, after fixing the degrees d_1 and d_2, the set of points $(\dots, a_I, \dots, b_J, \dots) \in K^M$ corresponding to the curves D_1 and D_2 satisfying the statement contains the complement of a thin set of K^M.

COROLLARY 3.10B. *Let $k = \mathbb{Q}$ or an imaginary quadratic field. If D_1 and D_2 are two general curves in $\mathbb{P}^2$ over k with* $\deg D_1 + \deg D_2 \geq 4$, *then $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ is arithmetically $\mathcal{O}_k$-hyperbolic.*

Proof of Theorem 3.9B. First, assume that one of (i), (ii), (iii), or (iv) holds. We need to show that $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ is arithmetically $\mathcal{O}_k$-hyperbolic. Let P_i, $i = 1, 2$, be irreducible homogeneous polynomials in three variables over k such that $D_i = [P_i = 0]$. Let $d_i = \deg P_i$ and d be the least common multiple of d_1 and d_2. Let R be a set of $\mathcal{O}_k$-integral points of $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$. It follows from the proof of Theorem 3.5B that R lies in a finite union of closed subsets defined by the form

$$P_2^{d/d_2} - cP_1^{d/d_1}, \qquad c \in k^*. \tag{7}$$

Let A be an irreducible factor of this polynomial over $k[X, Y, Z]$. If the curve $C = [A = 0]$ is (geometrically) reducible, then the k-rational points of C are contained in the intersection of the components of C, and hence the number of points in $C(k)$ is finite. Therefore, we may assume that the curve $C = [A = 0]$ is (geometrically) irreducible. Then it suffices to show that $C \setminus \{D_1 \cup D_2\}$ contains only finitely many $\mathcal{O}_k$-integral points.

We first make an observation. If $\mathfrak{p} \in C \cap \{D_1 \cup D_2\}$, then it follows from (7) that $\mathfrak{p} \in D_1 \cap D_2$. Since D_1 and D_2 are smooth and intersect transversally, and as A is a factor of (7), the local expansions of P_1 and P_2 around $\mathfrak{p}$ indicate that (i) when $d_1 = d_2$, C is smooth at $\mathfrak{p}$ and C intersects D_1 and D_2 transversally; (ii) when $d_1 < d_2$, C intersects D_2 at $\mathfrak{p}$ tangentially and does not have a common tangent with D_1 at $\mathfrak{p}$.

We now return to our assertions. By Siegel's theorem, we only need to consider the case when C intersects $D_1 \cup D_2$ in either a single point or two points. Assume that $C \cap (D_1 \cup D_2) = \{\mathfrak{p}\}$. From our observation, C and D_1 do not have any common tangent at $\mathfrak{p}$. Together with the assumption that D_1 is smooth, we have

$$(C, D_1)_{\mathfrak{p}} = m_{\mathfrak{p}}(C), \tag{8}$$

where $(C, D_1)_{\mathfrak{p}}$ is the intersection multiplicity of C and D_1 at $\mathfrak{p}$ and $m_{\mathfrak{p}}(C)$ is the multiplicity of C at $\mathfrak{p}$. On the other hand, Bézout's theorem implies that

$$(C, D_1)_{\mathfrak{p}} = \deg C \cdot d_1. \tag{9}$$

Since $m_{\mathfrak{p}}(C) \leq \deg C$ and equality only holds when C is a line, (8) and (9) imply that $d_1 = 1$ and C is a line. When $d_1 = 1$, we only need to consider when $d_2 \geq 3$. Since, in this case, we have assumed that $\mathfrak{p}$ is not a maximal inflexion point of D_2, it follows that C intersects D_2 at a second point, contrary to our assumptions.

We next consider the case when $C \cap (D_1 \cup D_2)$ consists of exactly two distinct points, say $\mathfrak{p}$ and $\mathfrak{q}$. If both points are k-rational, then $C \setminus \{\mathfrak{p}, \mathfrak{q}\}$ contains only finitely many $\mathcal{O}_k$-integral points by Theorem 3.3B(c). Therefore, we may assume that $[k(\mathfrak{p}) : k] \geq 2$. Since C and D_i are over k, it is easy to verify that $k(\mathfrak{p}) = k(\mathfrak{q})$ and $[k(\mathfrak{p}) : k] = 2$ (otherwise, $C \cap (D_1 \cup D_2)$ would contain more than two points). If $d_1 = d_2$, our geometric observation implies that $\mathfrak{p}$ and $\mathfrak{q}$ are smooth points on C and C intersects D_1 and D_2 transversally at $\mathfrak{p}$ and $\mathfrak{q}$. Therefore, $(C, D_i)_{\mathfrak{p}} = (C, D_i)_{\mathfrak{q}} = 1$ and Bézout's theorem implies that $d_i \cdot \deg C = 2$. This shows that $d_1 = d_2 = 2$ and $\deg C = 1$. In this case, we have assumed that $k = \mathbb{Q}$ and $k(\mathfrak{p}) = k(\mathfrak{q})$ is imaginary quadratic. Therefore, $C \setminus \{\mathfrak{p}, \mathfrak{q}\}$ contains only finitely many $\mathcal{O}_k$-integral points by Theorem 3.3B(c). If $d_1 < d_2$, similarly to (8), we have $(C, D_1)_{\mathfrak{p}} = m_{\mathfrak{p}}(C)$ and $(C, D_1)_{\mathfrak{q}} = m_{\mathfrak{q}}(C)$. Together with Bézout's theorem, this yields

$$d_1 \cdot \deg C = m_{\mathfrak{p}}(C) + m_{\mathfrak{q}}(C) \leq 2 \cdot \deg C, \tag{10}$$

and equality holds only if C is a line. Hence, we have either (i) $d_1 = 1$, or (ii) $d_1 = 2$, and C is a line. In the second case, we have also assumed that $k = \mathbb{Q}$ and $k(\mathfrak{p}) = k(\mathfrak{q})$ is imaginary quadratic. Therefore, in the second case, $C \setminus \{\mathfrak{p}, \mathfrak{q}\}$ contains only finitely many $\mathcal{O}_k$-integral points by Theorem 3.3B(c). It remains to consider the case when $d_1 = 1$. In this case, since C is a component of the curve defined by $P_2 - cP_1^{d_2}$ and D_2 is a smooth curve, we have $m_{\mathfrak{p}}(C) = m_{\mathfrak{q}}(C) = 1$. Therefore, (10) implies that $\deg C = 2$. For $d_2 \geq 2$, $C \setminus \{\mathfrak{p}, \mathfrak{q}\}$ contains only finitely many $\mathcal{O}_k$-integral points since, in this case, we have assumed that either $k = \mathbb{Q}$ and $k(\mathfrak{p}) = k(\mathfrak{q})$ is imaginary quadratic or C has no k-rational points (and $d_2 \geq 3$).

For the converse part, we need to consider the following cases:

(a) $d_1 = d_2 = 1$,
(b) $d_1 = 1$, $d_2 = 2$, and $D_1 \cap D_2$ consists of two k-rational points,
(c) $d_1 = 1$, $d_2 \geq 3$ and D_1 intersects D_2 at a k-rational maximal inflexion point of D_2,
(d) there is a point $\mathfrak{p} \in D_1 \cap D_2$ with $[k(\mathfrak{p}) : k] = 2$ and either k is imaginary quadratic or $k = \mathbb{Q}$ and $k(\mathfrak{p})$ is real quadratic, and
  (i) $d_1 = 1$, $d_2 = 2$, or
  (ii) $d_1 = 1$, $d_2 \geq 3$, and there is a conic C intersecting D_2 only at $\mathfrak{p}$ and its conjugate point and moreover $C(k)$ is not empty, or
  (iii) $d_1 = 2$, and the line L passing through $\mathfrak{p}$ and its conjugate point $\mathfrak{q}$ only intersects $D_1 \cap D_2$ at $\mathfrak{p}$ and $\mathfrak{q}$.

A regular function on $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ is of the form $G[X,Y,Z]/(P_1^{n_1}P_2^{n_2})$ where $G[X,Y,Z]$ is a homogeneous polynomial over k of degree $n_1d_1+n_2d_2$. Let T be a k-linear transformation in $\mathbb{P}^2$ and R a set of $\mathcal{O}_k$-integral points on $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$. It follows from Definition 2.3 that $T(R)$ is also a set of $\mathcal{O}_k$-integral points on $\mathbb{P}^2 \setminus \{D_1^T \cup D_2^T\}$, where D_1^T and D_2^T are the images of D_1 and D_2, respectively, under the transformation T.

In (a), (b) and (c), the intersection of D_1 and D_2 contains a k-rational point $\mathfrak{p}$. Then we can take a linear transformation over k and assume that $\mathfrak{p} = (0,0,1)$, $P_1 = X$, and the tangent line of D_2 at $\mathfrak{p} = (0,0,1)$ is $Y = 0$. Then $P_2 = Y$ in case (a), $P_2 = YG(X,Y,Z) + aX^{d_2}$, where $a \in k^*$ and $G(X,Y,Z)$ is a homogeneous polynomial over k of degree d_2-1, in cases (b) and (c). It is easy to check that for any $m \in \mathbb{Z}$, $P_1(1,1,m) = P_2(1,1,m) = 1$ in case (a) and $P_1(1,0,m) = 1$ and $P_2(1,0,m) = a \neq 0$ in cases (b) and (c). Therefore, $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ is not arithmetically $\mathcal{O}_k$-hyperbolic.

For the three cases in (d), we may make k-linear transformations and assume that $\mathfrak{p} = [0,\alpha,1]$ where $\alpha \notin k$ and $\alpha^2 = u \in \mathcal{O}_k$. We recall that the Pell equation $Y^2 - uZ^2 = 1$ has infinitely many integer solutions in $\mathcal{O}_k$ when k is imaginary quadratic or when $k = \mathbb{Q}$ and $\mathbb{Q}(\alpha)$ is real quadratic (cf. Theorem 4 in [9]). Let $\{(y_i,z_i)\}$ be an infinite set of $\mathcal{O}_k$-integral solutions to this Pell equation.

For case (i), we may again assume that $P_1 = X$. We also note that a conic over k passing though $\mathfrak{p}$ must be of the form $Y^2-uZ^2+X(aX+bY+cZ) = 0$. Therefore, we may make a linear transformation such that $P_2 = Y^2 - uZ^2 + aX^2$, $a \in k^*$. Then $P_1(1,y_i,z_i) = 1$ and $P_2(1,y_i,z_i) = 1+a$, which is not zero if $a \neq -1$. If $a = -1$, we have $P_1(1,2y_i,2z_i) = 1$ and $P_1(1,2y_i,2z_i) = 3$. Therefore, $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ is not arithmetically $\mathcal{O}_k$-hyperbolic.

For case (ii), we may again assume that $P_1 = X$ and the equation of the conic C intersecting $D_1 \cup D_2$ at $\mathfrak{p}$ and $\mathfrak{q}$ is $Q(X,Y,Z) = Y^2-uZ^2+aX^2 = 0$, $a \in k$. Since C intersects D_2 only at $\mathfrak{p}$ and its conjugate $\mathfrak{q}$, P_2 is of the form $cX^d + Q(X,Y,Z)G(X,Y,Z)$ where $G(X,Y,Z)$ is a polynomial over k and $c \in k^*$. Since $C(k) \neq \emptyset$ and C intersects $D_1 = [X = 0]$ only at $\mathfrak{p}$ and $\mathfrak{q}$, we may assume there exists a k-rational point $(1,y,z) \in C(k)$, i.e. $y^2 - uz^2 = -a$. Since the norm map $k(\alpha) \to k$ is multiplicative, it is easy to verify that the norm of $(y_i+\alpha z_i)(y+\alpha z) = y_iy + uz_iz + \alpha z_iy + \alpha y_iz$ equals $-a$. Therefore, $(y_iy + uz_iz)^2 - u(z_iy + y_iz)^2 + a = 0$. Then $P_1(1, y_iy + uz_iz, z_iy + y_iz) = 1$ and $P_2(1, y_iy + uz_iz, z_iy + y_iz) = c \neq 0$. Therefore, $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ is not arithmetically $\mathcal{O}_k$-hyperbolic.

For case (iii), we can also assume similarly that $P_1 = Y^2 - uZ^2 + aX^2$, $a \in k^*$, and the line L passing through $\mathfrak{p}$ and $\mathfrak{q}$ is defined by $X = 0$. Since the line L intersects $D_1 \cap D_2$ only at $\mathfrak{p}$ and $\mathfrak{q}$, P_2 is of the form $(Y^2-uZ^2)^{d_2/2} + XG(X,Y,Z)$, where $G(X,Y,Z)$ is a polynomial over k and

d_2 is an even integer. Then $P_1(0, y_i, z_i) = 1$ and $P_2(0, y_i, z_i) = 1$. Therefore, $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ is not arithmetically $\mathcal{O}_k$-hyperbolic. ∎

Proof of Corollary 3.10B. It is clear that the geometric conditions given in Theorem 3.9B hold for two general curves. Therefore, we only need to consider the algebraic assumptions for (ii) and (iv) in Theorem 3.9B. For (ii), we may make a k-linear transformation and assume that the line through $\mathfrak{p}$ and $\mathfrak{q}$ is given by $X = 0$. Then D_1 is defined by

$$P_1 = a_1Y^2 + a_2YZ + a_3Z^2 + X(a_4X + a_5Y + a_6Z),$$

and $\mathfrak{p}$ and $\mathfrak{q}$ are solutions of $a_1Y^2+a_2YZ+a_3Z^2 = 0$. Since D_2 only intersects $[X = 0]$ at $\mathfrak{p}$ and $\mathfrak{q}$, the polynomial defining D_2 must be the form

$$P_2 = b_1(a_1Y^2 + a_2YZ + a_3Z^2)^{d_2/2} + XG(X, Y, Z)$$

where $b_1 \neq 0$ and d_2 must be even. Clearly, the coefficients of D_1 and D_2 satisfy some algebraic equations which give a basic thin set of the first type.

We now consider (iv) in Theorem 3.9B. We may make a k-linear transformation and assume that $D_1 = [X = 0]$. Then, to find intersection points of D_1 and D_2, we consider $P_2(0, Y, 1) = a_nY^n + \cdots + a_1Y + a_0$. Clearly, it is an irreducible polynomial over $k(a_n, \ldots, a_0)$, and hence by Lemma 3.11, $\tau_nY^n + \cdots + \tau_1Y + \tau_0$ is irreducible over k for $(\tau_0, \ldots, \tau_n) \in k^{n+1}$ outside a thin set. Since the degree of D_2 is at least 3, this easily shows that for any intersection point $\mathfrak{p} \in D_1(\bar{k}) \cap D_2(\bar{k})$, we have $[k(\mathfrak{p}) : k] = \deg D_2 \geq 3$ if D_1 and D_2 are general curves in $\mathbb{P}^2$. ∎

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