ACTA ARITHMETICA
166.2 (2014)

Horizontal monotonicity of the modulus of the zeta function, L-functions, and related functions

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1. Introduction. Starting from the work of Riemann [16], the zeta function $\zeta(s)$ (as a function of the complex variable $s = \sigma + it$) has been primarily investigated in the vertical sense, especially in the critical strip $0 \leq \sigma \leq 1$ and on the critical line $\sigma = 1/2$. Questions related to the horizontal behaviour of $|\zeta(s)|$ have been considered by Saidak and Zvengrowski [17], and earlier by Spira [19]. Indeed, the opening page of the article on the Riemann zeta function in the Wolfram MathWorld [24] has a plot showing horizontal "ridges" of $|\zeta(\sigma + it)|$ for $0 < \sigma < 1$ and $1 < t < 100$. To quote from that article, "the fact that the ridges decrease monotonically for $0 < \sigma < 1/2$ is not a coincidence since it turns out that monotone decrease implies the Riemann hypothesis" (cf. [17] and [2]). Of course, strict decrease of the modulus of any continuous complex function f along any curve in the complex plane clearly implies that f can have no zero along that curve. In this note, among other things, we shall prove that the assertion that $|\zeta(\sigma + it)|$ is strictly decreasing in σ for $0 < \sigma < 1/2$ (with the minor additional condition $|t| \geq 8$) is in fact *equivalent* to the Riemann hypothesis, and also show (without the Riemann hypothesis) that $|\zeta(s)|$ is decreasing in σ in the region $\sigma < 0$, again for $|t| \geq 8$.

Recently a paper by Sondow and Dumitrescu [18] explores this question for the related Riemann ξ function, defined by

$$\xi(s) := (s-1) \cdot \Gamma(s/2+1) \cdot \pi^{-s/2} \cdot \zeta(s). \tag{1}$$

Here we shall consider this question for $\zeta(s)$ (as mentioned above), as well as for $\xi(s)$ and Euler's function η (cf. [5]); this function is also known as the Dedekind η function and is defined by $\eta(s) := (1 - 2^{1-s})\zeta(s)$ or, for $\sigma > 0$, by the alternating Dirichlet series $\eta(s) = \sum_{n\geq 1}(-1)^{n+1}/n^s$. Furthermore,

2010 *Mathematics Subject Classification*: Primary 11M06; Secondary 11M26.
Key words and phrases: Riemann zeta function, Dirichlet L-functions, monotonicity.

DOI: 10.4064/aa166-2-4

we shall also consider the same question for Dirichlet characters χ that are primitive (all relevant definitons are given in §2) and the corresponding functions $L(s,\chi)$ and $\xi(s,\chi)$. A recent paper of Srinivasan and Zvengrowski [20] also examines this question, for the Γ function, and another recent paper of Alzer [1], titled "Monotonicity properties of the Riemann zeta function," concerns monotonicity of a function related to the zeta function, but only along the real line. For completeness, let us quote the results in [18] and [20].

THEOREM 1.1 (Sondow–Dumitrescu). *The ξ function is increasing in modulus along every horizontal half-line lying in any open right half-plane that contains no zeros of ξ. Similarly, the modulus decreases along each horizontal half-line lying in any zero-free, open, left half-plane.*

THEOREM 1.2 (Srinivasan–Zvengrowski). *For any fixed t with $|t| > 5/4$, $|\Gamma(s)|$ is increasing in σ.*

Section 2 starts by quoting an elementary lemma from [20] that relates the horizontal increase or decrease of $|f(s)|$, for any holomorphic function f, to $\Re(f'(s)/f(s))$, the real part of the logarithmic derivative of f. Using this lemma we give a very short proof of the Sondow–Dumitrescu theorem (Theorem 2.5 and Corollary 2.6). We also show how a portion of this theorem was implicitly anticipated in a paper of Pólya [15] written in 1927. It is also related to work of Lagarias [12], Haglund [7], and others; again this is briefly discussed in Section 2. A short introduction is then given to Dirichlet characters and L-functions. The same results are then proved for the corresponding $\xi(s,\chi)$ function whenever the character χ is primitive and non-unit (see Theorem 2.5L and Corollary 2.6L).

Theorem 2.5 and Corollary 2.6 are equivalent to the theorem of Sondow and Dumitrescu, stated as Theorem 1.1 above. The second part of the corollary, which is the same as Corollary 1 in [18], is actually implicit, after a suitable interpretation, in Pólya's paper [15] which discusses the "Nachlass" of J. L. W. V. Jensen. Namely, following I′ on p. 18 of [15], and using $z = x+iy$ as in that reference, we consider the holomorphic function $F(z) = \xi(1/2-iz)$ $= \xi(1/2+y-ix)$. Note that $|F(z)| = |\xi(1/2+y-ix)| = |\xi(1/2+y+ix)|$, since $\xi(\overline{s}) = \overline{\xi(s)}$. The condition that all zeros of F are real is precisely the Riemann hypothesis; in fact, this was Riemann's original formulation. According to condition I′, this is equivalent to $\partial^2|\xi(1/2+y+ix)|^2/\partial y^2 \geq 0$. This implies that $|\xi(1/2+y+ix)|^2$ is a convex function of y. By symmetry it has zero derivative at $y = 0$, hence it is increasing for $y \geq 0$ and decreasing for $y \leq 0$. The same is then also true for $|\xi(1/2+y+ix)|$. And conversely, as already remarked before Corollary 2.6, these monotonicity properties imply the Riemann hypothesis.

The fact that $\Re(\xi'(s)/\xi(s)) > 0$ when $\sigma > 1$, and that the Riemann hypothesis is equivalent to the same statement for $\sigma > 1/2$ (cf. [18] or

Theorem 2.5 and Corollary 2.6 below), also appears in the 1999 paper of Lagarias [12] and the 1997 paper of Hinkkanen [8]. Combining this with Lemma 2.3 gives an immediate proof of Theorem 1.1. Another version of the Sondow–Dumitrescu result appears as a "known result" at the beginning of Section 6 of [7], this time for the related Ξ function (the horizontal monotonicity of ξ being equivalent to vertical monotonicity of Ξ), but no reference or proof is given.

In Section 3 we prove our next main result, namely

THEOREM 1.3. *For $|t| \geq 8$ and $\sigma < 1$, one has*

$$\Re\left(\frac{\eta'(s)}{\eta(s)}\right) < \Re\left(\frac{\zeta'(s)}{\zeta(s)}\right) < \Re\left(\frac{\xi'(s)}{\xi(s)}\right). \tag{2}$$

This relates the horizontal growth rates of all three functions under consideration. The subsequent main result, which follows as a corollary of (2) together with the results in Section 2, is now stated.

THEOREM 1.4. *The moduli of all three functions $\eta(s)$, $\zeta(s)$, and $\xi(s)$ are decreasing with respect to σ in the region $\sigma \leq 0$, $|t| \geq 8$. Extending this region to $\sigma \leq 1/2$, for any of the three functions, is equivalent to the Riemann hypothesis.*

The corresponding results for L-functions are stated below and also proved in Section 3, except that the Euler η function has no standard analogue here so is omitted. This is due to the fact that ζ has a pole at $s = 1$ which is removed by multiplication by $1-2^{s-1}$, thereby producing η, whereas the L-functions we are considering have no poles.

THEOREM 1.3L. *For $|t| \geq 8$, $\sigma < 1$, and any primitive Dirichlet character χ, one has*

$$\Re\left(\frac{L'(s,\chi)}{L(s,\chi)}\right) < \Re\left(\frac{\xi'(s,\chi)}{\xi(s,\chi)}\right).$$

THEOREM 1.4L. *With χ as above, the moduli of $L(s,\chi)$ and $\xi(s,\chi)$ are decreasing with respect to σ in the region $\sigma \leq 0$, $|t| \geq 8$. Extending this region to $\sigma \leq 1/2$, for either of these functions, is equivalent to the generalized Riemann hypothesis.*

Returning to $\zeta(s)$, the inequality (1) seems to indicate that in order to seek further results on monotonicity for $\sigma < 1/2$, the most promising of the three functions is η, and the least promising ξ. On the other hand, combining the monotonicity results for ζ with the Voronin Universality Theorem [23] for ζ (or for $\log\zeta$) seems to offer an approach to possibly showing that the Riemann hypothesis is false. We also note that the inequality $|t| \geq 8$ is essential. Slightly smaller numbers than 8 will also work, but for $|t| < 6.2897$ the conclusion of Theorem 3.4 is false for both ζ and η. Also for $\sigma > 1/2$, neither $|\zeta|$ nor $|\eta|$ is monotone (by "monotone" we always mean monotone with respect to σ).

Finally, in Section 4, we show that the main results, Theorems 1.4 and 1.4L, can be combined into a single omnibus theorem about monotonicity for the degree 1 Selberg class.

2. Monotonicity of $|\xi(s)|$ and $|\xi(s,\chi)|$. To measure the rate of change of $|f(s)|$ with respect to σ, the following elementary lemma is useful. For a proof see [20].

LEMMA 2.1. *For any holomorphic function f with $f(s) \neq 0$ in some open domain $\mathcal{D}$,*

$$\Re\left(\frac{f'(s)}{f(s)}\right) = \frac{1}{|f(s)|} \cdot \frac{\partial |f(s)|}{\partial \sigma}, \qquad s \in \mathcal{D}.$$

COROLLARY 2.2. *For $s \in \mathcal{D}$,*

$$\operatorname{sgn}\left(\frac{\partial |f(s)|}{\partial \sigma}\right) = \operatorname{sgn}\left(\Re\left(\frac{f'(s)}{f(s)}\right)\right).$$

The fact that Lemma 2.1 does not apply at a zero of f is not a problem towards our main objectives, as the next lemma shows.

LEMMA 2.3.

(a) *Let f be holomorphic in an open domain $\mathcal{D}$ and not identically zero. Suppose $\Re(f'(s)/f(s)) < 0$ for all $s \in \mathcal{D}$ such that $f(s) \neq 0$. Then $|f(s)|$ is strictly decreasing with respect to σ in $\mathcal{D}$, i.e. for each $s_0 \in \mathcal{D}$ there exists a $\delta > 0$ such that $|f(s)|$ is strictly decreasing with respect to σ on the horizontal interval from $s_0 - \delta$ to $s_0 + \delta$.*

(b) *Conversely, if $|f(s)|$ is decreasing with respect to σ in $\mathcal{D}$, then $\Re(f'(s)/f(s)) \leq 0$ for all $s \in \mathcal{D}$ such that $f(s) \neq 0$.*

Proof. (a) From Lemma 2.1 and Corollary 2.2 it clearly suffices to show this for those $s_0 = \sigma_0 + it_0 \in \mathcal{D}$, where $f(s_0) = 0$. Thanks to f being holomorphic and not identically 0, there exists $\delta > 0$ with $\{s : |s - s_0| < \delta\} \subset \mathcal{D}$ and with no further zeros of f in this open disc. Then using the next part of the hypothesis and Corollary 2.2, $|f(s)|$ is strictly decreasing with respect to σ on the two horizontal open intervals from $\sigma_0 - \delta + it_0$ to $\sigma_0 + it_0$, and from $\sigma_0 + it_0$ to $\sigma_0 + \delta + it_0$. Since $|f|$ is continuous in $\mathcal{D}$, a simple continuity argument shows that it must be strictly decreasing on the entire horizontal interval from $\sigma_0 - \delta + it_0$ to $\sigma_0 + \delta + it_0$.

(b) Conversely, we are assuming $\partial |f(s)|/\partial \sigma \leq 0$ in $\mathcal{D}$, so Lemma 2.1 implies that $\Re(f'(s)/f(s)) \leq 0$ at any $s \in \mathcal{D}$ for which $f(s) \neq 0$. ∎

Of course, analogous results hold for monotonic increase and $\Re(f'(s)/f(s)) > 0$. Combining Lemma 2.3 with the fact that a function can have no zeros in an open domain in which its modulus is strictly decreasing (increasing) with respect to σ gives the next result.

COROLLARY 2.4. *With the same hypotheses as in Lemma* 2.3(a), *f has no zeros in $\mathcal{D}$.*

Let us now apply the above to the Riemann ξ function and thereby give a short proof of Theorem 1.1. It is well known that $\xi(1-s) = \xi(s)$ and that $\xi(\overline{s}) = \overline{\xi(s)}$. Hence $|\xi(1/2-\sigma+it)| = |\xi(1/2+\sigma-it)| = |\xi(1/2+\sigma+it)|$, which shows that $|\xi|$ is symmetrical about the critical line $\sigma = 1/2$. So showing that $|\xi|$ is decreasing in a domain to the left of the critical line is equivalent to showing that it is increasing in the reflection of the same domain about the point $s = 1/2$, and this is what we shall show.

THEOREM 2.5. *Let σ_0 be greater than or equal to the real part of any zero of ξ. Then $|\xi(s)|$ is strictly increasing in the half-plane $\sigma > \sigma_0$.*

Proof. We start with the formula due to Hadamard [6] and von Mangoldt [13] (cf. also [11], (36), or simply take the logarithmic derivative of the final formula given in [4, §2.8])

$$\frac{\xi'(s)}{\xi(s)} = \sum_{\rho} \frac{1}{s-\rho}, \tag{3}$$

where the summation is taken over all zeros ρ of ξ (which, as is well known, lie in the critical strip $0 < \Re(\rho) < 1$), in conjugate pairs and in order of increasing $\Im(\rho)$. If any such zero is written as $\rho = \alpha + i\beta$, then by hypothesis $\sigma > \alpha$. It is then trivial to check that

$$\Re(1/(s-\rho)) = (\sigma-\alpha)/[(\sigma-\alpha)^2 + (t-\beta)^2] > 0,$$

hence $\Re(\xi'(s)/\xi(s)) > 0$ and by Corollary 2.4 $|\xi(s)|$ is increasing in σ, in the given half-plane $\sigma > \sigma_0$. ∎

Combining this theorem with well known facts about the zeros of ξ, and then employing the same reasoning used to obtain Corollary 2.4, gives the next result.

COROLLARY 2.6. *In the right* (*resp. left*) *half-plane $\sigma \geq 1$* (*resp. $\sigma \leq 0$*), *$|\xi|$ is increasing* (*resp. decreasing*). *The same is true for the right* (*resp. left*) *half-plane $\sigma \geq 1/2$* (*resp. $\sigma \leq 1/2$*) *if and only if the Riemann hypothesis is true.*

We shall now establish results similar to Theorem 2.5 and Corollary 2.6 for Dirichlet L-functions. For the definition of Dirichlet L-functions, the associated Dirichlet characters modulo q, and standard related definitions such as the "unit" (or "principal") character, "primitive" character, "even, odd" character, etc., we refer the reader to [14] or any other standard references on this material.

In particular, following [14, §5.3, 5.4], the corresponding symmetrized function $\xi(s,\chi)$ is defined by

$$\xi(s,\chi)=\left(\frac{q}{\pi}\right)^{(s+a)/2}\Gamma\left(\frac{s+a}{2}\right)L(s,\chi),\quad \text{where}\quad a=\begin{cases}0, & \chi \text{ even},\\ 1, & \chi \text{ odd}.\end{cases}\tag{4}$$

We assume henceforth that χ is a primitive character modulo $q>1$. Then just like the Riemann ξ function, $\xi(s,\chi)$ satisfies a simple functional equation

$$\xi(s,\chi)=w_\chi \xi(1-s,\overline{\chi}),\tag{5}$$

where the complex number w_χ is defined in [14, §5.3] and depends on a certain Gauss sum, also on whether χ is even or odd. For our purposes it suffices to simply quote [14, 5.3.2, 5.3.3], which establishes that $|w_\chi|=1$ for the characters under consideration, a simple fact but a key element of the proof of Corollary 2.6L.

Theorem 2.5L. *Let χ be any primitive Dirichlet character modulo $q>1$, let $\xi(s,\chi)$ be as described above, and let σ_0 be greater than or equal to the real part of any zero of $\xi(s,\chi)$. Then $|\xi(s,\chi)|$ is strictly increasing in the half-plane $\sigma>\sigma_0$.*

Proof. Following [14, §6.4], we have the Hadamard product formula

$$\xi(s,\chi)=e^{A+Bs}\prod_\rho\left(1-\frac{s}{\rho}\right)e^{s/\rho},\tag{6}$$

where the product is taken over all zeros ρ of $\xi(s,\chi)$ (which again lie in the critical strip $0<\Re(\rho)<1$), in conjugate pairs and in order of increasing $\Im(\rho)$. Furthermore $e^A=\xi(0,\chi)$ and $\Re(B)=-\sum_\rho \Re(1/\rho)$. Taking the logarithmic derivative yields

$$\frac{\xi'(s,\chi)}{\xi(s,\chi)}=B+\sum_\rho\frac{1}{s-\rho}+\sum_\rho\frac{1}{\rho}.\tag{7}$$

Now taking the real parts gives

$$\Re\left(\frac{\xi'(s,\chi)}{\xi(s,\chi)}\right)=\sum_\rho\Re\left(\frac{1}{s-\rho}\right),\tag{8}$$

and from this point on, the remainder of the proof is formally identical to that of Theorem 1.1, starting from (2). ∎

We now wish to also consider the left half-plane and prove the analogue of Corollary 2.6 for $\xi(s,\chi)$, but a little extra care is needed because whereas ξ has the same modulus at the four points $s,\overline{s},1-s,1-\overline{s}$, the same is not true in general for $\xi(s,\chi)$ (it does hold for real characters χ). Noting that $\overline{L(s,\chi)}=L(\overline{s},\overline{\chi})$, whence also $\overline{\xi(s,\chi)}=\xi(\overline{s},\overline{\chi})$, and recalling that $|w_\chi|=1$, we deduce from (4) that

$$|\xi(s,\chi)| = 1\cdot|\xi(1-s,\overline{\chi})| = |\overline{\xi(1-s,\overline{\chi})}| = |\xi(1-\overline{s},\chi)| = |\xi(1-\sigma+it,\chi)|.$$

The same reasoning used to obtain Corollary 2.6 can now be followed to obtain

COROLLARY 2.6L. *Let χ be as above. In the right* (*resp. left*) *half-plane $\sigma \geq 1$* (*resp. $\sigma \leq 0$*), *$|\xi(s,\chi)$ is increasing* (*resp. decreasing*). *The same is true for the right* (*resp. left*) *half-plane $\sigma \geq 1/2$* (*resp. $\sigma \leq 1/2$*) *if and only if the generalized Riemann hypothesis is true.*

3. Proofs of Theorems 1.3, 1.4, and 1.3L, 1.4L. For convenience we label the first inequality of Theorem 1.3 (formula (2)) as (A), and the second (B). To prove either of these we shall take the logarithmic derivatives of the formulae given for η,ξ in the Introduction, and then look at the real parts of these logarithmic derivatives. Again, for convenience, we will divide the proof into corresponding parts (A) and (B), and separately give two lemmas that will be of use.

LEMMA 3.1. *For $\sigma < 1$, one has*

$$\Re\left(\frac{1}{2^{s-1}-1}\right) < 0.$$

Proof. First note that $2^{s-1}-1 = 0$ if and only if $s = 1 + 2n\pi i/\log 2$, $n \in \mathbb{Z}$. In particular $2^{s-1}-1 \neq 0$ for $\sigma < 1$. Now

$$\Re\left(\frac{1}{2^{s-1}-1}\right) = \frac{2^{\sigma-1}\cos(t\log 2)-1}{|2^{s-1}-1|^2}. \tag{9}$$

The denominator of (9) is strictly positive since $\sigma < 1$. As for the numerator, one has $|2^{\sigma-1}\cos(t\log 2)| < |\cos(t\log 2)| \leq 1$, so the numerator is strictly negative. ∎

Proof of (A). From the formula for η (following (1)) given in the Introduction, it follows that $\log(\eta(s)) = \log(1-2^{1-s}) + \log(\zeta(s))$. Differentiating gives

$$\frac{\eta'(s)}{\eta(s)} = \frac{2^{1-s}\log 2}{1-2^{1-s}} + \frac{\zeta'(s)}{\zeta(s)} = \frac{\log 2}{2^{s-1}-1} + \frac{\zeta'(s)}{\zeta(s)}. \tag{10}$$

Taking the real parts of (11) and using Lemma 3.1 as well as $\log 2 > 0$ completes the proof. ∎

For the second inequality (B) it will be necessary to recall the digamma function $\psi(s) := \Gamma'(s)/\Gamma(s)$. A few of its properties that will be needed are listed in the next lemma. We also remark here that, as the proof will show, (B) actually holds for $s = \sigma + it$, $|t| \geq 8$, for all $\sigma \in \mathbb{R}$.

Lemma 3.2.

(i) $\psi(s) - \psi(1-s) = -\pi\cot(\pi s)$.
(ii) $|\Re(\psi(s)) - \Re(\psi(1-s))| < 3\pi e^{-2\pi t}$ *for* $t \geq 0.1$.
(iii) *For given* $0 < \theta < \pi$, *in the sector* $-\theta < \arg(s) < \theta$ *one has*

$$\psi(s) = \log s - \frac{1}{2s} + R_0'(s), \quad \textit{where} \quad |R_0'(s)| \leq \sec^3(\theta/2) \cdot \frac{B_2}{2|s|^2}, \tag{11}$$

with $B_2 = 1/6$ *being the second Bernoulli number.*
(iv) $|x/(x^2+t^2)| \leq 1/(2|t|)$ *for any* $x, t \in \mathbb{R}$, $t \neq 0$.
(v) *For any* $\sigma \in \mathbb{R}$ *and* $|t| \geq 4$, *we have* $\Re(\psi(s)) > 1.3091$.

Proof. Formula (i) is a simple consequence of Euler's reflection formula for the Γ function; it can be found e.g. in [21, p. 14].

Formula (ii) follows from (i) and an elementary estimate of $\Re(\cot z)$, since (i) implies

$$|\Re(\psi(s)) - \Re(\psi(1-s))| = |\Re(\psi(s) - \psi(1-s))| = \pi|\Re(\cot z)|, \tag{12}$$

where for convenience we set $\pi s = z = x + iy$. We outline the remaining steps towards proving (ii), which are essentially an exercise in calculus. First recall that

$$\cot z = \frac{\cos x \cosh y - i \sin x \sinh y}{\sin x \cosh y + i \cos x \sinh y}. \tag{13}$$

From (13) it is easy to derive

$$\Re(\cot z) = \frac{\sin(2x)}{b - \cos(2x) + 1} =: g_b(x), \quad \text{where} \quad b = 2\sinh^2 y > 0. \tag{14}$$

We claim that $|g_b(x)| < 3e^{-2y}$ when $y > (\log 3)/4$. Indeed, using elementary calculus one shows that $|g_b(x)|_{\max} = 1/\sqrt{b^2+2b}$, hence proving the claim reduces to showing $1/\sqrt{b^2+2b} < 3e^{-2y}$. Using the definition of b, this inequality reduces to $y > (\log 3)/4$ and the claim is thus proved. Finally, substituting $z = \pi s$, we obtain (ii) with $y = \pi t > (\log 3)/4$, i.e. $t > (\log 3)/(4\pi) = .08742\ldots$.

Formula (iii) is a special case ($n = 0$) of the Stirling series

$$\psi(s) = \log s - \frac{1}{2s} - \sum_{k=1}^{n} \frac{B_{2k}}{2ks^{2k}} + R_{2n}' \tag{15}$$

for the digamma function, together with the Stieltjes estimate for the error term (cf. [4, p. 114], or the original manuscript of Stieltjes [22])

$$|R_{2n}'| \leq \left(\sec\frac{\theta}{2}\right)^{2n+3} \left|\frac{B_{2n+2}}{(2n+2)s^{2n+2}}\right|.$$

Formula (iv) is equivalent to $0 \leq (|x| - |t|)^2$.

Finally, to prove (v) first consider $\sigma > 0$, $|t| \geq 4$. Then applying (iii) to the sector $-\theta = -\pi/2 < \arg(s) < \pi/2 = \theta$, we have $\theta/2 = \pi/4$, and thus

$$\Re(\psi(s)) = \log|s| - \frac{\sigma}{2|s|^2} + \Re(R_0'(s)), \tag{16}$$

where $|\Re(R_0'(s))| \leq |R_0'(s)| < 2\sqrt{2}/(2 \cdot 6 \cdot |s|^2)$. Using this estimate for the remainder term, as well as $|s| > 4$, (10) now gives (to five significant digits)

$$\Re(\psi(s)) \geq \log 4 - \frac{1}{16} - \frac{\sqrt{2}}{6 \cdot 16} = 1.3091, \tag{17}$$

where (iv) was used to give the $1/16$ estimate for the second term. To extend this result from $\sigma > 0$ to all $\sigma \in \mathbb{R}$, simply apply (i), which shows that $\Re(\psi(s))$ and $\Re(\psi(1-s))$ differ here by less than $3\pi e^{-8\pi} = 1.1462 \cdot 10^{-10}$, which is negligible to within the accuracy of five significant digits. This completes the proof of Lemma 3.2. ∎

We remark that the estimate obtained in Lemma 3.2(v) is close to best possible, which (using experimental evidence from MAPLE) equals 1.3837.

Proof of (B). The logarithmic derivative of formula (1) for $\xi(s)$ gives

$$\frac{\xi'(s)}{\xi(s)} = \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} + \frac{1}{2}\psi\left(\frac{s}{2}+1\right) - \frac{1}{2}\log\pi. \tag{18}$$

Hence, to complete the proof of (B), it suffices to show that

$$\Re\left(\frac{1}{s-1} + \frac{1}{2}\psi\left(\frac{s}{2}+1\right)\right) - \frac{1}{2}\log\pi > 0, \qquad |t| \geq 8. \tag{19}$$

Now, by Lemma 3.2(iv), the first term is greater than or equal to $-1/16$. By Lemma 3.2(v), we have $(1/2)\Re(\psi(z)) > 0.6545$. Thus the sum in question is greater than $-1/16 + 0.6545 - (\log\pi)/2 = .01964 > 0$. ∎

We can now prove Theorem 1.4 rather easily.

Proof of Theorem 1.4. For $\sigma \leq 0$, we have seen in the proof of Theorem 2.5 that $\Re(\xi'(s)/\xi(s)) < 0$. Combining this with the inequalities (2) shows that the same is true for ζ and η, thus all three are decreasing in modulus for $\sigma \leq 0$, $|t| \geq 8$. And the same argument used in Corollary 2.6 shows that extending this to the larger region $\sigma \leq 1/2$, $|t| \geq 8$, is equivalent to the Riemann hypothesis. ∎

Let us now turn to extending the above results to the Dirichlet L-function $L(s,\chi)$, for any primitive character χ modulo $q > 1$. We start with Theorem 1.3L, and remark that the proof runs quite parallel to that of inequality (B) above (e.g. compare (18) and (20)).

Proof of Theorem 1.3L. Taking the logarithmic derivative of (5) gives

$$\frac{\xi'(s,\chi)}{\xi(s,\chi)} = \frac{1}{2}\log q - \frac{1}{2}\log\pi + \frac{1}{2}\psi\left(\frac{s+a}{2}\right) + \frac{L'(s,\chi)}{L(s,\chi)}. \tag{20}$$

So, taking the real parts, to prove the theorem it suffices to show that

$$\log q - \log\pi + \Re\left(\psi\left(\frac{s+a}{2}\right)\right) > 0, \qquad |t| \geq 8. \tag{21}$$

Since $(1/2)(s+a) = (\sigma+a)/2 + it/2$, we can use Lemma 3.2(v) to obtain $\log q - \log\pi + \Re\big(\psi\big(\frac{s+a}{2}\big)\big) > \log 2 - \log\pi + 1.3091 > 0$, completing the proof. ∎

Theorem 1.4L follows from Theorems 2.5L and 1.3L in the same way as Theorem 1.4 followed from Theorems 2.5 and 1.3; it is not necessary to repeat the proof.

Finally, for any $q > 1$ consider the unit character χ_1 modulo q. Write the prime factorization of q as $q = \prod p_i^{n_i}$. It is easy to calculate that $L(s,\chi_1) = \prod(1-p_i^{-s})\cdot\zeta(s)$. Then it is easy to see that $L(s,\chi_1)$ will have infinitely many zeros along the vertical line $\sigma = 0$, of the form $s_{n,i} = t_{n,i} = 2\pi n/\log p_i$. Hence $|L(s,\chi_1)|$ cannot be monotone in σ for $\sigma < 1/2$ along any horizontal line $t = t_{n,i}$. However, it is decreasing for $\sigma < 0$, and this is easily proved from the above formula for $L(s,\chi_1)$ with the techniques we have been using. The situation for non-primitive characters is analogous and proved similarly.

4. Relation to the Selberg class. The definition of the Selberg class $\mathcal{S}$ of functions and their basic properties can be found in [3], [9], [14], and a number of other sources. For our purposes we simply recall that all such functions are meromorphic with at most a pole at $s = 1$, and each such function has a degree $d \geq 0$ (defined in terms of the Γ factors in the functional equation it satisfies). The Riemann zeta function and the Dirichlet functions $L(s,\chi)$, for χ a primitive character modulo $q > 1$, are well known to be in the Selberg class. Indeed the functional equations (1) and (5) show that they are in $\mathcal{S}_1$, the degree 1 Selberg class.

Since the functions $L(s,\chi)$ with χ as above have in fact no pole at all, i.e. are entire, the vertically shifted functions $L(s+i\theta,\chi)$, for any constant $\theta \in \mathbb{R}$, will also be in $\mathcal{S}_1$. This is not true for the Riemann zeta function since it does have a pole at $s = 1$. Referring to Theorem 5.2 in [9] and the material in the same section preceding this theorem, or to the original papers of Conrey and Ghosh [3], and Kaczorowski and Perelli [10], we see that $\mathcal{S}_1$ is in fact identical to the class of functions consisting of ζ and the $L(s+i\theta,\chi)$ (θ and χ as above). Since any horizontal monotonicity properties of a function are obviously unaffected by a vertical shift, we can now combine our main results, Theorems 1.4 and 1.4L, into the following omnibus theorem.

THEOREM 4.1. *Let f be any function in $\mathcal{S}_1$. Using the above notation, $|f(s)|$ is strictly decreasing in σ along any horizontal line $s = \sigma + it$, for any fixed t, with $|t| > 8$ for $\zeta(s)$ and $|t + \theta| > 8$ for $L(s + i\theta, \chi)$, and $\sigma < 0$. The continuation of this property for $\sigma < 1/2$ is equivalent to the Riemann hypothesis and the generalized Riemann hypothesis.*

We remark that it is also proved in [3, Theorem 3.1] and in [10] that the only function f in $\mathcal{S}$ with degree $0 \le d < 1$ is the constant function $f = 1$.

Acknowledgements. The authors wish to thank Professor Amir Akbary for many stimulating discussions related to this work, in particular about the Selberg class.

The first named author was supported by grant NSh-3856.2014.1 of the President of the Russian Federation.

The third named author was supported by a Discovery Grant from the Natural Sciences and Engineering Research Council of Canada.

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Received on 16.1.2014
and in revised form on 21.6.2014 (7702)