ACTA ARITHMETICA
130.1 (2007)

Hybrid mean value for a generalization of a problem of D. H. Lehmer

by

Huaning Liu and Wenpeng Zhang (Xi'an)

1. Introduction. Let $q > 2$ be an odd number and c be any integer with $(c, q) = 1$. For each integer a with $1 \le a \le q$ and $(a, q) = 1$, we know that there exists one and only one b with $1 \le b \le q$ such that $ab \equiv c \bmod q$. Let $N(q, c)$ be the number of cases in which a and b are of opposite parity, that is,

$$N(q,c) = \sideset{}{'}\sum_{a=1}^{q} \sideset{}{'}\sum_{\substack{b=1 \\ ab\equiv c \bmod q \\ 2\nmid a+b}}^{q} 1,$$

where $\sum_{a=1}^{\prime q}$ denotes summation over all a such that $(a, q) = 1$. For $c = 1$ and $q = p$ an odd prime, D. H. Lehmer [3] asked to find $N(p, 1)$ or at least to say something nontrivial about it. In [6] and [7], the second author proved that

$$N(q,1) = \frac{1}{2}\,\phi(q) + O(q^{1/2} d^2(q) \ln^2 q), \tag{1}$$

where $\phi(q)$ is the Euler function, and $d(q)$ is the divisor function. For any nonnegative integer n, let

$$N(q,1,n) = \sideset{}{'}\sum_{a=1}^{q} \sideset{}{'}\sum_{\substack{b=1 \\ ab\equiv 1 \bmod q \\ 2\nmid a+b}}^{q} (a-b)^{2n}.$$

2000 *Mathematics Subject Classification*: Primary 11L05.

Key words and phrases: Lehmer problem, Cochrane sums.

Research partially supported by the National Natural Science Foundation of China under Grants No. 60472068 and No. 10671155; Natural Science Foundation of Shaanxi province of China under Grant No. 2006A04; and the Natural Science Foundation of the Education Department of Shaanxi province of China under Grant No. 06JK168.

The second author [8] proved the following asymptotic formula:

$$N(q,1,n)=\frac{1}{(2n+1)(2n+2)}\,\phi(q)q^{2n}+O(4^n q^{2n+1/2}d^2(q)\ln^2 q).$$

For any fixed positive integer c with $(c,q)=1$, let

$$E(q,c)=N(q,c)-\frac{1}{2}\,\phi(q).$$

The second author [11] showed that for any odd prime p,

$$\sum_{c=1}^{p-1}|E(p,c)|^2=\frac{3}{4}\,p^2+O\left(p\exp\left(\frac{3\ln p}{\ln\ln p}\right)\right).$$

This proves the error term in (1) is the best possible. In [12], he found a close relation between the error terms $E(q,c)$ and the classical Kloosterman sums

$$K(m,n;q)=\sideset{}{'}\sum_{b=1}^{q} e\left(\frac{mb+n\overline{b}}{q}\right),$$

where $e(y)=e^{2\pi iy}$, and obtained the following hybrid mean value formula:

$$\sideset{}{'}\sum_{c=1}^{q} E(q,c)K(\overline{4}c,1;q)=\frac{4}{\pi^2}\,q\phi(q)\prod_{p\|q}\left(1-\frac{1}{p(p-1)}\right)+O(q^{3/2+\varepsilon}),$$

where $4\overline{4}\equiv 1 \bmod q$, ε is any fixed positive number, and $\prod_{p\|q}$ denotes the product over all prime divisors of q with $p\,|\,q$ and $p^2\nmid q$.

It is natural and interesting to study the hybrid mean value between the error terms in this problem of D. H. Lehmer and other arithmetic functions. Professor Todd Cochrane introduced a sum analogous to the Dedekind sum as follows:

$$C(h,q)=\sideset{}{'}\sum_{a=1}^{q}\left(\left(\frac{\overline{a}}{q}\right)\right)\left(\left(\frac{ah}{q}\right)\right),$$

where $\overline{a}$ is defined by the equation $a\overline{a}\equiv 1 \bmod q$ and

$$((x))=\begin{cases} x-[x]-1/2 & \text{if } x \text{ is not an integer,}\\ 0 & \text{if } x \text{ is an integer.}\end{cases}$$

Then he suggested studying the arithmetical properties and mean value distribution properties of $C(h,q)$. The second author and Y. Yi [14] proved that $|C(h,k)|\ll\sqrt{k}\,d(k)\ln^2 k$, and obtained some interesting hybrid mean value formulae between Cochrane sums and Kloosterman sums (see [9], [10] and [13]).

Now we consider a generalization of this problem of D. H. Lehmer. For any integer $k \geq 1$, let

$$M(p,k,c) = \sum_{\substack{a_1=1 \\ a_1\cdots a_k b \equiv c \bmod p \\ 2 \nmid a_1+\cdots+a_k+b}}^{p-1} \cdots \sum_{a_k=1}^{p-1} \sum_{b=1}^{p-1} (a_1+\cdots+a_k-b)^2,$$

$$E(p,k,c) = M(p,k,c) - \frac{(3k^2-5k+4)p-2k-2}{24}\, p(p-1)^k.$$

In this paper, we use the Fourier expansion for character sums and the mean value theorems for Dirichlet L-functions to study the hybrid mean value of $E(p,k,c)$ and the k-dimensional Cochrane sums

$$C(c,k,p) = \sum_{a_1=1}^{p-1} \cdots \sum_{a_k=1}^{p-1} \left(\left(\frac{\overline{a}_1}{p}\right)\right) \cdots \left(\left(\frac{\overline{a}_k}{p}\right)\right) \left(\left(\frac{a_1\cdots a_k c}{p}\right)\right),$$

and give two interesting asymptotic formulae. Namely, we prove the following:

Theorem 1. *For any prime $p \geq 3$, we have the asymptotic formula*

$$\sum_{c=1}^{p-1} E(p,1,c)C(c,1,p) = -\frac{5}{144}\, p^4 + O(p^{7/2+\varepsilon}),$$

where ε is any fixed positive number.

Theorem 2. *For any prime $p \geq 3$ and integer $k \geq 2$, we have*

$$\begin{aligned}
&\sum_{c=1}^{p-1} E(p,k,c)C(c,k,p) \\
&= -\frac{2^{k-2}(k^2-k+2)\zeta^{2k+1}(2)p^{k+3}}{\pi^{2k+2}} \Big[\sum_{i=0}^{k+1} C_{k+1}^i (-2)^i C_k(i)\Big] \\
&\quad \times \prod_{p_1 \nmid 2p} \left(1 - \frac{1-C_{2k}^k}{p_1^2}\right) \\
&\quad + \frac{2^{k-1}(k+1)p^{k+3}}{\pi^{2k+4}} \Big[\sum_{i=0}^{k} C_k^i (-2)^i \frac{D_k(2,i)}{2^{3i+1}} + \sum_{j=0}^{k} C_k^j (-2)^{j+3} \frac{D_k(2,j+1)}{2^{3j+4}}\Big] \\
&\quad \times \prod_{p_1 \nmid 2p} D_k(p_1,0) + O(p^{k+5/2+\varepsilon}),
\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function, $\prod_{p_1 \nmid 2p}$ denotes the product over *all different primes* p_1 *with* $p_1 \nmid 2p$,

$$C_k(i) = \frac{1}{2^{i+1}}\left[\left(1-\frac{1}{2^2}\right)C_{k+i}^k + \frac{C_{2k}^k}{2^2}\right],$$

$$D_k(p,i) = \sum_{n=0}^{\infty} \frac{C_{k+n}^k}{p^{4n}} \sum_{j=0}^{i+n} p^{2j} C_{k+j-1}^{k-1},$$

and $C_m^n = m!/n!(m-n)!$.

2. Some lemmas. To prove the theorems, we need the following lemmas.

LEMMA 1. *Let* χ *be a primitive character modulo* q *with* $\chi(-1) = -1$, *and* $q \geq 3$ *an odd number. Then*

$$\sum_{a=1}^{q} a\chi(a) = \frac{qi}{\pi}\,\tau(\chi)L(1,\overline{\chi}), \tag{2}$$

$$\sum_{a=1}^{q} (-1)^a \chi(a) = \frac{2(1-2\chi(2))i}{\pi}\,\tau(\chi)L(1,\overline{\chi}), \tag{3}$$

$$\sum_{a=1}^{q} (-1)^a a\chi(a) = \frac{q(1-2\chi(2))i}{\pi}\,\tau(\chi)L(1,\overline{\chi}), \tag{4}$$

$$\sum_{a=1}^{q} (-1)^a a^2\chi(a) = \frac{q^2(1-2\chi(2))i}{\pi}\,\tau(\chi)L(1,\overline{\chi}) + \frac{q^2(8\chi(2)-1)i}{\pi^3}\,\tau(\chi)L(3,\overline{\chi}) + O(q^2), \tag{5}$$

where $L(s,\chi)$ *is the Dirichlet L-function corresponding to* χ, $\tau(\chi) = \sum_{a=1}^{q} \chi(a)e(a/q)$ *is the Gauss sum, and* $|\tau(\chi)| = \sqrt{q}$.

Proof. For any odd character χ modulo q, it is easy to show that

$$\begin{aligned}\sum_{a=1}^{q} (-1)^a a\chi(a) &= \frac{q}{2}\sum_{a=1}^{q} (-1)^a \chi(a),\\ \sum_{b=1}^{q} (-1)^b \chi(b) &= 2\chi(2)\sum_{b=1}^{(q-1)/2} \chi(b).\end{aligned} \tag{6}$$

From [2] we have the identity

$$(1-2\chi(2))\sum_{c=1}^{q} c\chi(c) = \chi(2)q\sum_{c=1}^{(q-1)/2} \chi(c). \tag{7}$$

On the other hand, from Theorems 12.11 and 12.20 of [1] we know that if χ is a primitive character modulo q with $\chi(-1) = -1$, then

$$\frac{1}{q}\sum_{b=1}^{q} b\chi(b) = \frac{i}{\pi}\,\tau(\chi)L(1,\overline{\chi}). \tag{8}$$

This yields formulae (2)–(4).

For any odd primitive character χ modulo q, we also have

$$\begin{aligned}\sum_{a=1}^{q}(-1)^a a^2\chi(a) &= \sum_{\substack{a=1\\ 2|a}}^{q} a^2\chi(a) - \sum_{\substack{a=1\\ 2\nmid a}}^{q} a^2\chi(a)\\ &= \sum_{\substack{a=1\\ 2|a}}^{q} a^2\chi(a) - \sum_{\substack{a=1\\ 2|a}}^{q} (q-a)^2\chi(q-a)\\ &= 2\sum_{\substack{a=1\\ 2|a}}^{q} a^2\chi(a) - 2q\sum_{\substack{a=1\\ 2|a}}^{q} a\chi(a) + q^2\sum_{\substack{a=1\\ 2|a}}^{q}\chi(a).\end{aligned} \tag{9}$$

Noting that

$$\sum_{a=1}^{q} a\chi(a) = \sum_{\substack{a=1\\ 2|a}}^{q} a\chi(a) + \sum_{\substack{a=1\\ 2|a}}^{q}(q-a)\chi(q-a) = 2\sum_{\substack{a=1\\ 2|a}}^{q} a\chi(a) - q\sum_{\substack{a=1\\ 2|a}}^{q}\chi(a),$$

from (8) and (9) we get

$$\begin{aligned}\sum_{a=1}^{q}(-1)^a a^2\chi(a) &= 2\sum_{\substack{a=1\\ 2|a}}^{q} a^2\chi(a) - q\sum_{a=1}^{q} a\chi(a)\\ &= 8\chi(2)\sum_{a=1}^{q/2} a^2\chi(a) - \frac{q^2 i}{\pi}\,\tau(\chi)L(1,\overline{\chi}).\end{aligned} \tag{10}$$

Noticing the Fourier expansion for character sums which was first given by Pólya [5]:

$$\sum_{0<n\le qy}\chi(n) = \begin{cases} \dfrac{\tau(\chi)}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)\sin(2\pi ny)}{n} + O(1) & \text{if } \chi(-1)=1,\\ \dfrac{\tau(\chi)L(1,\overline{\chi})}{\pi i} - \dfrac{\tau(\chi)}{\pi i}\displaystyle\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)\cos(2\pi ny)}{n} + O(1) & \\ & \text{if } \chi(-1)=-1,\end{cases}$$

where χ is a primitive character modulo q, and $y > 0$ is a real number, by

Abel's identity we have

$$\sum_{n=1}^{q/2} n^2\chi(n) = \frac{q^2}{4}\sum_{0<n\le q/2}\chi(n) - 2\int_0^{q/2} u \sum_{0<n\le u}\chi(n)\,du$$

$$= \frac{q^2(\overline{\chi}(2)-2)i}{4\pi}\,\tau(\chi)L(1,\overline{\chi}) - 2\int_0^{q/2} u\sum_{0<n\le u}\chi(n)\,du$$

$$= \frac{q^2(\overline{\chi}(2)-2)i}{4\pi}\,\tau(\chi)L(1,\overline{\chi}) - 2q^2\int_0^{1/2} s\sum_{0<n\le qs}\chi(n)\,ds$$

$$= \frac{q^2(\overline{\chi}(2)-2)i}{4\pi}\,\tau(\chi)L(1,\overline{\chi})$$

$$- 2q^2\int_0^{1/2} s\left[\frac{\tau(\chi)L(1,\overline{\chi})}{\pi i} - \frac{\tau(\chi)}{\pi i}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)\cos(2\pi n s)}{n} + O(1)\right]ds$$

$$= \frac{q^2(\overline{\chi}(2)-1)i}{4\pi}\,\tau(\chi)L(1,\overline{\chi}) + 2q^2\,\frac{\tau(\chi)}{\pi i}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)}{n}\int_0^{1/2} s\cos(2\pi n s)\,ds + O(q^2)$$

$$= \frac{q^2(\overline{\chi}(2)-1)i}{4\pi}\,\tau(\chi)L(1,\overline{\chi}) + \frac{q^2\tau(\chi)}{2\pi^3 i}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)(\cos(\pi n)-1)}{n^3} + O(q^2)$$

$$= \frac{q^2(\overline{\chi}(2)-1)i}{4\pi}\,\tau(\chi)L(1,\overline{\chi}) - \frac{q^2\tau(\chi)}{\pi^3 i}\sum_{\substack{n=1\\ 2\nmid n}}^{\infty}\frac{\overline{\chi}(n)}{n^3} + O(q^2)$$

$$= \frac{q^2(\overline{\chi}(2)-1)i}{4\pi}\,\tau(\chi)L(1,\overline{\chi}) + \frac{q^2(\overline{\chi}(2)-8)}{8\pi^3 i}\,\tau(\chi)L(3,\overline{\chi}) + O(q^2),$$

so from (10) we obtain (5).

Lemma 2. *Let p be an odd prime. Then*

$$E(p,1,c) = \frac{1}{p-1}\sum_{\substack{\chi(-1)=1\\ \chi\ne\chi_0}}\overline{\chi}(c)\Big(\sum_{a=1}^{p}(-1)^a a\chi(a)\Big)^2$$

$$- \frac{1}{p-1}\sum_{\chi(-1)=-1}\overline{\chi}(c)\Big[\Big(\sum_{a=1}^{p} a\chi(a)\Big)^2$$

$$+ \Big(\sum_{a=1}^{p}(-1)^a a^2\chi(a)\Big)\Big(\sum_{b=1}^{p}(-1)^b\chi(b)\Big) - \Big(\sum_{a=1}^{p}(-1)^a a\chi(a)\Big)^2\Big]$$

$$+ O(p),$$

where $\sum_{\chi(-1)=1,\, \chi\neq\chi_0}$ *denotes summation over all nonprincipal even characters modulo* p, *and* $\sum_{\chi(-1)=-1}$ *denotes the summation over all odd characters modulo* p.

Proof. From the definition of $M(p,1,c)$ we get

$$M(p,1,c) = \sum_{\substack{a=1 \\ ab\equiv c \bmod p \\ 2\nmid a+b}}^{p-1} \sum_{b=1}^{p-1} (a-b)^2 = \frac{1}{2} \sum_{\substack{a=1 \\ ab\equiv c \bmod p}}^{p-1} \sum_{b=1}^{p-1} [1-(-1)^{a+b}](a-b)^2$$

$$= \frac{1}{2} \sum_{\substack{a=1 \\ ab\equiv c \bmod p}}^{p-1} \sum_{b=1}^{p-1} (a-b)^2 - \frac{1}{2} \sum_{\substack{a=1 \\ ab\equiv c \bmod p}}^{p-1} \sum_{b=1}^{p-1} (-1)^{a+b}(a-b)^2.$$

By the orthogonality relation for character sums modulo p we easily deduce

$$M(p,1,c) = \frac{p-2}{12}\, p(p-1) - \frac{1}{p-1} \sum_{\chi\neq\chi_0} \overline{\chi}(c) \Big(\sum_{a=1}^{p} a\chi(a)\Big)^2 + \frac{p-1}{4}$$
$$- \frac{1}{p-1} \sum_{\chi\neq\chi_0} \overline{\chi}(c)$$
$$\times \Big[\Big(\sum_{a=1}^{p} (-1)^a a^2 \chi(a)\Big)\Big(\sum_{b=1}^{p} (-1)^b \chi(b)\Big) - \Big(\sum_{a=1}^{p} (-1)^a a\chi(a)\Big)^2\Big].$$

Note that

$$\sum_{a=1}^{p} (-1)^b \chi(b) = 0 \quad \text{if } \chi(-1) = 1 \tag{11}$$

and

$$\sum_{a=1}^{p} a\chi(a) = 0 \quad \text{if } \chi(-1) = 1 \text{ and } \chi \neq \chi_0. \tag{12}$$

This yields the conclusion of Lemma 2.

LEMMA 3. *Let* p *be an odd prime and* $k \geq 2$ *be an integer. Then*

$$E(p,k,c)$$
$$= \frac{-1}{2(p-1)} \sum_{\chi(-1)=-1} \overline{\chi}(c) \Big[(k+1) \Big(\sum_{a=1}^{p} (-1)^a \chi(a)\Big)^k \Big(\sum_{b=1}^{p} (-1)^b b^2 \chi(b)\Big)$$
$$+ k(k-3) \Big(\sum_{a=1}^{p} (-1)^a a\chi(a)\Big)^2 \Big(\sum_{b=1}^{p} (-1)^b \chi(b)\Big)^{k-1}\Big].$$

Proof. From the definition of $M(p,k,c)$ we get

$$\begin{aligned}
M(p,k,c) &= \sum_{\substack{a_1=1 \\ a_1\cdots a_k b\equiv c \bmod p \\ 2\nmid a_1+\cdots+a_k+b}}^{p-1}\cdots\sum_{a_k=1}^{p-1}\sum_{b=1}^{p-1}(a_1+\cdots+a_k-b)^2 \\
&= \frac{1}{2}\sum_{\substack{a_1=1 \\ a_1\cdots a_k b\equiv c \bmod p}}^{p-1}\cdots\sum_{a_k=1}^{p-1}\sum_{b=1}^{p-1}[1-(-1)^{a_1+\cdots+a_k+b}](a_1+\cdots+a_k-b)^2 \\
&=: P(p,k,c)+E(p,k,c).
\end{aligned}$$

By the orthogonality relation for character sums modulo p we deduce

$$\begin{aligned}
P(p,k,c) &= \frac{1}{2}\sum_{\substack{a_1=1 \\ a_1\cdots a_k b\equiv c \bmod p}}^{p-1}\cdots\sum_{a_k=1}^{p-1}\sum_{b=1}^{p-1}(a_1+\cdots+a_k-b)^2 \\
&= \frac{1}{2(p-1)}\sum_{\chi \bmod p}\overline{\chi}(c)\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\sum_{b=1}^{p-1}\chi(a_1)\cdots\chi(a_k)\chi(b)(a_1+\cdots+a_k-b)^2 \\
&= \frac{1}{2(p-1)}\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\sum_{b=1}^{p-1}(a_1+\cdots+a_k-b)^2 \\
&= \frac{k+1}{2(p-1)}\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\sum_{b=1}^{p-1}b^2+\frac{k(k-3)}{2(p-1)}\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\sum_{b=1}^{p-1}a_1 b \\
&= \frac{k+1}{12}\,p(2p-1)(p-1)^k+\frac{k(k-3)}{8}\,p^2(p-1)^k \\
&= \frac{(3k^2-5k+4)p-2k-2}{24}\,p(p-1)^k.
\end{aligned}$$

On the other hand, from (11) we have

$$\begin{aligned}
E(p,k,c) &= -\frac{1}{2}\sum_{\substack{a_1=1 \\ a_1\cdots a_k b\equiv c \bmod p}}^{p-1}\cdots\sum_{a_k=1}^{p-1}\sum_{b=1}^{p-1}(-1)^{a_1+\cdots+a_k+b}(a_1+\cdots+a_k-b)^2 \\
&= -\frac{1}{2(p-1)}\sum_{\chi \bmod p}\overline{\chi}(c)\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\sum_{b=1}^{p-1}(-1)^{a_1+\cdots+a_k+b} \\
&\qquad\times\chi(a_1)\cdots\chi(a_k)\chi(b)(a_1+\cdots+a_k-b)^2
\end{aligned}$$

$$= \frac{-1}{2(p-1)} \sum_{\chi(-1)=-1} \overline{\chi}(c) \Big[(k+1)\Big(\sum_{a=1}^{p}(-1)^a\chi(a)\Big)^k\Big(\sum_{b=1}^{p}(-1)^b b^2\chi(b)\Big)$$
$$+ k(k-3)\Big(\sum_{a=1}^{p}(-1)^a a\chi(a)\Big)^2\Big(\sum_{b=1}^{p}(-1)^b\chi(b)\Big)^{k-1}\Big].$$

This proves Lemma 3.

Lemma 4. *Let p be an odd prime and h be an integer with $(h,p)=1$. Then*

$$C(h,k,p) = \frac{i^{k+1}}{\pi^{k+1}(p-1)} \sum_{\chi(-1)=-1} \overline{\chi}(h)\tau^{k+1}(\chi)L^{k+1}(1,\overline{\chi}).$$

Proof. From the orthogonality relation for characters modulo p we have

$$C(h,k,p) = \sum_{a_1=1}^{p-1} \cdots \sum_{a_k=1}^{p-1} \left(\left(\frac{\overline{a}_1}{p}\right)\right) \cdots \left(\left(\frac{\overline{a}_k}{p}\right)\right)\left(\left(\frac{a_1\cdots a_k h}{p}\right)\right) \tag{13}$$

$$= \frac{1}{p-1} \sum_{\chi \bmod p} \Big\{\sum_{a_1=1}^{p-1}\chi(a_1)\left(\left(\frac{a_1}{p}\right)\right)\Big\} \cdots \Big\{\sum_{a_k=1}^{p-1}\chi(a_k)\left(\left(\frac{a_k}{p}\right)\right)\Big\}$$
$$\times \Big\{\sum_{a_{k+1}=1}^{p-1}\chi(a_{k+1})\left(\left(\frac{a_{k+1}h}{p}\right)\right)\Big\}$$
$$= \frac{1}{p-1} \sum_{\chi \bmod p} \Big\{\sum_{a=1}^{p-1}\chi(a)\Big(\frac{a}{p}-\frac{1}{2}\Big)\Big\}^k \Big\{\overline{\chi}(h)\sum_{a_{k+1}=1}^{p-1}\chi(a_{k+1}h)\left(\left(\frac{a_{k+1}h}{p}\right)\right)\Big\}$$
$$= \frac{1}{p-1} \sum_{\chi \bmod p} \overline{\chi}(h)\Big\{\sum_{a=1}^{p-1}\chi(a)\Big(\frac{a}{p}-\frac{1}{2}\Big)\Big\}^{k+1}.$$

For $\chi(-1)=1$, it is easy to prove that

$$\sum_{a=1}^{p-1}\chi(a)\Big(\frac{a}{p}-\frac{1}{2}\Big) = 0. \tag{14}$$

If $\chi(-1)=-1$, then

$$\sum_{a=1}^{p-1}\chi(a)\Big(\frac{a}{p}-\frac{1}{2}\Big) = \frac{1}{p}\sum_{a=1}^{p-1}a\chi(a). \tag{15}$$

Combining (8), (13), (14) and (15) we immediately obtain the conclusion of Lemma 4.

LEMMA 5. *Let q, k, i be integers with $q \geq 3$, $k \geq 1$ and $i \geq 0$. Then we have the asymptotic formulae*

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(2^i)|L(1,\overline{\chi})|^{2k+2}$$
$$= C_k(i)\zeta^{2k+1}(2)\phi(q)\prod_{p|q}\left(1-\frac{1}{p^2}\right)^{2k+1}\prod_{p\nmid 2q}\left(1-\frac{1-C_{2k}^k}{p^2}\right)+O(q^{\varepsilon})$$

and

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(2^i)L^{k+1}(1,\chi)L^k(1,\overline{\chi})L(3,\overline{\chi}) = \frac{\phi(q)}{2^{3i+1}}D_k(2,i)\prod_{p\nmid 2q}D_k(p,0)+O(q^{\varepsilon}).$$

Proof. We only prove the second formula, since the first one can be deduced similarly. Let $d_k(n)$ be the kth divisor function (i.e., the number of positive integer solutions of the equation $n_1\cdots n_k = n$), and define $\tau_k(n) = \sum_{d|n} d^2 d_k(d)$. For any nonprincipal character χ modulo q, and parameter $N \geq q$, applying Abel's identity we have

$$L(s,\chi) = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s} = \sum_{1\leq n\leq N}\frac{\chi(n)}{n^s} + s\int_N^{\infty}\frac{\sum_{N<n\leq y}\chi(n)}{y^{s+1}}\,dy$$
$$= \sum_{1\leq n\leq N}\frac{\chi(n)}{n^s} + O\left(\frac{\sqrt{q}\log q}{N^s}\right).$$

Noting that for $(ab,q)=1$, by the orthogonality relations for character sums modulo q we get

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(a)\overline{\chi}(b) = \begin{cases} \frac{1}{2}\phi(q) & \text{if } a \equiv b \bmod q,\\ -\frac{1}{2}\phi(q) & \text{if } a \equiv -b \bmod q,\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\Sigma := \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(2^i)L^{k+1}(1,\chi)L^k(1,\overline{\chi})L(3,\overline{\chi})$$
$$= \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(2^i)\left[\sum_{1\leq n\leq N}\frac{\chi(n)}{n} + O\left(\frac{\sqrt{q}\log q}{N}\right)\right]^{k+1}$$
$$\times\left[\sum_{1\leq m_1\leq N}\frac{\overline{\chi}(m_1)}{m_1} + O\left(\frac{\sqrt{q}\log q}{N}\right)\right]^k\left[\sum_{1\leq m_2\leq N}\frac{\overline{\chi}(m_2)}{m_2^3} + O\left(\frac{\sqrt{q}\log q}{N^3}\right)\right]$$

$$= \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(2^i)\Bigg(\sum_{1\le n\le N}\frac{\chi(n)}{n}\Bigg)^{k+1}\Bigg(\sum_{1\le m_1\le N}\frac{\overline{\chi}(m_1)}{m_1}\Bigg)^{k}\Bigg(\sum_{1\le m_2\le N}\frac{\overline{\chi}(m_2)}{m_2^3}\Bigg)$$

$$+ O\bigg(\frac{q^{3/2}\log^{2k} N\log q}{N}\bigg)$$

$$= \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(2^i)\sum_{1\le n\le N^{k+1}}\frac{\chi(n)d_{k+1}(n)}{n}\sum_{1\le m\le N^{k+1}}\frac{\overline{\chi}(m)}{m}\sum_{d\mid m}\frac{d_k(m/d)}{d^2}$$

$$+ O\bigg(\frac{q^{3/2}\log^{2k} N\log q}{N}\bigg)$$

$$= \frac{\phi(q)}{2}\sum_{\substack{1\le n\le N^{k+1}\\ (n,q)=1}}\ \sum_{\substack{1\le m\le N^{k+1}\\ (m,q)=1\\ m\equiv 2^i n \bmod q}}\frac{d_{k+1}(n)}{n}\,\frac{\sum_{d\mid m} d_k(m/d)/d^2}{m}$$

$$- \frac{\phi(q)}{2}\sum_{\substack{1\le n\le N^{k+1}\\ (n,q)=1}}\ \sum_{\substack{1\le m\le N^{k+1}\\ (m,q)=1\\ m\equiv -2^i n \bmod q}}\frac{d_{k+1}(n)}{n}\,\frac{\sum_{d\mid m} d_k(m/d)/d^2}{m}$$

$$+ O\bigg(\frac{q^{3/2}\log^{2k} N\log q}{N}\bigg)$$

$$= \frac{\phi(q)}{2}\sum_{\substack{1\le n\le N^{k+1}\\ (n,q)=1}}\frac{d_{k+1}(n)}{2^i n^2}\sum_{d\mid 2^i n}\frac{d_k(2^i n/d)}{d^2} + O\bigg(\frac{q^{3/2}\log^{2k} N\log q}{N}\bigg)$$

$$+ O\bigg(\phi(q)\sum_{1\le n\le N^{k+1}}\frac{d_{k+1}(n)}{n}\,\varSigma_1\bigg)$$

$$+ O\bigg(\phi(q)\sum_{1\le n\le N^{k+1}}\frac{d_{k+1}(n)}{n}\,\varSigma_2\bigg)$$

where

$$\varSigma_1 = \sum_{\substack{\frac{1-2^i n}{q}\le l\le \frac{N^{k+1}-2^i n}{q}\\ l\neq 0}}\frac{\sum_{d\mid lq+2^i n} d_k((lq+2^i n)/d)/d^2}{lq+2^i n},$$

$$\varSigma_2 = \sum_{\frac{1+2^i n}{q}\le l\le \frac{N^{k+1}+2^i n}{q}}\frac{\sum_{d\mid lq-2^i n} d_k((lq-2^i n)/d)/d^2}{lq-2^i n}.$$

Hence

$$\Sigma = \frac{\phi(q)}{2^{3i+1}} \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{d_{k+1}(n)\tau_k(2^i n)}{n^4} + O\left(\frac{q^{3/2}\log^{2k} N \log q}{N}\right) + O(N^{\varepsilon}).$$

Now taking $N = q^{3/2}$, we immediately get

$$\Sigma = \frac{\phi(q)}{2^{3i+1}} \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{d_{k+1}(n)\tau_k(2^i n)}{n^4} + O(q^{\varepsilon}). \tag{16}$$

Noting that $d_k(p^r) = C_{k+r-1}^{k-1}$ (see [4, 6.4.12]), from the properties of multiplicative functions we have

$$\begin{aligned}\sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{d_{k+1}(n)\tau_k(2^i n)}{n^4} &= \sum_{\substack{n=1 \\ (n,2q)=1}}^{\infty} \frac{d_{k+1}(n)\tau_k(2^i n)}{n^4} \\ &+ \sum_{\substack{n=1 \\ (n,2q)=1}}^{\infty} \frac{d_{k+1}(2n)\tau_k(2^{i+1} n)}{(2n)^4} + \cdots + \sum_{\substack{n=1 \\ (n,2q)=1}}^{\infty} \frac{d_{k+1}(2^j n)\tau_k(2^{i+j} n)}{(2^j n)^4} + \cdots \\ &= \Bigg[\sum_{\substack{n=1 \\ (n,2q)=1}}^{\infty} \frac{d_{k+1}(n)\tau_k(n)}{n^4}\Bigg] \\ &\times \left[\tau_k(2^i) + \frac{d_{k+1}(2)\tau_k(2^{i+1})}{2^4} + \cdots + \frac{d_{k+1}(2^m)\tau_k(2^{i+m})}{2^{4m}} + \cdots\right] \\ &=: S_1 \cdot S_2.\end{aligned} \tag{17}$$

By applying Euler products we get

$$\begin{aligned} S_1 &= \prod_{p \nmid 2q} \left[1 + \frac{d_{k+1}(p)\tau_k(p)}{p^4} + \cdots + \frac{d_{k+1}(p^n)\tau_k(p^n)}{p^{4n}} + \cdots\right] \\ &= \prod_{p \nmid 2q} \left[1 + \cdots + \frac{C_{k+n}^k \sum_{j=0}^n p^{2j} C_{k+j-1}^{k-1}}{p^{4n}} + \cdots\right] \\ &= \prod_{p \nmid 2q} \left[\sum_{n=0}^{\infty} \frac{C_{k+n}^k}{p^{4n}} \sum_{j=0}^{n} p^{2j} C_{k+j-1}^{k-1}\right] \\ &= \prod_{p \nmid 2q} D_k(p,0). \end{aligned} \tag{18}$$

On the other hand, it is not hard to show that

$$
\begin{aligned}
(19)\qquad S_2 &= \tau_k(2^i) + \frac{d_{k+1}(2)\tau_k(2^{i+1})}{2^4} + \cdots + \frac{d_{k+1}(2^m)\tau_k(2^{i+m})}{2^{4m}} + \cdots \\
&= \sum_{j=0}^{i} 2^{2j} C_{k+j-1}^{k-1} + \frac{C_{k+1}^{k} \sum_{j=0}^{i+1} 2^{2j} C_{k+j-1}^{k-1}}{2^4} \\
&\quad + \cdots + \frac{C_{k+m}^{k} \sum_{j=0}^{i+m} 2^{2j} C_{k+j-1}^{k-1}}{2^{4m}} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{C_{k+n}^{k}}{2^{4n}} \sum_{j=0}^{i+n} 2^{2j} C_{k+j-1}^{k-1} = D_k(2,i).
\end{aligned}
$$

Then from (16)–(19) we get the second formula of Lemma 5.

LEMMA 6. *Let p be an odd prime. Then*

$$
\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1,\overline{\chi})|^4 = \frac{5\pi^4}{144}\, p + O(p^{\varepsilon}),
$$

$$
\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2)|L(1,\overline{\chi})|^4 = \frac{\pi^4}{36}\, p + O(p^{\varepsilon}),
$$

$$
\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(4)|L(1,\overline{\chi})|^4 = \frac{11\pi^4}{576}\, p + O(p^{\varepsilon}),
$$

$$
\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} L^2(1,\chi)L(1,\overline{\chi})L(3,\overline{\chi}) = \frac{7\pi^6}{4320}\, p + O(p^{\varepsilon}),
$$

$$
\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2)L^2(1,\chi)L(1,\overline{\chi})L(3,\overline{\chi}) = \frac{\pi^6}{1080}\, p + O(p^{\varepsilon}),
$$

$$
\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(4)L^2(1,\chi)L(1,\overline{\chi})L(3,\overline{\chi}) = \frac{11\pi^6}{23040}\, p + O(p^{\varepsilon}).
$$

Proof. Taking $k = 1$ in Lemma 5, we have

$$
\begin{aligned}
\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^i)|L(1,\overline{\chi})|^4 &= C_1(i)\zeta^3(2)p \prod_{p_1 \nmid 2p} \left(1 - \frac{1 - C_2^1}{p_1^2}\right) + O(p^{\varepsilon}) \\
&= C_1(i)\zeta^3(2)p \prod_{p_1 \nmid 2p} \left(1 + \frac{1}{p_1^2}\right) + O(p^{\varepsilon})
\end{aligned}
$$

$$= \frac{4}{5}\,\zeta^3(2)C_1(i)p\prod_{p_1}\left(1+\frac{1}{p_1^2}\right)+O(p^\varepsilon)$$

$$= \frac{4}{5}\,\frac{\zeta^4(2)}{\zeta(4)}\,C_1(i)p+O(p^\varepsilon)=\frac{1}{18}\,\pi^4C_1(i)p+O(p^\varepsilon).$$

Noting that

$$C_1(i)=\frac{1}{2^{i+1}}\left[\left(1-\frac{1}{2^2}\right)C_{1+i}^1+\frac{C_2^1}{2^2}\right]$$

$$=\frac{1}{2^{i+1}}\left[\frac{3(1+i)}{4}+\frac{2}{4}\right]=\frac{5+3i}{2^{i+3}},$$

we obtain the first three formulas of the conclusion.

On the other hand, from the definition of $D_k(p,i)$ we have

$$D_1(p,i)=\sum_{n=0}^{\infty}\frac{C_{1+n}^1}{p^{4n}}\sum_{j=0}^{i+n}p^{2j}$$

$$=\sum_{n=0}^{\infty}\frac{n+1}{p^{4n}}\,\frac{p^{2(n+i+1)}-1}{p^2-1}$$

$$=\frac{1}{p^2-1}\left[p^{2i+2}\sum_{n=0}^{\infty}\frac{n+1}{p^{2n}}-\sum_{n=0}^{\infty}\frac{n+1}{p^{4n}}\right]$$

$$=\frac{1}{p^2-1}\left[p^{2i+2}\,\frac{1}{(1-1/p^2)^2}-\frac{1}{(1-1/p^4)^2}\right].$$

Therefore

$$D_1(p_1,0)=\frac{1}{p_1^2-1}\left[p_1^2\,\frac{1}{(1-1/p_1^2)^2}-\frac{1}{(1-1/p_1^4)^2}\right]$$

$$=\frac{1}{(1-1/p_1^2)^3}\left[1-\frac{1}{p_1^2(1+1/p_1^2)^2}\right]$$

$$=\frac{1-1/p_1^6}{(1-1/p_1^2)^4(1+1/p_1^2)^2}$$

and

$$D_1(2,i)=\frac{1}{2^2-1}\left[2^{2i+2}\frac{1}{(1-1/2^2)^2}-\frac{1}{(1-1/2^4)^2}\right]$$

$$=\frac{2^{2i}\cdot 1600-256}{675}.$$

Then we get

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2^i)L^2(1,\chi)L(1,\overline{\chi})L(3,\overline{\chi}) = \frac{p}{2^{3i+1}}\, D_1(2,i) \prod_{p_1\nmid 2p} D_1(p_1,0) + O(p^{\varepsilon})$$
$$= \frac{2^{2i}\cdot 1600 - 256}{675\cdot 2^{3i+1}}\, p \prod_{p_1\nmid 2p} \frac{1-1/p_1^6}{(1-1/p_1^2)^4(1+1/p_1^2)^2} + O(p^{\varepsilon})$$
$$= \frac{2^{2i}\cdot 25 - 4}{2^{3i}\cdot 42}\, p \prod_{p_1} \frac{1-1/p_1^6}{(1-1/p_1^2)^4(1+1/p_1^2)^2} + O(p^{\varepsilon})$$
$$= \frac{2^{2i}\cdot 25 - 4}{2^{3i}\cdot 42}\, \frac{\zeta^2(2)\zeta^2(4)}{\zeta(6)}\, p + O(p^{\varepsilon}) = \frac{(2^{2i}\cdot 25-4)\pi^6}{2^{3i}\cdot 6\cdot 2160}\, p + O(p^{\varepsilon}).$$

This yields the last three formulas of Lemma 6.

3. Proof of the theorems. In this section, we complete the proof of the theorems. For any prime $p \ge 3$, note that

$$\tau(\chi)\tau(\overline{\chi}) = p\chi(-1).$$

Then from Lemmas 1, 2, 4, 6 and the orthogonality relation for character sums modulo p we get

$$\sum_{c=1}^{p-1} E(p,1,c)C(c,1,p)$$
$$= \frac{1}{\pi^2(p-1)} \sum_{\chi(-1)=-1} \tau^2(\overline{\chi})L^2(1,\chi)\Big[\Big(\sum_{a=1}^{p} a\chi(a)\Big)^2$$
$$+ \Big(\sum_{a=1}^{p}(-1)^a a^2\chi(a)\Big)\Big(\sum_{b=1}^{p}(-1)^b\chi(b)\Big) - \Big(\sum_{a=1}^{p}(-1)^a a\chi(a)\Big)^2\Big]$$
$$+ O(p^{5/2+\varepsilon})$$
$$= \frac{p^4}{\pi^4(p-1)} \sum_{\chi(-1)=-1} (-2+4\chi(2)-4\chi(4))|L(1,\overline{\chi})|^4$$
$$+ \frac{p^4}{\pi^6(p-1)} \sum_{\chi(-1)=-1} (2-20\chi(2)+32\chi(4))L^2(1,\chi)L(1,\overline{\chi})L(3,\overline{\chi})$$
$$+ O(p^{7/2+\varepsilon})$$
$$= -\frac{5}{144}\, p^4 + O(p^{7/2+\varepsilon}).$$

This proves Theorem 1.

For any prime $p \geq 3$ and integer $k \geq 2$, from Lemmas 1 and 3–5 we have

$$\begin{aligned}
\sum_{c=1}^{p-1} E(p,k,c)C(c,k,p) &= -\frac{i^{k+1}}{2\pi^{k+1}(p-1)} \sum_{\chi(-1)=-1} \tau^{k+1}(\overline{\chi})L^{k+1}(1,\chi) \\
&\quad \times \Big[(k+1)\Big(\sum_{a=1}^{p}(-1)^a\chi(a)\Big)^k \Big(\sum_{b=1}^{p}(-1)^b b^2\chi(b)\Big) \\
&\quad + k(k-3)\Big(\sum_{a=1}^{p}(-1)^a a\chi(a)\Big)^2 \Big(\sum_{b=1}^{p}(-1)^b\chi(b)\Big)^{k-1}\Big] \\
&= -\frac{2^{k-2}(k^2-k+2)p^{k+3}}{\pi^{2k+2}(p-1)} \\
&\quad \times \sum_{\chi(-1)=-1} (1-2\chi(2))^{k+1}|L(1,\overline{\chi})|^{2k+2} + O(p^{k+5/2+\varepsilon}) \\
&\quad + \frac{2^{k-1}(k+1)p^{k+3}}{\pi^{2k+4}(p-1)} \\
&\quad \times \sum_{\chi(-1)=-1} (1-2\chi(2))^k(1-8\chi(2))L^{k+1}(1,\chi)L^k(1,\overline{\chi})L(3,\overline{\chi}) \\
&= -\frac{2^{k-2}(k^2-k+2)p^{k+3}}{\pi^{2k+2}(p-1)} \sum_{i=0}^{k+1} C_{k+1}^i(-2)^i \sum_{\chi(-1)=-1} \chi(2^i)|L(1,\overline{\chi})|^{2k+2} \\
&\quad + O(p^{k+5/2+\varepsilon}) \\
&\quad + \frac{2^{k-1}(k+1)p^{k+3}}{\pi^{2k+4}(p-1)} \sum_{\chi(-1)=-1} \Big[\sum_{i=0}^{k} C_k^i(-2)^i\chi(2^i) + \sum_{j=0}^{k} C_k^j(-2)^{j+3}\chi(2^{j+1})\Big] \\
&\quad \times L^{k+1}(1,\chi)L^k(1,\overline{\chi})L(3,\overline{\chi}) \\
&= -\frac{2^{k-2}(k^2-k+2)\zeta^{2k+1}(2)p^{k+3}}{\pi^{2k+2}} \Big[\sum_{i=0}^{k+1} C_{k+1}^i(-2)^i C_k(i)\Big] \prod_{p_1 \nmid 2p}\Big(1-\frac{1-C_{2k}^k}{p_1^2}\Big) \\
&\quad + \frac{2^{k-1}(k+1)p^{k+3}}{\pi^{2k+4}} \\
&\quad \times \Big[\sum_{i=0}^{k} C_k^i(-2)^i \,\frac{D_k(2,i)}{2^{3i+1}} + \sum_{j=0}^{k} C_k^j(-2)^{j+3}\,\frac{D_k(2,j+1)}{2^{3j+4}}\Big] \prod_{p_1 \nmid 2p} D_k(p_1,0) \\
&\quad + O(p^{k+5/2+\varepsilon}).
\end{aligned}$$

This completes the proof of Theorem 2.

Acknowledgments. The authors express their gratitude to the referee for his very helpful and detailed comments on improving this paper.

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, New York, 1976.

[2] T. Funakura, *On Kronecker's limit formula for Dirichlet series with periodic coefficients*, Acta Arith. 55 (1990), 59–73.

[3] R. K. Guy, *Unsolved Problems in Number Theory*, Springer, New York, 1994.

[4] Chengdong Pan and Chengbiao Pan, *Elements of the Analytic Number Theory*, Science Press, Beijing, 1991.

[5] G. Pólya, *Über die Verteilung der quadratischen Reste und Nichtreste*, Göttinger Nachr. 1918, 21–29.

[6] W. P. Zhang, *A problem of D. H. Lehmer and its generalization (I)*, Compos. Math. 86 (1993), 307–316.

[7] —, *A problem of D. H. Lehmer and its generalization (II)*, ibid. 91 (1994), 47–56.

[8] —, *On the difference between a D. H. Lehmer number and its inverse modulo q*, Acta Arith. 68 (1994), 255–263.

[9] —, *On a Cochrane sum and its hybrid mean value formula*, J. Math. Anal. Appl. 267 (2002), 89–96.

[10] —, *On a Cochrane sum and its hybrid mean value formula (II)*, ibid. 276 (2002), 446–457.

[11] —, *A problem of D. H. Lehmer and its mean square value formula*, Japan. J. Math. 29 (2003), 109–116.

[12] —, *On a problem of D. H. Lehmer and Kloosterman sums*, Monatsh. Math. 139 (2003), 247–257.

[13] —, *A sum analogous to Dedekind sums and its hybrid mean value formula*, Acta Arith. 107 (2003), 1–8.

[14] W. P. Zhang and Y. Yi, *On the upper bound estimate of Cochrane sums*, Soochow J. Math. 28 (2002), 297–304.

Department of Mathematics
Northwest University
Xi'an, Shaanxi, P.R. China
E-mail: hnliumath@hotmail.com
wpzhang@nwu.edu.cn

Received on 16.12.2003
and in revised form on 15.9.2006 (4680)