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Hecke action and the degree of the modular parameterization

by

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1. Introduction. Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian group of the first kind acting on the complex upper half plane $\mathcal{H}$ and X the resulting modular curve, i.e. the complex points of X can be described analytically as a compactification of the quotient $\Gamma \setminus \mathcal{H}$ by adding cusps of Γ. Further, suppose we have a non-constant map $\varphi : X \to E$, where E is some elliptic curve defined over $\mathbb{Q}$, i.e. E is modular. The pull-back of the unique (up to scalar multiplication) holomorphic differential on E is the differential $2\pi i f(\tau)d\tau$, where $f : \mathcal{H} \to \mathbb{C}$ is a holomorphic cuspform of weight 2 on Γ (we assume that the Manin constant is 1). Recent results ([1], [2], [6]–[8]) show that every elliptic curve E defined over $\mathbb{Q}$ is modular, i.e. there exist $\varphi : X_0(N) \to E$ and f related to E as above, the integer N being the conductor of E and f is a newform of level N. We refer to the map φ as "the modular parameterization."

In [9], Zagier computes the Petersson norm of the cusp-form f, related to E as above, in two different ways. He thus obtains a formula for the degree of the modular parameterization in terms of period integrals on the modular curve. In this paper we extend the ideas presented in [9] to obtain another formula for the degree of the modular parameterization and one for the Hecke eigenvalues (Fourier coefficients) of the the cusp form f. The degree of the map $\varphi : X_0(N) \to E$ is of interest for it was shown by Frey [3] that Szpiro's conjecture (relating the conductor and discriminant of elliptic curves over $\mathbb{Q}$) is equivalent to the degree of the map φ having polynomial growth in the level N. The Hurwitz formula gives no information about the degree since the Euler characteristic of E is zero.

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2. Fundamental domains and the homomorphism C_f. Let $\mathit{\Gamma} \subset \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian group of the first kind acting on the complex upper half plane $\mathcal{H}$ and $\mathcal{F} = \mathit{\Gamma} \setminus \mathcal{H}^*$ be a fundamental domain for the action ($\mathcal{H}^* = \mathcal{H} \cup \{\text{cusps of } \mathit{\Gamma}\}$). We view $\mathcal{F}$ as a hyperbolic polygon with vertices being interior or boundary points of $\mathcal{H}$ and having a finite number of edges which are identified in pairs in $\mathcal{F}$. The vertices $\{P_j\}_{j\in J}$ are labeled in such a way that P_{j+1} is the successor of P_j in the natural orientation. Let e_j denote the edge P_jP_{j+1}, e_{j^*} the edge with which it gets identified and $\gamma_j \in \mathit{\Gamma}$ the element that identifies them. The map $* : J \to J$ is an involution on J and the matrices γ_j are generators of $\mathit{\Gamma}$ satisfying $\gamma_{j^*} = \gamma_j^{-1}$. Henceforth we fix these matrices $\{\gamma_j\}_{j\in J}$ to be our preferred set of generators for $\mathit{\Gamma}$.

We have $\gamma_j(P_j) = P_{j^*+1}$ and the map $T : J \to J$ sending $j \mapsto j^* + 1$ breaks J up into finitely many orbits $[j] = \{j = T^t j, Tj, T^2 j, \ldots, T^{t-1} j\}$ in such a way that two vertices P_j and $P_{j'}$ are identified in $\mathcal{F}$ iff j and j' are in the same orbit. We pick a base point j_0 in each orbit and define a partial order on J by $j \prec j'$ if j and j' belong to the same orbit and $j = T^a j_0, j' = T^b j_0$ with $0 \le a < b < t =$ the size of the orbit.

The following construction allows us to pass from holomorphic cusp-forms of weight 2 for $\mathit{\Gamma}$ (denoted $S_2(\mathit{\Gamma})$) to homomorphisms from $\mathit{\Gamma}$ to the additive complex numbers $\mathbb{C}$. Fix $f \in S_2(\mathit{\Gamma})$, $\tau_0 \in \mathcal{H}^*$ and put

$$\mathit{\Phi}_f(\tau) = 2\pi i \int_{\tau_0}^{\tau} f(\xi)\, d\xi.$$

Since f is holomorphic $\mathit{\Phi}_f(\tau)$ is well defined and for $\gamma \in \mathit{\Gamma}$, the invariance of $f(\tau)d\tau$ gives

$$\mathit{\Phi}_f(\gamma(\tau)) - \mathit{\Phi}_f(\tau) = 2\pi i \int_{\tau_0}^{\gamma(\tau_0)} f(\xi)\, d\xi = C_f(\gamma).$$

It is easy to show (see [4]) that $C_f(\gamma)$ is independent of base-point τ_0 and it defines a homomorphism $C_f : \mathit{\Gamma} \to \mathbb{C}$. Its kernel contains all elliptic and parabolic elements of $\mathit{\Gamma}$ as well as commutators.

3. The Petersson inner product. For f and $g \in S_2(\mathit{\Gamma})$ their *Petersson inner product* $\langle f, g\rangle$ is defined to be

$$\langle f, g\rangle = \frac{i}{2} \iint_{\mathcal{F}} f(\tau)\overline{g(\tau)}\, d\tau \wedge \overline{d\tau}.$$

In [9], Zagier computes the Petersson norm of f in terms of the values of $C_f(\gamma_j)$ for the preferred generators $\{\gamma_j\}_{j\in J}$ of (see §2) $\mathit{\Gamma}$. We show that his method works for arbitrary f and $g \in S_2(\mathit{\Gamma})$ and in so doing obtain

Theorem 1. *With the above notation, the Petersson inner product of f and g in $S_2(\Gamma)$ is*

$$\langle f,g\rangle = \frac{i}{16\pi^2}\Big(\sum_{j\in J} C_f(\gamma_j)\overline{C_g(\gamma_j)} + 2\sum_{\substack{j,j'\in J\\ j\prec j'}} C_f(\gamma_j)\overline{C_g(\gamma_{j'})}\Big). \tag{1}$$

Remark. Let $f = g$ in (1) and keeping in mind that $\|f\|^2$ is real we recover the formula in [9], i.e.

$$\|f\|^2 = \frac{1}{8\pi^2}\sum_{\substack{j,j'\in J\\ j\prec j'}} \operatorname{Im}(C_f(\gamma_j)\overline{C_f(\gamma_j')}). \tag{2}$$

On the other hand, assuming that we have $\varphi : X \to E = \mathbb{C}/\Lambda$ (see §1) for some lattice $\Lambda \subset \mathbb{C}$ then the holomorphic differential dz on $\mathbb{C}$ being Λ-invariant defines a form on E. The pull-back $\varphi^*(dz)$ is of the form $2\pi i f(\tau)d\tau$ for some $f \in S_2(\Gamma)$ and we have

$$\|f\|^2 = \frac{i}{2}\int_{\Gamma_0(N)\backslash\mathcal{H}} f(\tau)\,d\tau \wedge \overline{f(\tau)}\,\overline{d\tau}$$

$$= \frac{i}{8\pi^2}\int_{\Gamma_0(N)\backslash\mathcal{H}} \varphi^*(dz)\wedge\overline{\varphi^*(dz)} = \frac{i}{8\pi^2}\deg(\varphi)\int_E dz\wedge\overline{dz},$$

$$\|f\|^2 = \frac{\deg(\varphi)}{4\pi^2}\operatorname{Vol}(E), \tag{3}$$

where $\operatorname{Vol}(E)$ is the area of the fundamental period parallelogram for the lattice Λ.

Equations (2) and (3) are combined in [9] to give the following formula:

$$\deg(\varphi) = \frac{1}{2\operatorname{Vol}(E)}\sum_{\substack{j,j'\in J\\ j\prec j'}} \operatorname{Im}(C_f(\gamma_j)\overline{C_f(\gamma_j')}).$$

4. The action of the Hecke operators on C_f and explicit formulas for the degree of the modular parameterization and Hecke eigenvalues of f. Fix $f \in S_2(\Gamma_0(N))$ with Fourier expansion $f(\tau) = \sum_{n\ge1} a_n e^{2\pi i n\tau}$, and for each prime p with $(p,N)=1$, let $T(p)$ be the weight 2 Hecke operator. The operator $T(p)$ acts on the cusp-form f as follows:

$$T(p)(f(\tau)) = \frac{1}{p}\sum_{l=0}^{p-1} f\Big(\frac{\tau+l}{p}\Big) + pf(p\tau).$$

Further, let the matrices $\{\gamma_j\}_{j\in J}$ be the preferred generators of the Hecke congruence subgroup $\Gamma_0(N)$. Then, for a fixed prime p not dividing N, we have

THEOREM 2.

$$C_{T(p)f}(\gamma_j) = \sum_{k\in J} n_{p,k,j} C_f(\gamma_k).$$

Here $n_{p,k,j} = \sum_{l=0}^{p-1}(m_{k,l} + m'_{k,j,l}) + m''_{k,j}$ *and the integers* $m_{k,l}$, $m'_{k,j,l}$ *and* $m''_{k,j}$ *are defined as follows. For a fixed* $j \in J$, *let* A_l, $B_{j,l}$ *and* D_j *denote the matrices in* $\Gamma_0(N)$ *that map* 0 *to* l/p, $(\gamma_j(0)+l)/p$ *and* $p\gamma_j(0)$ *respectively. Write these matrices as a product of the standard generators*, *i.e.*

$$A_l = \prod_{\substack{i=1\\(k\in J)}} \gamma_{k_i}^{\alpha_{k_i,l}}, \qquad B_{j,l} = \prod_{\substack{i=1\\(k\in J)}} \gamma_{k_i}^{\beta_{k_i,j,l}} \quad \text{and} \quad D_j = \prod_{\substack{i=1\\(k\in J)}} \gamma_{k_i}^{\delta_{k_i,j}}$$

and set

$$m_{k,l} = \sum_{k_i=k} \alpha_{k_i,l}, \qquad m'_{k,j,l} = \sum_{k_i=k} \beta_{k_i,j,l} \quad \text{and} \quad m''_{k,j} = \sum_{k_i=k} \delta_{k_i,j}.$$

REMARK. Note that the coefficients $n_{p,k,j}$ do not depend on the cusp-form f.

Combining Theorem 2 and equation (2), with the added assumption that the function f is a newform of level N, results in

$$\langle T(p)f, f\rangle = \frac{1}{8\pi^2} \sum_{\substack{j,j'\in J\\ j\prec j'}} \operatorname{Im}(C_{T(p)f}(\gamma_j)\overline{C_f(\gamma'_j)}),$$

$$a_p\|f\|^2 = \frac{1}{8\pi^2} \sum_{\substack{j,j'\in J\\ j\prec j'}} \sum_{k\in J} n_{p,k,j} \operatorname{Im}(C_f(\gamma_k)\overline{C_f(\gamma'_j)}). \tag{4}$$

Now equation (3) coupled with (4) results in the following two corollaries.

COROLLARY 1. *The pth Hecke eigenvalue* (*Fourier coefficient*) *of the cusp-form* f *is*

$$a_p = \frac{1}{2\deg(\varphi)\mathrm{Vol}(E)} \sum_{\substack{j,j'\in J\\ j\prec j'}} \sum_{k\in J} n_{p,k,j} \operatorname{Im}(C_f(\gamma_k)\overline{C_f(\gamma_{j'})}). \tag{5}$$

COROLLARY 2. *For each prime* p *we have*

$$\deg(\varphi) = \frac{1}{2a_p\mathrm{Vol}(E)} \sum_{\substack{j,j'\in J\\ j\prec j'}} \sum_{k\in J} n_{p,k,j} \operatorname{Im}(C_f(\gamma_k)\overline{C_f(\gamma_{j'})}). \tag{6}$$

REMARKS. In [5] Goldfeld gives an algorithm for computing the numbers $C_f(\gamma)$ for $f \in S_2(\Gamma_0(N))$ and $\gamma \in \Gamma_0(N)$ if the first N^2 Fourier coefficients of f are known. Moreover, one can compute the degree of the map φ without knowledge of the Fourier coefficients and vice-versa.

5. A proof of Theorem 1. Starting with the definition of the Petersson inner product we compute

$$\begin{aligned}\langle f,g\rangle &= \frac{i}{2}\iint_{\mathcal{F}} f(\tau)\overline{g(\tau)}\,d\tau\wedge\overline{d\tau}\\ &= \frac{1}{4\pi}\iint_{\mathcal{F}} d[\Phi_f(\tau)\overline{g(\tau)}\,\overline{d\tau}] \qquad \Big(\Phi_f(\tau)=2\pi i\int_{\tau_0}^{\tau} f(\xi)\,d\xi\Big)\\ &= \frac{1}{4\pi}\int_{\partial\mathcal{F}} \Phi_f(\tau)\overline{g(\tau)}\,\overline{d\tau} \qquad \text{(by Stokes' theorem)}\\ &= \frac{1}{4\pi}\sum_j\int_{e_j} \Phi_f(\tau)\overline{g(\tau)}\,\overline{d\tau}\\ &= \frac{1}{8\pi}\sum_j\Big(\int_{e_j}+\int_{e_{j^*}}\Big)\Phi_f(\tau)\overline{g(\tau)}\,\overline{d\tau}.\end{aligned}$$

Since e_{j^*} is the image of e_j under γ_j with orientation reversed and $g(\tau)d\tau$ is γ_j-invariant, we have

$$\int_{e_{j^*}} \Phi_f(\tau)\overline{g(\tau)}\,\overline{d\tau} = -\int_{e_j} \Phi_f(\gamma_j\tau)\overline{g(\tau)}\,\overline{d\tau}.$$

Consequently,

$$\begin{aligned}\langle f,g\rangle &= \frac{1}{8\pi}\sum_j\int_{e_j}[\Phi_f(\tau)-\Phi_f(\gamma_j\tau)]\overline{g(\tau)}\,\overline{d\tau}\\ &= -\frac{1}{8\pi}\sum_j\int_{e_j} C_f(\gamma_j)\overline{g(\tau)}\,\overline{d\tau}\\ &= -\frac{i}{16\pi^2}\sum_j C_f(\gamma_j)[\overline{\Phi_g(P_{j+1})}-\overline{\Phi_g(P_j)}].\end{aligned}$$

Recall that $\gamma_{j^*}=\gamma_j^{-1}$ and that C_f is a homomorphism. Replace j by j^* in the first summation to obtain

$$\begin{aligned}\sum C_f(\gamma_j)\,\overline{\Phi_g(P_{j+1})} &= \sum C_f(\gamma_{j^*})\overline{\Phi_g(P_{j^*+1})} = \sum C_f(\gamma_j^{-1})\,\overline{\Phi_g(P_{j^*+1})}\\ &= -\sum C_f(\gamma_j)\,\overline{\Phi_g(\gamma_j(P_j))}\\ &= -\sum C_f(\gamma_j)[\overline{C_g(\gamma_j)}+\overline{\Phi_g(P_j)}].\end{aligned}$$

Inserting this in the last line of the previous computation and simplifying gives

$$\langle f,g\rangle = \frac{i}{16\pi^2}\sum_j(C_f(\gamma_j)\,\overline{C_g(\gamma_j)}+2C_f(\gamma_j)\overline{\Phi_g(P_j)}).$$

Finally, we break up the second sum into orbits under T (see §2). Let $[j_0] = \{j_0, Tj_0, \dots, T^{t-1}j_0\}$ be a typical orbit with $T^t j_0 = j_0$. Now observe that $\sum_{j\in[j_0]} C_f(\gamma_j) = 0$ because $\prod_{j\in[j_0]} \gamma_j$ fixes P_{j_0} hence is the identity or an element of finite order. This observation proves fruitful as the next computation will show:

$$\begin{aligned}\sum_{j\in[j_0]} C_f(\gamma_j)\,\overline{\Phi_g(P_j)} &= \sum_{j\in[j_0]} C_f(\gamma_j)[\overline{\Phi_g(P_j)} - \overline{\Phi_g(P_{j_0})}] \\ &= \sum_{j\in[j_0]} C_f(\gamma_j) \sum_{j'\prec j} \overline{C_g(\gamma_{j'})}\end{aligned}$$

since $P_j = (\prod_{j'\prec j} \gamma_{j'})P_{j_0}$. Combining all of this gives Theorem 1.

6. A proof of Theorem 2. Once again we start with the definition of the action of the Hecke operator $T(p)$ on the cusp-form f and compute

$$\begin{aligned} C_{T(p)f}(\gamma_j) &= 2\pi i \int_{\tau_0}^{\gamma_j(\tau_0)} T(p)f(\xi)\,d\xi \\ &= 2\pi i\Big(\sum_{l=0}^{p-1} \int_{\tau_0}^{\gamma_j(\tau_0)} \underbrace{f\Big(\frac{\xi+l}{p}\Big)\frac{d\xi}{p}}_{\xi\mapsto p\xi'-l} + \int_{\tau_0}^{\gamma_j(\tau_0)} \underbrace{f(p\xi)p\,d\xi}_{\xi\mapsto\xi'/p}\Big) \\ &= 2\pi i\Big(\sum_{l=0}^{p-1} \int_{(\tau_0+l)/p}^{(\gamma_j(\tau_0)+l)/p} f(\xi)\,d\xi + \int_{p\tau_0}^{p\gamma_j(\tau_0)} f(\xi)\,d\xi\Big).\end{aligned}$$

We know that C_f is independent of base-point so pick $\tau_0 = 0$ to obtain

$$C_{T(p)f}(\gamma_j) = 2\pi i\Big(\sum_{l=0}^{p-1}\Big(\int_{l/p}^{0} + \int_{0}^{(\gamma_j(0)+l)/p}\Big) f(\xi)\,d\xi + \int_0^{p\gamma_j(0)} f(\xi)\,d\xi\Big).$$

Since $(p,N)=1$ it is easy to show that the rational numbers l/p, $(\gamma_j(0)+l)/p$ and $p\gamma_j(0)$ are all $\Gamma_0(N)$ equivalent to 0. Hence there exist matrices A_l, $B_{j,l}$ and D_j (all in $\Gamma_0(N)$) with $A_l(0) = l/p$, $B_{j,l}(0) = (\gamma_j(0)+l)/p$ and $D_j(0) = p\gamma_j(0)$. This gives

$$\begin{aligned} C_{T(p)f}(\gamma_j) &= 2\pi i\Big(\sum_{l=0}^{p-1}\Big(\int_{A_l(0)}^{0} + \int_0^{B_{j,l}(0)}\Big) f(\xi)\,d\xi + \int_0^{D_j(0)} f(\xi)\,d\xi\Big) \\ &= \Big(\sum_{l=0}^{p-1} [-C_f(A_l) + C_f(B_{j,l})] + C_f(D_j)\Big).\end{aligned} \tag{7}$$

Like any other element of $\Gamma_0(N)$, each of these matrices can be written in terms of the preferred generators, i.e.

$$A_l = \prod_{\substack{i=1 \\ (k\in J)}} \gamma_{k_i}^{\alpha_{k_i,l}}, \qquad B_{j,l} = \prod_{\substack{i=1 \\ (k\in J)}} \gamma_{k_i}^{\beta_{k_i,j,l}} \quad \text{and} \quad D_j = \prod_{\substack{i=1 \\ (k\in J)}} \gamma_{k_i}^{\delta_{k_i,j}}.$$

Since C_f is a homomorphism this allows us to express each term in the expression (4) as a $\mathbb{Z}$-linear combination of $\{C_f(\gamma_k)\}_{k\in J}$. More precisely, for a fixed j and l we have

$$C_f(A_l) = C_f\Big(\prod_{\substack{i=1 \\ (k\in J)}} \gamma_{k_i}^{\alpha_{k_i,l}}\Big) = \sum_{k\in J} m_{k,l} C_f(\gamma_k),$$

$$C_f(B_{j,l}) = C_f\Big(\prod_{\substack{i=1 \\ (k\in J)}} \gamma_{k_i}^{\beta_{k_i,j,l}}\Big) = \sum_{k\in J} m'_{k,j,l} C_f(\gamma_k),$$

$$C_f(D_j) = C_f\Big(\prod_{\substack{i=1 \\ (k\in J)}} \gamma_{k_i}^{\delta_{k_i,j}}\Big) = \sum_{k\in J} m''_{k,j} C_f(\gamma_k).$$

Here the coefficients of $C_f(\gamma_k)$ on the right of any of these equations is the sum of all numbers that occur as an exponent of γ_k in the factorization of the corresponding matrix on the left (i.e., $m_{k,l} = \sum_{k_i=k} \alpha_{k_i,l}$ etc.). The summation on the right can be taken over hyperbolic generators only since $C_f(\gamma)$ is zero for parabolic and elliptic elements. Combining all of this gives

$$C_{T(p)f}(\gamma_j) = \sum_{k\in J} \Big(\sum_{l=0}^{p-1} (m_{k,l} + m'_{k,j,l}) + m''_{k,j}\Big) C_f(\gamma_k).$$

Setting $n_{p,k,j} = \sum_{l=0}^{p-1}(m_{k,l} + m'_{k,j,l}) + m''_{k,j}$ establishes Theorem 2.

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