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On the sets of uniqueness of a distribution function of $\{\xi(p/q)^n\}$

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1. Introduction. Let $\xi > 0$ and $\theta > 1$ be real numbers. The distribution of the sequence $(\{\xi\theta^n\})_{n\geq 0}$, where $\{x\}$ denotes the fractional part of x, is one of the intriguing problems in number theory.

For a fixed θ, it is known (Weyl [13]) that the sequence $\{\xi\theta^n\}$ is uniformly distributed in $[0,1]$ for almost all ξ. Similarly, it is well known (Koksma [6], see also Boyd [1]) that for a fixed ξ, the sequence $\{\xi\theta^n\}$ is uniformly distributed in $[0,1]$ for almost all real $\theta > 1$.

On the other hand, it is still not known whether $\{(3/2)^n\}$ is dense in $[0,1]$, let alone whether this sequence is uniformly distributed or not. However, Vijayaraghavan [12] showed that for any two integers $p > q \geq 2$ with $\gcd(p,q) = 1$, $\{(p/q)^n\}$ has infinitely many limit points. But, as was remarked by him, a remark which is still valid today, it is striking that one cannot even decide whether $[0,1/2)$ (or $[1/2,1)$) contains infinitely many limit points of the sequence $\{(3/2)^n\}$. Therefore, it would be interesting to have the following result:

$$\limsup_{n\to\infty}\{(3/2)^n\} - \liminf_{n\to\infty}\{(3/2)^n\} > 1/2,$$

which would imply that $\{(3/2)^n\}$ has limit points in both of the intervals $[0,1/2)$ and $[1/2,1)$.

In [3], Flatto, Lagarias and Pollington showed that for integers $p > q \geq 2$ with $\gcd(p,q) = 1$,

$$\limsup_{n\to\infty}\{\xi(p/q)^n\} - \liminf_{n\to\infty}\{\xi(p/q)^n\} \geq 1/p$$

for all real $\xi > 0$.

An idea taken from a paper of Mahler [9] is an essential ingredient of the above paper. Mahler [9] had proved that the set of Z-numbers is at most countable, where a *Z-number* α is defined by the requirement that

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$0 \leq \{\alpha(3/2)^n\} < 1/2$ for all $n \geq 0$. The question of existence of a Z-number is still open and is closely related (see [9], [8]) to the $3x+1$ problem. The connection between the sequence $\{(3/2)^n\}$ and Waring's problem is also well known (see [4], for example).

The questions taken up in this paper are mainly motivated by a paper of Strauch [10]. We refer to this paper for statements of several related conjectures and some results of Choquet [2] and Tijdeman [11] in those directions.

In order to state the problems considered in the present paper and their connection to Z-numbers, we shall need the concept of distribution functions of a sequence. We refer to [7] for details. A *distribution function* $g(x)$ is a real-valued, non-decreasing function defined on $[0,1]$ for which $g(0) = 0$ and $g(1) = 1$. Let $\varDelta = (x_n)_{n=1}^{\infty}$ be a sequence with $x_n \in [0,1)$. For any positive integer N and a subinterval $I \subset [0,1]$, we define the following counting function:

$$A(I; N; \varDelta) = \#\{x_n \mid 1 \leq n \leq N,\ x_n \in I\}.$$

A distribution function g is called a *distribution function of the sequence* $\varDelta$ if there exists an increasing sequence of positive integers $N_1, N_2, \ldots$ such that

$$\lim_{k\to\infty} \frac{A([0,x); N_k; \varDelta)}{N_k} = g(x) \quad \text{for every } x \in [0,1].$$

The sequence $\varDelta$ is said to have the *asymptotic distribution function* g if

$$\lim_{k\to\infty} \frac{A([0,x); k; \varDelta)}{k} = g(x) \quad \text{for every } x \in [0,1].$$

We note that in terms of distribution functions, the following statement will clearly imply non-existence of Z-numbers:

Suppose $g(x)$ is a distribution function of $\{\xi(3/2)^n\}$. If $g(x)$ is constant for all x in an interval $I \subset [0,1]$, then $|I| < 1/2$.

Thus the study of distribution functions of $\{\xi(p/q)^n\}$ becomes relevant. In [10], Strauch studied distribution functions of $\{\xi(3/2)^n\}$. For a distribution function g of a given sequence, we call a set $X \subset [0,1]$ a *set of uniqueness* of g if defining g on X uniquely determines it in all of $[0,1]$. In other words, if g_1 and g_2 are any two distribution functions of the sequence which agree on X, then they agree on $[0,1]$. Strauch [10] proved the following theorem.

THEOREM A. *Let*

$$I_1 = [0, 1/3], \qquad I_2 = [1/3, 2/3], \qquad I_3 = [2/3, 1].$$

Then for any i, j with $1 \leq i \neq j \leq 3$, the set $X = I_i \cup I_j$ is a set of uniqueness of any distribution function g of $\{\xi(3/2)^n\}$.

As an application of this result, Strauch showed (see Section 5 of [10]) that the distribution function g given by

$$g(x) = \begin{cases} x & \text{for } x \in [0, 2/3], \\ x^2 - (2/3)x + 2/3 & \text{for } x \in [2/3, 1], \end{cases}$$

is not a distribution function of $\{\xi(3/2)^n\}$ for any ξ.

In this paper, in our first theorem, we extend Theorem A as follows.

THEOREM 1. *Suppose g is a distribution function of $\{\xi(p/q)^n\}$ where $p > q > 1$ are positive integers with $\gcd(p, q) = 1$. Let*

$$I_i = \left(\frac{i-1}{p}, \frac{i}{p}\right) \quad \textit{and} \quad J_i = [0, 1] - I_i \quad \textit{for } 1 \le i \le p.$$

Further, assume that $p \ge q^2 - q$ if $j/q \in I_i$ for some j with $1 \le j < q$. Then $X = J_i$ for $1 \le i \le p$ is a set of uniqueness of g.

It follows that if $q = 2$, then for every odd integer p, $X = J_i$ is a set of uniqueness of any distribution function of $\{\xi(p/2)^n\}$.

As a consequence, in the spirit of the example given by Strauch, we obtain (see Section 4) a whole class of distribution functions which are not distribution functions of the sequence $\{\xi(p/q)^n\}$ for any $\xi > 0$.

Determining the existence of the asymptotic distribution function of sequences of the form $\{\xi\theta^n\}$ is rather difficult. As we have mentioned before, work of Weyl [13] establishes that for almost all ξ, $\{\xi\theta^n\}$ is uniformly distributed in $[0, 1]$ and hence has the asymptotic distribution function $g(x) = x$. In the other direction, Helson and Kahane [5] established the existence of uncountably many ξ such that $\{\xi\theta^n\}$ does not have an asymptotic distribution function where $\theta > 1$ is any fixed real number. Therefore, for positive integers p, q as in Theorem 1, the sequence $\{\xi(p/q)^n\}$, for uncountably many ξ, has no asymptotic distribution function and hence is not uniformly distributed. However, for each such ξ, Theorem 1 (with notations as in the theorem) rules out the possibility of all but finitely many elements of the sequence $\{\xi(p/q)^n\}$ lying in a single interval I_i for some fixed i, $1 \le i \le p$. Indeed, otherwise any distribution function (there exists at least one, by Helly's selection principle; see [7, Theorem 7.1], for instance) $g(x)$ of such $\{\xi(p/q)^n\}$ satisfies

$$g(x) = \begin{cases} 0 & \text{for } x \in [0, (i-1)/p], \\ 1 & \text{for } x \in [i/p, 1], \end{cases}$$

and therefore by our Theorem 1, $\{\xi(p/q)^n\}$ will then have exactly one distribution function which will have to be its asymptotic distribution function (see [7], for instance).

Our next theorem is the following.

Theorem 2. *Suppose g is a distribution function of $\{\xi(p/q)^n\}$ with p, q as in Theorem 1. Then any interval $[a, a+(p-1)/p] \subset [0,1]$ of length $(p-1)/p$ is a set of uniqueness of g.*

We observe that, restricting to the case $p = 3$, $q = 2$, the above theorem describes a different class of sets of uniqueness of distribution functions of $\{\xi(3/2)^n\}$ not covered by Theorem A of Strauch.

2. Preliminaries. Let $\Delta_{\theta,\xi} = \{\xi\theta^n\}$ be any sequence as described in the introduction. As remarked before, the set D of distribution functions of $\Delta_{\theta,\xi}$ is non-empty. Let $\varphi : [0,1] \to [0,1]$ be such that for every $x \in [0,1]$, $\varphi^{-1}([0,x))$ is expressible as the union of finitely many disjoint subintervals $I_i(x)$ of $[0,1]$ with endpoints $\alpha_i(x) \le \beta_i(x)$. For example, if $\varphi(x) = \{2x\}$, then

$$\varphi^{-1}([0,x)) = [0, x/2) \cup \left[\frac{1}{2}, \frac{x+1}{2}\right).$$

For any distribution function $g(x)$ we put

$$g_\varphi(x) = \sum_i (g(\beta_i(x)) - g(\alpha_i(x))).$$

For any sequence $\Delta = (x_n)_{n=1}^{\infty}$, $x_n \in [0,1]$ and $\varphi : [0,1] \to [0,1]$ as above, if $\varphi(\Delta)$ denotes the sequence $(\varphi(x_n))_{n=1}^{\infty}$, then we have (see [10, Proposition]):

Lemma 1. *Let $g(x)$ be a distribution function of Δ associated with the sequence of indices $N_1, N_2, \ldots$. Suppose each term x_n is repeated only finitely many times. Then $\varphi(\Delta)$ has the distribution function g_φ for the same sequence of indices $N_1, N_2, \ldots$. Further, every distribution function of $\varphi(\Delta)$ has this form.*

In this paper, we take $\varphi(x) = \varphi_t(x) = \{tx\}$ with t an integer > 1. Then

$$g_\varphi(x) = g\left(\frac{x}{t}\right) + g\left(\frac{x+1}{t}\right) + \cdots + g\left(\frac{x+t-1}{t}\right) - g\left(\frac{1}{t}\right) - \cdots - g\left(\frac{t-1}{t}\right).$$

The next lemma is analogous to Theorem 1 of [10].

Lemma 2. *Every distribution function g of $\{\xi(p/q)^n\}$ satisfies $g_{\varphi_p}(x) = g_{\varphi_q}(x)$ for $x \in [0,1]$.*

Proof. We have $\{q\{x\}\} = \{qx\}$. Hence

$$\{q\{\xi(p/q)^n\}\} = \{\xi(p^n/q^{n-1})\} = \{p\xi(p/q)^{n-1}\} = \{p\{\xi(p/q)^{n-1}\}\}.$$

Thus $\varphi_q(\{\xi(p/q)^n\})$ and $\varphi_p(\{\xi(p/q)^{n-1}\})$ form the same sequence and the conclusion follows by Lemma 1. ∎

3. Proof of the theorems

Proof of Theorem 1. We assume that $g(x)$ is a distribution function of $\{\xi(p/q)^n\}$ which is known on J_i for some i, $1 \le i \le p$. We need to show that $g(x)$ can be determined on I_i. From Lemma 2, we have

$$\sum_{i=0}^{q-1} g\left(\frac{x+i}{q}\right) - \sum_{i=1}^{q-1} g\left(\frac{i}{q}\right) = \sum_{i=0}^{p-1} g\left(\frac{x+i}{p}\right) - \sum_{i=1}^{p-1} g\left(\frac{i}{p}\right). \tag{1}$$

We consider the following two cases.

Case I: *The interval I_i does not contain j/q for any j, $1 \le j \le q-1$.* There exists j, $1 \le j \le q-1$, such that

$$\frac{j-1}{q} < \frac{i-1}{p} < \frac{i}{p} < \frac{j}{q}.$$

We note that for any $x \in [0,1]$, on the left hand side of (1) all the summands other than $g((x+j-1)/q)$ are known and similarly, all the summands on the right hand side of (1) are known except $g((x+i-1)/p)$. Let $r = pj - qi$, so that $0 < r < p-q$. If

$$x \in S_1 := \left[0, \frac{p-q-r}{p}\right],$$

then

$$\frac{x+j-1}{q} \le \frac{i-1}{p}$$

and so for such x, $g((x+j-1)/q)$ is known. Now, from (1), $g((x+i-1)/p)$ gets known when $x \in [0, (p-q-r)/p]$. Thus, $g(x)$ gets known in

$$R_1 := \left[\frac{i-1}{p}, \frac{i-1}{p} + \frac{p-q-r}{p^2}\right].$$

Recursively, in the nth step we take

$$x \in S_n := \left[0, \frac{p-q-r}{p} + \frac{q(p-q-r)}{p^2} + \cdots + \frac{q^{n-1}(p-q-r)}{p^n}\right]$$

so that $g(x)$ gets known in

$$R_n := \left[\frac{i-1}{p}, \frac{i-1}{p} + \frac{p-q-r}{p^2}\left(1 + \frac{q}{p} + \cdots + \frac{q^{n-1}}{p^{n-1}}\right)\right].$$

Letting $n \to \infty$, we see that $g(x)$ gets known in

$$\left[\frac{i-1}{p}, \frac{i-1}{p} + \frac{p-q-r}{p(p-q)}\right]. \tag{2}$$

Similarly, we observe that for

$$x \in S_1' := \left[\frac{p-r}{p}, 1\right],$$

we have

$$\frac{x+j-1}{q} \geq \frac{i}{p}$$

and hence by (1), $g(x)$ gets known in

$$R_1' := \left[\frac{i-1}{p} + \frac{p-r}{p^2}, \frac{i}{p}\right].$$

Therefore, by a similar recursive argument, we take x in

$$S_n' := \left[\left(\frac{p-q-r}{p} + \frac{q(p-q-r)}{p^2} + \cdots + \frac{q^{n-2}(p-q-r)}{p^{n-1}}\right) + \frac{(p-r)q^{n-1}}{p^n}, 1\right],$$

at the nth step for $n \geq 2$, so that $g(x)$ gets known in

$$R_n' := \left[\frac{i-1}{p} + \frac{p-q-r}{p^2}\left(1 + \frac{q}{p} + \cdots + \frac{q^{n-2}}{p^{n-2}}\right) + \frac{q^{n-1}(p-r)}{p^{n+1}}, \frac{i}{p}\right].$$

Thus, letting $n \to \infty$, we see that $g(x)$ gets known in

$$\left[\frac{i-1}{p} + \frac{p-q-r}{p(p-q)}, \frac{i}{p}\right]. \tag{3}$$

From (2) and (3), now $g(x)$ is known over I_i.

CASE II: *I_i contains j/q for some j, $1 \leq j \leq q-1$.* We have

$$\frac{i-1}{p} < \frac{j}{q} < \frac{i}{p}.$$

We assume that $p \geq q^2 - q$. Let $r = qi - pj$, so that $0 < r < q$. First, we wish to determine $g(j/q)$. We note that for any $x \in [0,1]$,

$$g\left(\frac{x+l}{p}\right) \quad \text{for } 0 \leq l \leq i-2,\ i \leq l \leq p-1$$

and

$$g\left(\frac{x+l}{q}\right) \quad \text{for } 0 \leq l \leq j-2,\ j+1 \leq l \leq q-1$$

are all known. Thus we need to know

$$g\left(\frac{x+i-1}{p}\right), \quad g\left(\frac{x+j-1}{q}\right), \quad g\left(\frac{x+j}{q}\right).$$

We put $x = 1 - r/q$. Then

$$g\left(\frac{x+i-1}{p}\right) = g\left(\frac{qi-r}{pq}\right) = g\left(\frac{j}{q}\right). \tag{4}$$

Next we take

$$g\left(\frac{x+j-1}{q}\right) = g\left(\frac{j}{q} - \frac{r}{q^2}\right).$$

Since $j/q < i/p$, we have $j/q \leq i/p - 1/pq$. Hence using the assumption that $p \geq q^2 - q$, we get

$$\frac{j}{q} - \frac{r}{q^2} \leq \frac{i}{p} - \frac{1}{pq} - \frac{1}{p+q} \leq \frac{i}{p} - \frac{p+q+pq}{pq(p+q)} \leq \frac{i-1}{p}.$$

Thus, $g((x+j-1)/q)$ is known. Next, we consider

$$g\left(\frac{x+j}{q}\right) = g\left(\frac{j+1}{q} - \frac{r}{q^2}\right).$$

Since $j/q > (i-1)/p$, we have $j/q \geq i/p - 1/p + 1/pq$. Hence

$$\frac{j+1}{q} - \frac{r}{q^2} \geq \frac{i}{p} - \frac{1}{p} + \frac{1}{pq} + \frac{1}{q} - \frac{q-1}{q^2} \geq \frac{i}{p} + \frac{p+q-q^2}{pq^2} \geq \frac{i}{p}.$$

Thus, $g((x+j)/q)$ is also known and hence from (4) and (1), $g(j/q)$ is determined. Let

$$R = \left[\frac{r}{p}, 1 - \frac{q-r}{p}\right].$$

We note that for any $x \in R$, all the summands appearing in (1) other than $g((x+i-1)/p)$ are known and hence $g((x+i-1)/p)$ gets determined. Hence we find that $g(x)$ is determined in

$$S := \left[\frac{i-1}{p} + \frac{r}{p^2}, \frac{i}{p} - \frac{q-r}{p^2}\right]. \tag{5}$$

We check that $j/q \in S$ since $p \geq q^2 - q$. Next, we consider x lying in the interval

$$R_1^{(0)} := \left[1 + \frac{qr}{p^2} - \frac{q-r}{p}, 1\right],$$

so that $g(x)$ gets determined in

$$S_1^{(0)} := \left[\frac{i}{p} + \frac{qr}{p^3} - \frac{q-r}{p^2}, \frac{i}{p}\right], \tag{6}$$

since $g(x)$ is determined over $S \cup J_i$.

Similarly, if we consider x lying in the interval

$$R_1^{(1)} := \left[0, \frac{r}{p} - \frac{q(q-r)}{p^2}\right],$$

$g(x)$ gets determined in

$$S_1^{(1)} := \left[\frac{i-1}{p}, \frac{i-1}{p} + \frac{r}{p^2} - \frac{q(q-r)}{p^3}\right]. \tag{7}$$

Now, we proceed recursively as follows. For $n \geq 1$, let

$$(8)\qquad \begin{aligned} R_{2n}^{(0)} &:= qS_{2n-1}^{(0)} - j, & S_{2n}^{(0)} &:= \frac{i-1}{p} + \frac{1}{p} R_{2n}^{(0)}, \\ R_{2n+1}^{(0)} &:= qS_{2n}^{(0)} - j + 1, & S_{2n+1}^{(0)} &:= \frac{i-1}{p} + \frac{1}{p} R_{2n+1}^{(0)}, \end{aligned}$$

and

$$(9)\qquad \begin{aligned} R_{2n}^{(1)} &:= qS_{2n-1}^{(1)} - j + 1, & S_{2n}^{(1)} &:= \frac{i-1}{p} + \frac{1}{p} R_{2n}^{(1)}, \\ R_{2n+1}^{(1)} &:= qS_{2n}^{(1)} - j, & S_{2n+1}^{(1)} &:= \frac{i-1}{p} + \frac{1}{p} R_{2n+1}^{(1)}. \end{aligned}$$

Letting $n \to \infty$, we see that the sequences

$$(S_{2n+1}^{(1)}), \quad (S, S_{2n}^{(0)}), \quad (S_{2n}^{(1)}), \quad (S_{2n+1}^{(0)})$$

cover respectively the intervals

$$\left[\frac{i-1}{p}, \frac{i-1}{p} + \frac{pr - q(q-r)}{p(p^2-q^2)}\right], \quad \left[\frac{i-1}{p} + \frac{pr - q(q-r)}{p(p^2-q^2)}, \frac{i}{p} - \frac{q-r}{p^2}\right],$$
$$\left[\frac{i}{p} - \frac{q-r}{p^2}, \frac{i}{p} - \frac{p(q-r) - qr}{p(p^2-q^2)}\right], \quad \left[\frac{i}{p} - \frac{p(q-r) - qr}{p(p^2-q^2)}, \frac{i}{p}\right]. \ \blacksquare$$

Proof of Theorem 2. Suppose $g(x)$ is known for $x \in [a, a + (p-1)/p]$. We have

$$0 \leq a \leq \frac{1}{p} < \frac{p-1}{p} \leq a + \frac{p-1}{p}.$$

For all $x \in [0,1]$, since

$$\frac{i}{p} \leq \frac{x+i}{p} \leq \frac{i+1}{p},$$

$g((x+i)/p)$ is known for $1 \leq i \leq p-2$. Also $g(i/p)$ is known for all $i = 1, \ldots, p-1$. Similarly, on the left hand side of (1), all the summands are known except $g(x/q)$ and $g((x+q-1)/q)$. Let

$$x \in A_1 := [qa, pa].$$

Then, for such an x,

$$a \leq \frac{x}{q} \leq \frac{pa}{q} = a + \frac{(p-q)a}{q} \leq a + \frac{p-1}{p}.$$

Hence for $x \in [qa, pa]$, $g(x/q)$ is known. Further, for $x \in [qa, pa]$, since $a \leq 1/p$, we have

$$\frac{x+q-1}{q} \leq \frac{ap+q-1}{q} = a + a\,\frac{p-q}{q} + \frac{q-1}{q} \leq a + \frac{p-1}{p},$$

so that

$$a \leq \frac{x+q-1}{q} \leq a + \frac{p-1}{p}$$

and hence $g((x+q-1)/q)$ is known.

Finally, for $x \in [qa, pa]$, $g((x+p-1)/p)$ is known since

$$a \le \frac{qa+p-1}{p} \le \frac{x+p-1}{p} \le a + \frac{p-1}{p}.$$

Thus, for $x \in [qa, pa]$, all the entries in (1) are known except for $g(x/p)$. Hence by (1), $g(x/p)$ gets known when $x \in [qa, pa]$. But $x \in [qa, pa]$ implies $x/p \in [qa/p, a]$. Thus $g(x)$ is now known in the interval $B_1 := [qa/p, a + (p-1)/p]$. Recursively, after n steps, taking $x \in A_n := [(q/p)^{n-1}qa, pa]$, $g(x)$ gets known for any x in the interval $B_n = [(q/p)^n a, a + (p-1)/p]$. Since $(q/p)^n a \to 0$ as $n \to \infty$, we see that by this process $g(x)$ gets known over the interval $[0, a + (p-1)/p]$. Now, by using Theorem 1, $g(x)$ is known in $[0,1]$. ∎

4. Remarks. We note that by the technique which is used to prove Theorem 1, one can derive the following general result.

If $g_1(x)$ and $g_2(x)$ are any two distribution functions satisfying (1) *and $g_1(x) = g_2(x)$ for $x \in J_i$ for some $i, 1 \le i \le p$ (J_i is as defined in the statement of Theorem* 1), *then $g_1(x) = g_2(x)$ for all $x \in [0,1]$.*

Now, as was remarked in the introduction, we can construct a whole class of distribution functions which are not distribution functions of the sequence $\{\xi(p/q)^n\}$ for any $\xi > 0$. Indeed, if we consider any function

$$g_1(x) = \begin{cases} x & \text{for } x \in [0, (p-1)/p], \\ h(x) & \text{for } x \in [(p-1)/p, 1], \end{cases}$$

where $h : [(p-1)/p, 1] \to [(p-1)/p, 1]$ is any non-decreasing function other than the identity map with $h((p-1)/p) = (p-1)/p$ and $h(1) = 1$, then $g_1(x)$ is clearly a distribution function. However, $g_1(x)$ cannot be a distribution function for the sequence $\{\xi(p/q)^n\}$ for any $\xi > 0$, for the following reason. First of all, by the consequence of Lemma 2, to be a distribution function for $\{\xi(p/q)^n\}$, g_1 must satisfy (1). Therefore, by the above result, taking $g_2(x) = x$, $x \in [0,1]$ (which clearly satisfies (1)) and observing that g_1 and g_2 agree on the interval $[0, (p-1)/p]$, we have $g_1(x) = g_2(x)$ for all $x \in [0,1]$, a contradiction to the choice of h.

We now pose a question related to a conjecture of Strauch [10], which says that every measurable set $X \subset [0,1]$ having measure at least $2/3$ is a set of uniqueness of any distribution function of $\{\xi(3/2)^n\}$ for any $\xi > 0$. Since Strauch also showed that each of the sets $Y = [2/9, 1/3] \cup [1/2, 1]$ and $Z = [0, 1/2] \cup [2/3, 7/9]$ is a set of uniqueness of any such distribution function and both Y and Z are of measure $11/18 < 2/3$, in light of our Theorem 2, it would be interesting to know whether there exists an interval I of measure less than $2/3$ such that I is a set of uniqueness of any distribution function of $\{\xi(3/2)^n\}$.

Finally, we observe that the following generalization of the above mentioned result of Strauch is not difficult to establish.

Let $q < p$ *and* $pq > p^2 - q^2$ *(and hence* $p < 2q$*). Then* $Y_1 := [0, 1 - 1/q] \cup [1 - 1/p, 1 - q/p^2]$ *or* $Z_1 := [q/p^2, 1/p] \cup [1/q, 1]$ *is a set of uniqueness of any distribution function of* $\{\xi(p/q)^n\}$ *where the measure of each of the sets* Y_1 *and* Z_1 *is* $1 + 1/p - 1/q - q/p^2 < 1 - 1/p$.

We note that the above result as well as our Theorem 1 include the case $p = 3$, $q = 2$. However, when $q \geq 3$, the cases where we assume that $p \geq q(q-1)$ in Theorem 1 are mutually exclusive from those considered in the above statement which requires $p < 2q$.

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