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On certain statistical properties of continued fractions with even and with odd partial quotients

by

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1. Introduction. Let (a_n) (respectively, (q_n)) denote the sequence of digits (resp., denominators of the convergents) in the regular continued fraction (RCF) expansion of an irrational number. For each $R > 1$, consider the *renewal time* $n_R := \min\{n : q_n > R\}$, so that $q_{n_R-1} \le R < q_{n_R}$. As a consequence of their renewal-type theorem for the natural extension of the Gauss map associated with regular continued fractions, Sinai and Ulcigrai [15] proved the existence of the joint limiting distribution of $(q_{n_R-1}/R, R/q_{n_R}, a_{n_R-K}, \dots, a_{n_R+K})$, with K a fixed nonnegative integer, as $R \to \infty$. The classical Gauss–Kuz'min statistics give the probability of a random x in $[0,1]$ having a prescribed string of digits in its continued fraction expansion at the nth position, for large n; the joint limiting distribution studied in [15, 16] gives the probability of a random x in $[0,1]$ having a prescribed string of digits in its continued fraction expansion at the first place where the denominator of the convergent is larger than R, for large R. The joint limiting distribution may therefore be considered an analogue of Gauss–Kuz'min statistics. Employing an abstract characterization of denominators of successive convergents in the regular continued fraction expansion $\mathrm{RCF}(x)$ of x, Ustinov succeeded in explicitly computing this limiting distribution in the RCF case [16].

Sinai and Ulcigrai's result was subsequently extended to the situation of continued fractions with even partial quotients (ECF) by Cellarosi [3]. The ECF limiting distribution was further used in the renormalization of theta sums—that is, replacing the theta sum $\sum e^{\pi i \omega n^2}$ with a theta sum of

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the type $\sum e^{-\pi i n'^2/\omega}$ modulo a rescaling, rotation, and small error term—as the map $\omega \mapsto -1/\omega$ modulo 2 is closely related to the forward shift of even continued fractions. This has led to some new results about the distribution of normalized theta sums and geometrical properties of their associated curlicues [4, 14].

This paper studies this type of limiting distributions in the case of three types of continued fractions: ECF, OCF (continued fractions with odd partial quotients), and NCF_α (the Nakada α-expansions, which include NICF, or continued fraction to the nearest integer, as a special case). In the ECF case we provide a direct proof of the main result in [3] while making the limiting distribution explicit. The analogous problem is also solved in the OCF case, for which no ergodic-theoretical approach is known at this time. As in [16], the key tool is providing an abstract characterization for pairs of successive convergents in $\mathrm{ECF}(x)$ and $\mathrm{OCF}(x)$, which may be of independent interest. The OCF case is the most intricate, because the sequence of denominators of successive convergents in $\mathrm{OCF}(x)$ is not necessarily increasing as in the RCF, ECF, or NCF_α cases. Finally we provide an explicit relation between the NCF_α limiting joint distribution and the distribution computed in [16].

Concretely, for a given type of continued fraction expansion (ECF, OCF, or NCF_α), consider the *renewal time*

$$n_R = \min\{n \in \mathbb{N} : q_n > R\} = \min\{n \in \mathbb{N} : q_{n-1} \le R < q_n\}, \quad R > 1,$$

and the joint limiting distribution of $\big(q_{n_R-1}/R, R/q_{n_R}, \omega_{n_R-K}, \ldots, \omega_{n_R+K}\big)$ with $\omega_k = (a_k, e_k)$, for fixed K, as $R \to \infty$. Here again, ω_k denote the continued fraction digits and q_n denote the denominators of the convergents for a given type of CF expansion (see Section 2 for more details).

We will evaluate the Lebesgue measure $\mathcal{L}^{E/O,\pm}_{x_1,x_2,x_3,x_4}(R)$ of the set of numbers $x \in \Omega := [0,1] \setminus \mathbb{Q}$ for which there exist successive convergents $P/Q, P'/Q'$ in $\mathrm{ECF}(x)$ (respectively in $\mathrm{OCF}(x)$) such that for given x_1, x_2, x_3, x_4 the following conditions are satisfied:

$$\frac{Q}{R} \le x_1, \quad \frac{R}{Q'} \le x_2, \quad \frac{Q}{Q'} \le x_3, \tag{1.1}$$

$$0 \le \frac{Q'x - P'}{-Qx + P} \le x_4, \quad \text{respectively} \quad -x_4 \le \frac{Q'x - P'}{-Qx + P} \le 0, \tag{1.2}$$

depending on the choice of the $\pm$ sign. In both ECF and OCF situations, we take $x_1, x_2, x_3, x_4 \in (0,1]$ (1). In the OCF case, the ratio Q/Q' of successive denominators can in fact be any rational number in the interval $(0, G)$, but since in the definition of n_R we are interested only in $Q \le R < Q'$, we

(1) If any of the parameters equals 0, then $\mathcal{L}$ equals 0 as well, so we ignore this degenerate case.

can restrict to $x_3 \le 1$ in the definition of $\mathcal{L}^{O,\pm}$. The golden ratios $G = (1+\sqrt{5})/2$ and $g = 1/G = (-1+\sqrt{5})/2$ will be used often.

The terms q_{n_R-1}/R and R/q_{n_R} in the joint limiting distribution clearly relate to the parameters x_1 and x_2 in the function $\mathcal{L}$. Likewise, the digits ω_k in the joint limiting distribution relate to the parameters x_3 and x_4 in $\mathcal{L}$ due to equalities (2.4) and (2.6) below.

The main result of this paper shows that $\mathcal{L}^{E/O,\pm}(R)$ has an explicitly computable limiting distribution as $R \to \infty$.

THEOREM 1.1. *The joint distributions $\mathcal{L}^{E/O,\pm}_{x_1,x_2,x_3,x_4}(R)$ exist as $R \to \infty$ and*

$$\mathcal{L}^{E,\pm}_{x_1,x_2,x_3,x_4}(R) = \frac{2F_\pm}{3\zeta(2)} + O_\varepsilon(R^{-1+\varepsilon}), \tag{1.3}$$

$$\begin{aligned} \mathcal{L}^{O,+}_{x_1,x_2,x_3,x_4}(R) &= \frac{F_+ - D_1}{\zeta(2)} + O_\varepsilon(R^{-1/2+\varepsilon}), \\ \mathcal{L}^{O,-}_{x_1,x_2,x_3,x_4}(R) &= \frac{F_- - D_2 - D_3}{\zeta(2)} + O_\varepsilon(R^{-1/2+\varepsilon}), \end{aligned} \tag{1.4}$$

where $F_\pm = F_\pm(x_1,x_2,x_3,x_4)$ and $D_i = D_i(x_1,x_2,x_3,x_4)$ are given by [2]

$$F_\pm = \mp \begin{cases} \mathrm{Li}_2(\mp x_1x_2x_4) & \text{if } x_3 \ge x_1x_2, \\ \mathrm{Li}_2(\mp x_3x_4) - \log(1 \pm x_3x_4)\log\frac{x_1x_2}{x_3} & \text{if } x_3 < x_1x_2, \end{cases} \tag{1.5}$$

$$\begin{aligned} &D_2 = F_-(x_1,x_2,x_3,x_4) - F_-(x_1,x_2,\min\{x_3,g^2\},x_4), \\ &D_1 = \sum_{\ell\ge1} I_\ell^+, \qquad D_3 = \sum_{\ell\ge2} I_\ell^-, \end{aligned} \tag{1.6}$$

with

$$I_\ell^\pm = \int_{1/x_2}^{A_\ell} dx \int_{x/(2\ell+g)}^{B_\ell(x)} \frac{x_4\,dy}{y(y \pm x_4x)}, \tag{1.7}$$

where

$$A_\ell = (2\ell+g)x_1, \quad B_\ell(x) = B_{\ell,x_2,x_3}(x) = \min\left\{x_3x, x_1, \frac{x}{2\ell}, \frac{x-1}{2\ell-1}\right\}.$$

The integrals $I_\ell^\pm$ can be written explicitly as a combination of logarithms and dilogarithms.

Kraaikamp's metric theory for S-expansions [6] provides immediate characterizations of pairs of successive convergents for such continued fractions, which are obtained from RCF only by singularization (see the remark at the end of Section 3 for definition of singularization). In the last section we show how to compute the joint limiting distribution associated as above with

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[2] In this paper the convention is that $\int_a^b = 0$ when $a \ge b$.

Nakada's α-expansions [10] for $1/2 \le \alpha \le 1$. The cases $\alpha = 1$ and $\alpha = 1/2$ are best known, corresponding to the RCF and NICF (continued fraction to the nearest integer). The latter was introduced by Minnigerode [9] and was also studied in [1, 13, 18]. Our calculations show explicit connections with Ustinov's RCF distribution.

2. Basic ECF and OCF properties. For each $x \in \Omega$, the ECF (respectively, OCF) expansion of x is given by

$$x = \cfrac{1}{a_1 + \cfrac{e_1}{a_2 + \cfrac{e_2}{a_3 + \cfrac{e_3}{\ddots}}}} = [[(a_1, e_1), (a_2, e_2), (a_3, e_3), \ldots]], \tag{2.1}$$

where $e_n \in \{\pm 1\}$ and all a_n's are even positive integers (respectively, all a_n's are odd positive integers with $a_n + e_n \ge 2$). For more details see [5, 6, 8, 11, 12, 13]. As in [5, 8], consider the "flipped" continued fraction map $T_D : [0,1] \to [0,1]$ for a subset D of $[0,1]$, defined by $T_D(0) = 0$, $T_D(1) = 1$, and

$$T_D(x) = \begin{cases} \{1/x\} & \text{if } x \in (0,1) \setminus D, \\ 1 - \{1/x\} & \text{if } x \in D, \end{cases}$$

with auxiliary functions

$$e_D(x) = \begin{cases} 1 & \text{if } x \in [0,1] \setminus D, \\ -1 & \text{if } x \in D, \end{cases} \qquad a_D(x) = \begin{cases} [1/x] & \text{if } x \in [0,1] \setminus D, \\ 1 + [1/x] & \text{if } x \in D. \end{cases}$$

Note that

$$T_D(x) = e_D(x)\left(\frac{1}{x} - a_D(x)\right), \qquad \forall x \in (0,1).$$

Consider the sets

$$D_O := \bigcup_{n \in 2\mathbb{N}} \left[\frac{1}{n+1}, \frac{1}{n}\right), \qquad D_E := [0,1) \setminus D_O = \bigcup_{n \in 2\mathbb{N}-1} \left[\frac{1}{n+1}, \frac{1}{n}\right).$$

Denote $D = D_E$ in the ECF case, respectively $D = D_O$ in the OCF case. In both ECF or OCF situations the signs $e_n = e_n(x)$ and the digits $a_n = a_n(x)$ are given, for $x \in \Omega$, by

$$e_0 = 1, \quad e_n = e_D(t_{n-1}), \quad a_0 = 0, \quad a_n = a_D(t_{n-1}),$$

where $t_n = t_n(x) = T_D^n(x)$. On the D-continued fraction expansion the iterates of the Gauss type map T_D act as a shift map by

$$T_D^n[[(a_1, e_1), (a_2, e_2), \ldots]] = [[(a_{n+1}, e_{n+1}), (a_{n+2}, e_{n+2}), \ldots]], \quad \forall n \in \mathbb{N}_0.$$

The D-convergents p_n/q_n are defined by

$$
(2.2)\qquad \begin{cases} p_{-1}=1, & p_0=0, \quad p_n=a_np_{n-1}+e_{n-1}p_{n-2},\\ q_{-1}=0, & q_0=1, \quad q_n=a_nq_{n-1}+e_{n-2}q_{n-2}, \end{cases}
$$

or in equivalent formulation

$$
(2.3)\qquad \begin{pmatrix} p_{n-1} & p_n\\ q_{n-1} & q_n\end{pmatrix}=\begin{pmatrix} p_{n-2} & p_{n-1}\\ q_{n-2} & q_{n-1}\end{pmatrix}\begin{pmatrix} 0 & e_{n-1}\\ 1 & a_n\end{pmatrix}=\cdots
$$

$$
=\begin{pmatrix} 0 & e_0\\ 1 & a_1\end{pmatrix}\begin{pmatrix} 0 & e_1\\ 1 & a_2\end{pmatrix}\cdots\begin{pmatrix} 0 & e_{n-1}\\ 1 & a_n\end{pmatrix},\qquad \forall n\in\mathbb{N}.
$$

The following elementary fundamental relations are satisfied:

$$
\begin{aligned}
&p_{n-1}q_n-p_nq_{n-1}=(-1)^ke_0e_1\cdots e_{n-1}=:\delta_n,\\
&\frac{p_{n-1}}{q_{n-1}}-\frac{p_n}{q_n}=\frac{\delta_n}{q_{n-1}q_n},\qquad \forall n\in\mathbb{N}_0,\\
&x=\frac{p_n+p_{n-1}e_nt_n}{q_n+q_{n-1}e_nt_n},\qquad \forall n\in\mathbb{N}.
\end{aligned}
$$

The latter equation is equivalent to

$$
(2.4)\qquad e_nt_n=e_nT_D^n(x)=\frac{q_nx-p_n}{-q_{n-1}x+p_{n-1}},\qquad \forall n\in\mathbb{N}.
$$

Upon (2.4) we infer

$$
(2.5)\qquad 0<\left|\frac{q_nx-p_n}{-q_{n-1}x+p_{n-1}}\right|<1,\quad \forall x\in\Omega,\ \forall n\in\mathbb{N}.
$$

It is well-known and plain to check for every continued fraction that if x is as in (2.1), then

$$
(2.6)\qquad \frac{q_{n-1}}{q_n}=[[(a_n,e_{n-1}),(a_{n-1},e_{n-2}),\ldots,(a_2,e_1),(a_1,*)]],\quad \forall n\in\mathbb{N},
$$

where $(a_1,*)$ means that the finite expansion terminates with a_1.

3. Successive ECF and OCF convergents. In $\mathrm{GL}_2(\mathbb{Z})$ consider the matrices

$$
I=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad J=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad A=\begin{pmatrix}0&1\\1&1\end{pmatrix},\quad B=A^2=\begin{pmatrix}1&1\\1&0\end{pmatrix},
$$

and denote their images in $\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$ by $[I]$, $[J]$, $[A]$, $[B]$. Clearly $\{[I],[J]\}$ forms a subgroup on two elements and $\{[I],[A],[B]\}$ forms a subgroup on three elements of $\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$. Consider the sets

$$\mathcal{R} = \left\{ M = \begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix} : 0 \le P \le Q,\, 1 \le P' \le Q' \right\},$$
$$\mathcal{R}_E := \{ M \in \mathcal{R} : 1 \le Q \le Q',\, M \equiv I \text{ or } J \pmod 2 \},$$
$$\mathcal{R}_O := \{ M \in \mathcal{R} : \lambda_M > g,\, M \equiv I, A, \text{ or } B \pmod 2 \}.$$

For $M \in \mathcal{R}$ denote

$$\lambda_M = \frac{Q'}{Q}, \qquad E_M(x) = \frac{Q'x - P'}{-Qx + P}, \quad x \notin \mathbb{Q}. \tag{3.1}$$

3.1. Successive convergents for $\mathrm{ECF}(x)$

LEMMA 3.1. *In the ECF expansion, $q_k \ge q_{k-1} \ge 1$, $p_{k+1} \ge p_k \ge 1$, and $q_k - p_k \ge q_{k-1} - p_{k-1} \ge 1$ for every $k \ge 1$.*

Proof. Let (x_n) be a sequence defined by $x_n = a_n x_{n-1} + e_{n-1} x_{n-2}$ with a_n an even positive integer and $e_n \in \{\pm 1\}$. Suppose that $x_{k_0} \ge x_{k_0-1} \ge 1$ for some $k_0 \ge 1$. Then $x_{k_0+1} \ge 2x_{k_0} - x_{k_0-1} \ge x_{k_0}$. This shows inductively that $x_n \ge x_{n-1} \ge 1$ for every $n \ge k_0$. The statement follows by taking $(x_n, k_0) = (q_n, 1)$, $(x_n, k_0) = (p_n, 2)$, and respectively $(x_n, k_0) = (q_n - p_n, 1)$. ∎

Furthermore, since $p_{n-1}q_n - p_n q_{n-1} = \pm 1$, it follows that $q_n(x) > q_{n-1}(x)$ for all $n \ge 2$ and $x \in \Omega$.

PROPOSITION 3.2. *For each $x \in \Omega$ the following are equivalent:*

(i) *$P/Q, P'/Q'$ are successive convergents in* $\mathrm{ECF}(x)$.
(ii) *$M = \left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathcal{R}_E$ and $0 < |E_M(x)| < 1$.*

Proof. (i)⇒(ii). Suppose $M = \left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) = \left(\begin{smallmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{smallmatrix}\right)$ for some $n \ge 1$. From Lemma 3.1, $\left(\begin{smallmatrix} 0 & e_{k-1} \\ 1 & a_k \end{smallmatrix}\right) \equiv J \pmod 2$ and equality (2.3) we infer that $M \in \mathcal{R}_E$. The second condition in (ii) follows from (2.5).

(ii)⇒(i). Consider first the case $Q = 1$. Only the matrices $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & Q' \end{smallmatrix}\right)$ and $M = \left(\begin{smallmatrix} 1 & Q'-1 \\ 1 & Q' \end{smallmatrix}\right)$ may arise. Since $M \equiv I$ or $J \pmod 2$, only the former case can occur and Q' is necessarily an even positive integer. The corresponding inequality

$$0 < \left| \frac{Q'x - 1}{-x} \right| < 1 \quad \text{is equivalent to} \quad x \in \left(\frac{1}{Q'+1}, \frac{1}{Q'} \right) \cup \left(\frac{1}{Q'}, \frac{1}{Q'-1} \right)$$

or, according to the definition of a_1, to $a_1 = Q'$, showing that $0/1, 1/Q'$ are successive convergents of x.

When $Q > 1$, take (with $\ell \geq 1$):

$$
\begin{aligned}
&e_M = 1, && Q_0 = Q' - 2\ell Q, && P_0 = P' - 2\ell P && \text{if } [\lambda] = 2\ell,\\
&e_M = -1, && Q_0 = 2\ell Q - Q', && P_0 = 2\ell P - P' && \text{if } [\lambda] = 2\ell - 1,\\
&M_0 = \begin{pmatrix} P_0 & P \\ Q_0 & Q \end{pmatrix}.
\end{aligned}
$$

In both cases one has $0 < Q_0 < Q$, $M = M_0\left(\begin{smallmatrix} 0 & e_M \\ 1 & 2\ell \end{smallmatrix}\right)$, and so $M_0 \equiv I$ or $J \pmod 2$. Since $Q' > Q > Q_0$, the condition $0 < |E_M| < 1$ is equivalent to x lying between $\frac{P'+P}{Q'+Q}$ and $\frac{P'-P}{Q'-Q}$, while $0 < |E_{M_0}| < 1$ is equivalent to x lying between $\frac{P+P_0}{Q+Q_0}$ and $\frac{P-P_0}{Q-Q_0}$. When $P/Q < P'/Q'$ the former implies the latter because

$$
\begin{aligned}
\frac{P-P_0}{Q-Q_0} &= \frac{(2\ell+1)P-P'}{(2\ell+1)Q-Q'} < \frac{P}{Q} < \frac{P'+P}{Q'+Q} < \frac{P'}{Q'} < \frac{P'-P}{Q'-Q}\\
&\leq \frac{P+P_0}{Q+Q_0} = \frac{P'-(2\ell-1)P}{Q'-(2\ell-1)Q} < \frac{P_0}{Q_0} = \frac{P'-2\ell P}{Q'-2\ell Q} \quad \text{when } [\lambda] = 2\ell,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{P_0}{Q_0} &= \frac{2\ell P-P'}{2\ell Q-Q'} < \frac{P+P_0}{Q+Q_0} = \frac{(2\ell+1)P-P'}{(2\ell+1)Q-Q'} < \frac{P}{Q} < \frac{P'+P}{Q'+Q} < \frac{P'}{Q'}\\
&< \frac{P'-P}{Q'-Q} \leq \frac{P-P_0}{Q-Q_0} = \frac{P'-(2\ell-1)P}{Q'-(2\ell-1)Q} \quad \text{when } [\lambda] = 2\ell-1.
\end{aligned}
$$

When $P'/Q' < P/Q$, analogous inequalities show that $0 < |E_M| < 1$ implies $0 < |E_{M_0}| < 1$. Furthermore, the inequalities $0 \leq P_0 \leq P$ follow from $|P'Q - PQ'| = |PQ_0 - P_0Q| = 1$ and $P \geq 1$. ∎

3.2. Successive convergents for $\mathrm{OCF}(x)$**.** Denominators of successive convergents for $\mathrm{OCF}(x)$ satisfy ([11, Eq. 2.10])

$$
\begin{aligned}
r_n &:= q_n/q_{n-1}\\
&= a_n + e_{n-1}[[(a_{n-1}, e_{n-2}), (a_{n-2}, e_{n-3}), \ldots, (a_2, e_1), (a_1, *)]]\\
&\geq a_n - [[(3,-1), (3,-1), \ldots, (3,-1), (3,*)]]\\
&> a_n - [[(3,-1), (3,-1), (3,-1)\ldots]]\\
&= a_n - 1 + 1/G = a_n - 2 + G.
\end{aligned} \tag{3.2}
$$

In the opposite direction one has

$$
r_n = a_n + \frac{e_{n-1}}{r_{n-1}} < a_n + \frac{e_{n-1}}{a_{n-1} - 2 + G} \leq a_n + \frac{1}{G-1} = a_n + G. \tag{3.3}
$$

In particular (3.2) and (3.3) show that if $a_n \geq 3$, then $r_n > 1 + G$, proving

Lemma 3.3. *If $r_n \le 2+g$ then $a_n = 1$, and in particular $e_n = 1$ and*

$$0 < \frac{q_n x - p_n}{-q_{n-1}x + p_{n-1}} < 1.$$

Proposition 3.4. *For each $x \in \Omega$ the following are equivalent:*

(i) *$P/Q, P'/Q'$ are successive convergents in* OCF(x).
(ii) *$M = \left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathcal{R}_O$ and one of the following two conditions holds:*

($*$) *$\lambda_M := Q'/Q > 2+g$ and $0 < |E_M(x)| < 1$.*
($**$) *$g < \lambda_M \le 2+g$ and $0 < E_M(x) < 1$.*

Proof. (i)$\Rightarrow$(ii). Suppose that there is $n \ge 1$ such that

$$M = \begin{pmatrix} 0 & e_0 = 1 \\ 1 & a_1 \end{pmatrix}\begin{pmatrix} 0 & e_1 \\ 1 & a_2 \end{pmatrix}\cdots\begin{pmatrix} 0 & e_{n-1} \\ 1 & a_n \end{pmatrix} = \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix}. \tag{3.4}$$

Since $\left(\begin{smallmatrix} 0 & e_{i-1} \\ 1 & a_i \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right) = A \pmod 2$ and $\{[I],[A],[B]\}$ forms a subgroup of $SL_2(\mathbb{Z}/2\mathbb{Z})$, it follows that $M \equiv I, A$, or $B \pmod 2$. The inequality $GQ' > Q$ follows from (3.2), while $0 \le P = p_{n-1} \le Q = q_{n-1}$, $0 < P' = p_n \le Q' = q_n$ are well-known (they follow as a result of the RCF $\to$ OCF algorithm or can be directly deduced from $p_{n-1}q_n - p_n q_{n-1} = \pm 1$). Properties ($*$) and ($**$) follow from (2.4), (2.5), and from Lemma 3.3.

(ii)$\Rightarrow$(i). Consider the partition $(g,\infty) = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$, where

$$\begin{aligned} \mathcal{S}_1 &= (g,1) \cup (2+g,3) \cup (4+g,5) \cup \cdots, \\ \mathcal{S}_2 &= [1,2) \cup [3,4) \cup [5,6) \cup \cdots, \\ \mathcal{S}_3 &= [2,2+g) \cup [4,4+g) \cup [6,6+g) \cup \cdots. \end{aligned}$$

For each matrix $M = \left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathcal{R}_O$ with $\lambda = \lambda_M$, define

$$k_M = \begin{cases} 2\ell - 1 & \text{if } \lambda \in \mathcal{S}_2,\ [\lambda] = 2\ell - 1,\ \ell \ge 1, \\ 2\ell + 1 & \text{if } \lambda \in \mathcal{S}_1,\ [\lambda] = 2\ell,\ \ell \ge 0, \text{ and } \{\lambda\} > g, \\ 2\ell - 1 & \text{if } \lambda \in \mathcal{S}_3,\ [\lambda] = 2\ell,\ \ell \ge 1, \text{ and } \{\lambda\} < g. \end{cases}$$

Note that

$$k_M \ge 3 \iff \lambda > 2 + g = G^2.$$

We prove the following statement:

Lemma 3.5. *Let $x \in \Omega$ and $M = \left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathcal{R}_O$ with $\widetilde{Q} = \min\{Q, Q'\} > 1$ and M satisfying ($*$) or ($**$). There exist $e_M \in \{\pm 1\}$ and $M_0 = \left(\begin{smallmatrix} P_0 & P \\ Q_0 & Q \end{smallmatrix}\right) \in \mathcal{R}_O$ such that*

$$M = M_0 \begin{pmatrix} 0 & e_M \\ 1 & k_M \end{pmatrix}, \tag{3.5}$$

$e_M + k_{M_0} \geq 2$, M_0 satisfies the corresponding property $(*)$ *or* $(**)$*, and $\widetilde{Q}_0 = \min\{Q_0, Q\} \leq \widetilde{Q}$. Furthermore, if $\lambda = \lambda_M \in \mathcal{S}_1 \cup \mathcal{S}_2$, then we can take $\widetilde{Q}_0 < \widetilde{Q}$.*

Proof of Lemma 3.5. Consider the following integers:

$$e_M = \begin{cases} 1 & \text{if } \lambda \in \mathcal{S}_2 \cup \mathcal{S}_3, \\ -1 & \text{if } \lambda \in \mathcal{S}_1, \end{cases}$$

$$Q_0 = \begin{cases} Q' - (2\ell - 1)Q & \text{if } \lambda \in \mathcal{S}_3,\ [\lambda] = 2\ell,\ \ell \geq 1, \text{ and } \{\lambda\} < g, \\ Q' - (2\ell - 1)Q & \text{if } \lambda \in \mathcal{S}_2,\ [\lambda] = 2\ell - 1,\ \ell \geq 1, \\ (2\ell + 1)Q - Q' & \text{if } \lambda \in \mathcal{S}_1,\ [\lambda] = 2\ell,\ \ell \geq 0, \text{ and } \{\lambda\} > g, \end{cases}$$

$$= \begin{cases} (1 + \{\lambda\})Q & \text{if } \lambda \in \mathcal{S}_3, \\ \{\lambda\}Q & \text{if } \lambda \in \mathcal{S}_2, \\ (1 - \{\lambda\})Q & \text{if } \lambda \in \mathcal{S}_1, \end{cases}$$

$$P_0 = \begin{cases} P' - (2\ell - 1)P & \text{if } \lambda \in \mathcal{S}_3,\ [\lambda] = 2\ell,\ \ell \geq 1, \text{ and } \{\lambda\} < g, \\ P' - (2\ell - 1)P & \text{if } \lambda \in \mathcal{S}_2,\ [\lambda] = 2\ell - 1,\ \ell \geq 1, \\ (2\ell + 1)P - P' & \text{if } \lambda \in \mathcal{S}_1,\ [\lambda] = 2\ell,\ \ell \geq 0, \text{ and } \{\lambda\} > g. \end{cases}$$

Equality (3.5) holds in all cases with this choice of Q_0 and P_0. One plainly checks that

$$\lambda_0 := \frac{Q}{Q_0} \in \begin{cases} (2 + g, \infty) & \text{if } \lambda \in \mathcal{S}_1, \\ (1, \infty) & \text{if } \lambda \in \mathcal{S}_2, \\ (g, 1] & \text{if } \lambda \in \mathcal{S}_3. \end{cases}$$

In particular this shows that $\lambda_0 > g$. The inequality $e_M + k_{M_0} \geq 2$ is trivial when $\lambda \in \mathcal{S}_2 \cup \mathcal{S}_3$. When $\lambda \in \mathcal{S}_1$ we have $\lambda_0 > 2 + g$, hence $k_{M_0} \geq 3$ and $e_M + k_{M_0} \geq 2$.

Clearly $\left(\begin{smallmatrix} 0 & e_M \\ 1 & k_M \end{smallmatrix}\right) \equiv A \pmod 2$. The inequalities $0 \leq P_0 \leq Q_0$ follow immediately from $P_0 Q - P Q_0 = \pm 1$ and $P < Q$, the latter being a consequence of the assumption $\widetilde{Q} > 1$. The fact that M_0 satisfies either $(*)$ or $(**)$ follows from Lemma 3.6. ∎

Back to the proof of Proposition 3.4, note that when $\lambda \in (g, 1]$ one has $0 < Q_0 = Q - Q' < Q' < Q$ (the first inequality holds because $G < 2$), while for $\lambda \in (\mathcal{S}_1 \cup \mathcal{S}_2) \setminus (g, 1)$ it is plain that $0 < Q_0 < Q < Q'$. Hence whenever $\lambda \in \mathcal{S}_1 \cup \mathcal{S}_2$ one has $\min\{Q_0, Q\} < \min\{Q, Q'\}$.

When $\lambda \in \mathcal{S}_3$ one only has $\min\{Q_0, Q\} = \min\{Q, Q'\}$ (actually $Q < Q_0 < Q'$). However, in this case $e_M = -1$ so $k_{M_0} \geq 3$, and $\lambda_{M_0} = Q/Q_0 \in (g, 1)$. Thus one can apply the same procedure to M_0 and find $M_{-1} = \left(\begin{smallmatrix} P_{-1} & P_0 \\ Q_{-1} & Q_0 \end{smallmatrix}\right) \in \mathcal{R}_0$ that satisfies $(*)$ or $(**)$, and such that $M_0 = M_{-1}\left(\begin{smallmatrix} 0 & e_{M_0} \\ 1 & k_{M_0} \end{smallmatrix}\right)$, $e_{M_0} + k_{M_{-1}} \geq 2$, and $\widetilde{Q}_{-1} := \min\{Q_{-1}, Q_0\} < \widetilde{Q}_0 = \widetilde{Q}$ (this inequality is strict because $\lambda_0 \in (g, 1) \subseteq \mathcal{S}_1$).

We next discuss the case $\widetilde{Q} = 1$. When $Q' = 1 \leq Q$, the inequality $Q'/Q = 1/Q \geq g$ yields $Q = 1$. Hence $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, with $0/1, 1/1$ successive convergents of every $x \in (0,1)$ that satisfies $0 < \frac{x-1}{-x} < 1$, i.e. of every $x \in (1/2, 1)$. Suppose now $Q = 1 < Q'$. When $1/G < Q'/Q = Q' < 2+g$, one has $Q' = 2$ and only the matrices $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ and $M = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ may arise. But the former matrix is not admissible being $\equiv \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \pmod 2$, while the latter corresponds to $0 < \frac{2x-1}{-x+1} < 1$, hence $x \in (1/2, 2/3)$, $e_1 = 1$ and $a_1 = [1/x] = 1$, and indeed $1/1, 1/2$ are successive convergents in $\mathrm{OCF}(x)$ for every $x \in (1/2, 2/3)$. When $2 + g < Q'/Q = Q'$ the only matrices that may arise are $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & Q' \end{smallmatrix}\right)$ with $Q' \geq 3$ odd, and respectively $M = \left(\begin{smallmatrix} 1 & Q'-1 \\ 1 & Q' \end{smallmatrix}\right)$ with $Q' \geq 4$ even. The inequality for the former is

$$0 < \left| \frac{Q'x - 1}{-x} \right| < 1, \quad \text{which gives} \quad x \in \left(\frac{1}{Q'+1}, \frac{1}{Q'} \right) \cup \left(\frac{1}{Q'}, \frac{1}{Q'-1} \right)$$

with Q' odd, so that $a_1 = Q'$ (and $e_1 = 1$ respectively $e_1 = -1$). The inequality for the latter is

$$0 < \left| \frac{Q'x - Q' + 1}{-x+1} \right| < 1, \quad \text{giving} \quad \frac{Q'}{Q'+1} > x > \frac{Q'-2}{Q'-1} \geq \frac{2}{3},$$

so $e_1 = 1$, $a_1 = 1$. Furthermore one has

$$1/Q' < 1/x - 1 = T_D(x) < 1/(Q'-2)$$

with $Q' - 1 \geq 3$ odd integer, so $a_2 = Q' - 1$ and $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 1 & Q'-1 \end{smallmatrix}\right)$, showing that indeed $1/1, (Q'-1)/Q'$ are successive convergents in $\mathrm{OCF}(x)$ for every x with $\frac{Q'}{Q'+1} > x > \frac{Q'-2}{Q'-1}$ and $Q' \geq 4$ even.

This inductive process on $\widetilde{Q}$ now implies that (3.4) holds for some $e_1, \ldots, e_{n-1} \in \{\pm 1\}$ and $a_1, \ldots, a_n$ odd positive integers with $e_i + a_i \geq 2$ for all $i \in \{1, \ldots, n-1\}$. Conditions $(*)$ and $(**)$ show that x lies between $\frac{p_n - p_{n-1}}{q_n - q_{n-1}}$ and $\frac{p_n + p_{n-1}}{q_n + q_{n-1}}$ when $q_n > q_{n-1}$, and between $\frac{p_n}{q_n}$ and $\frac{p_n + p_{n-1}}{q_n + q_{n-1}}$ when $q_n < q_{n-1}$. So x is of the form $[[(a_1, e_1), (a_2, e_2), \ldots, (a_{n-1}, e_{n-1}), (a_n + t, *)]]$ for some $t \in (-1, 1)$ when $q_n > q_{n-1}$, and $t \in (0,1)$ when $q_n < q_{n-1}$. Therefore $p_{n-1}/q_{n-1} = P/Q$, $p_n/q_n = P'/Q'$ are successive convergents of x. ∎

Lemma 3.6. *With the definitions from the proof of the implication* (ii)⇒(i) *in Proposition* 3.4, *one has:*

(i) *If* $g < \lambda < 1$, *then* $0 < E_M(x) < 1 \Rightarrow |E_{M_0}(x)| < 1$.
(ii) *If* $1 \leq \lambda < 2 + g$, *then* $0 < E_M(x) < 1 \Rightarrow 0 < E_{M_0}(x) < 1$.
(iii) *If* $2\ell + g < \lambda < 2\ell + 1$, $\ell \geq 1$, *then* $|E_M(x)| < 1 \Rightarrow -1 < E_{M_0}(x) < 0$.
(iv) *If* $2\ell - 1 \leq \lambda < 2\ell + g$, $\ell \geq 2$, *then* $|E_M(x)| < 1 \Rightarrow 0 < E_{M_0}(x) < 1$.

Proof. In all cases $0 < E_M(x) = \frac{Q'x-P'}{-Qx+P} < 1$ is equivalent to x lying between $\frac{P'}{Q'}$ and $\frac{P'+P}{Q'+Q}$, while $0 < E_{M_0}(x) = \frac{Qx-P}{-Q_0x+P_0} < 1$ is equivalent to x lying between $\frac{P}{Q}$ and $\frac{P+P_0}{Q+Q_0}$.

(i) In this case $Q_0 = Q - Q' < Q$ and so $-1 < E_{M_0}(x) < 1$ is equivalent to x lying between $\frac{P+P_0}{Q+Q_0} = \frac{2P-P'}{2Q-Q'}$ and $\frac{P-P_0}{Q-Q_0} = \frac{P'}{Q'}$. The conclusion follows because

$$\frac{2P-P'}{2Q-Q'} < \frac{P}{Q} < \frac{P'+P}{Q'+Q} < \frac{P'}{Q'} \quad \text{when } \frac{P}{Q} < \frac{P'}{Q'}, \quad \text{and}$$
$$\frac{P'}{Q'} < \frac{P'+P}{Q'+Q} < \frac{P}{Q} < \frac{2P-P'}{2Q-Q'} \quad \text{when } \frac{P'}{Q'} < \frac{P}{Q}.$$

(ii) In this case $\frac{P+P_0}{Q+Q_0} = \frac{P'}{Q'}$ and x between $\frac{P'}{Q'}$ and $\frac{P'+P}{Q'+Q}$ implies x between $\frac{P}{Q}$ and $\frac{P'}{Q'}$.

(iii) In this case $0 < Q_0 = (2\ell+1)Q - Q' < Q < Q'$, and $-1 < E_M(x) < 1$ is equivalent to x lying between $\frac{P'+P}{Q'+Q}$ and $\frac{P'-P}{Q'-Q}$, while $-1 < E_{M_0}(x) < 0$ is equivalent to x lying between $\frac{P}{Q}$ and $\frac{P-P_0}{Q-Q_0} = \frac{P'-2\ell P}{Q'-2\ell Q}$. The implication follows because either

$$\frac{P}{Q} < \frac{P'+P}{Q'+Q} < \frac{P'}{Q'} < \frac{P'-P}{Q'-Q} < \frac{P'-2\ell P}{Q'-2\ell Q}$$

or

$$\frac{P'-2\ell P}{Q'-2\ell Q} < \frac{P'-P}{Q'-Q} < \frac{P'}{Q'} < \frac{P'+P}{Q'+Q} < \frac{P}{Q}.$$

(iv) In this case $Q' > Q$ and $\frac{P+P_0}{Q+Q_0} = \frac{P'-(2\ell-2)P}{Q'-(2\ell-2)Q}$. The implication follows because $-1 < E_M(x) < 1$ is equivalent to x lying between $\frac{P'+P}{Q'+Q}$ and $\frac{P'-P}{Q'-Q}$, $0 < E_{M_0}(x) < 1$ is equivalent to x lying between $\frac{P}{Q}$ and $\frac{P+P_0}{Q+Q_0}$, and either

$$\frac{P}{Q} < \frac{P'+P}{Q'+Q} < \frac{P'}{Q'} < \frac{P'-P}{Q'-Q} < \frac{P'-(2\ell-2)P}{Q'-(2\ell-2)Q}$$

or

$$\frac{P'-(2\ell-2)P}{Q'-(2\ell-2)Q} < \frac{P'-P}{Q'-Q} < \frac{P'}{Q'} < \frac{P'+P}{Q'+Q} < \frac{P}{Q}. \quad \blacksquare$$

The following statement will also be useful:

LEMMA 3.7. *Denominators of successive convergents in OCF satisfy*

(i) $q_{n+2} > q_n$.
(ii) $q_{n+3} > q_n$.
(iii) $q_{n+2} > \min\{q_n, q_{n+1}\}$.

Proof. By Proposition 3.4 and its proof $q_{n+2}/q_{n+1} > 2 \Rightarrow q_{n+2}/q_n > 2g > 1$, $q_{n+2}/q_{n+1} \in (1,2) \Rightarrow q_{n+1}/q_n > 1 \Rightarrow q_{n+2}/q_n > 1$, and $q_{n+2}/q_{n+1} \in$

$(g,1) \Rightarrow q_{n+1}/q_n > 2+g \Rightarrow q_{n+2}/q_n > g(2+g) > 1$. Thus in all possible cases $q_{n+2} > q_n$, which establishes (i).

(ii) follows from $q_{n+3}/q_{n+2} \in (g,1) \Rightarrow q_{n+2}/q_{n+1} > 2+g \Rightarrow q_{n+3}/q_n > (2+g)g^2 = 1$, $q_{n+3}/q_{n+2} = \lambda \in (1,2) \Rightarrow q_{n+2}/q_{n+1} = 1/(\lambda-1) \Rightarrow q_{n+3}/q_n > \lambda g/(\lambda-1) > 2g > 1$, $q_{n+3}/q_{n+2} \in (2,2+g) \Rightarrow q_{n+2}/q_{n+1} \in (g,1) \Rightarrow q_{n+1}/q_n > 2+g \Rightarrow q_{n+3}/q_n > 2g(2+g) > 1$, and $q_{n+3}/q_n > 2+g \Rightarrow q_{n+3}/q_n > (2+g)g^2 = 1$.

To prove (iii) suppose that $q_{n+2} \le q_{n+1}$. Then $q_{n+2}/q_{n+1} \in (g,1)$, which gives in turn $q_{n+1}/q_n > 2+g$, and therefore $q_{n+2}/q_n > g(2+g) > 1$. ∎

Remark. Proposition 3.2 was originally proved, using a different method, by Kraaikamp and Lopes [7], but Proposition 3.4 is, to the best of our research, new. Our proofs have an additional benefit of implying how to derive a_n and e_{n-1} (and hence q_{n-2}) if only q_{n-1} and q_n are known.

Our investigations yielded yet another method of proof, significantly longer but more direct, which we sketch here. Examples 1.8 in [8] explain how to algorithmically generate the OCF expansion of x from the RCF expansion of x using *insertion*,

$$\begin{aligned} \text{replacing} \quad & [[\ldots,(a_n,1),(a_{n+1},e_{n+1}),\ldots]] \\ \text{with} \quad & [[\ldots,(a_n+1,-1),(1,1),(a_{n+1}-1,e_{n+2}),\ldots]], \end{aligned}$$

and *singularization*,

$$\begin{aligned} \text{replacing} \quad & [[\ldots,(a_n,e_n),(1,1),(a_{n+2},e_{n+2}),\ldots]] \\ \text{with} \quad & [[\ldots,(a_n+e_n,-e_n),(a_{n+2}+1,e_{n+2}),\ldots]]. \end{aligned}$$

Both of these operations alter the sequence of convergents: insertion adds a new convergent, while singularization deletes one. Nevertheless, it can be shown that if $P/Q, P'/Q'$ are successive RCF convergents to some x, then either $P/Q, P'/Q'$ are successive OCF convergents to x, or $(Q-P)/Q$, $(Q'-P')/Q'$ are successive OCF convergents to $1-x$. (Only one of these pairs forms a matrix that is congruent to I, A, or B modulo 2.) By carefully following how insertion and singularization change the last e_{n-1} and a_n in the RCF expansion of P'/Q' into the last e_{m-1} and a_m of the OCF expansion of P'/Q', we can determine exactly what $e(M)$ and $a(M)$ must be and hence how to derive P_0 and Q_0. A similar proof works for the ECF case as well.

4. Estimating the limiting joint distribution for ECF and OCF. For each $M = \left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathcal{R}$ and $\xi \in (0,1]$ denote by $I_\xi^+(M)$ (respectively, $I_\xi^-(M)$) the set of solutions x of $0 \le E_M(x) \le \xi$ (respectively, of $-\xi \le E_M(x) \le 0$). The Lebesgue measure of $I_\xi^\pm(M)$ is

$$f_\xi^\pm(Q,Q') = \left| \frac{P' \pm \xi P}{Q' \pm \xi Q} - \frac{P'}{Q'} \right| = \frac{\xi}{Q'(Q' \pm \xi Q)}.$$

The integral

$$F_{\pm} = F_{\pm}(x_1, x_2, x_3, x_4) := \int_{R/x_2}^{\infty} dv \int_0^{\min\{x_3 v, x_1 R\}} du\, f_{x_4}^{\pm}(u, v)$$
$$= \pm \int_{R/x_2}^{\infty} \frac{dv}{v} \log \left| \frac{v \pm x_4 \min\{x_3 v, x_1 R\}}{v} \right|$$
$$= \pm \int_{x_3/x_2}^{\infty} \frac{dw}{w} \log \left| \frac{w \pm x_3 x_4 \min\{w, x_1\}}{w} \right|$$

can be expressed when $x_3 \geq x_1 x_2$ as

$$F_{\pm} = \pm \int_0^{x_1 x_2 x_4} \frac{dt}{t} \log(1 \pm t) = \mp \operatorname{Li}_2(\mp x_1 x_2 x_4),$$

and when $x_3 < x_1 x_2$ as

$$F_{\pm} = \int_{x_3/x_2}^{x_1} \frac{dw}{w} \log(1 \pm x_3 x_4) \pm \int_{x_1}^{\infty} \frac{dw}{w} \log \frac{w \pm x_1 x_3 x_4}{w}$$
$$= \pm \log(1 \pm x_3 x_4) \log \frac{x_1 x_2}{x_3} \mp \operatorname{Li}_2(\mp x_3 x_4),$$

so $F_{\pm}$ is as in (1.5).

4.1. The ECF case. By Lemma 3.1 and Proposition 3.2, for each $R > 1$ and $x \in \Omega$ there is a unique $M = \left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathcal{R}_E$ with $Q \leq R < Q'$ and $|E_M(x)| < 1$. Given $x_1, x_2, x_3, x_4 \in (0,1)$ consider $\mathcal{N}^{E,\pm}_{x_1,x_2,x_3,x_4}(x, R)$, the number of matrices $M \in \mathcal{R}_E$ that satisfy (1.1) and (1.2). One has

$$\mathcal{L}^{E,\pm}(R) = \mathcal{L}^{E,\pm}_{x_1,x_2,x_3,x_4}(R) = \int_0^1 \mathcal{N}^{E,\pm}_{x_1,x_2,x_3,x_4}(x, R)\, dx.$$

For $\Gamma \in \{I, J, A, B\}$ we shall estimate

$$\mathcal{L}^{\pm}_{\Gamma}(R) := \sum_{\substack{M = \left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathcal{R}_E \\ Q' \geq R/x_2 \\ Q \leq \min\{x_3 Q', x_1 R\} \\ M \equiv \Gamma \pmod 2}} f^{\pm}_{x_4}(Q, Q').$$

This can be done by Möbius summation, as in the following standard lemmas (for Lemma 4.2 see, e.g., [2, Lemma 2.1]).

Lemma 4.1. *For every interval J, every function $g \in C^1(J)$ of total variation $T_J g$, and every integer x, with σ_0 the divisor counting function,*

$$\sum_{\substack{a\in J,\, b\in[1,q]\\ ab\equiv x\,(\mathrm{mod}\,q)\\ (a,q)=1}} g(a) = \sum_{\substack{a\in J\\ (a,q)=1}} g(a) = \frac{\varphi(q)}{q}\int_J g(u)\,du + O(\sigma_0(q)(\|g\|_\infty + T_J g)).$$

LEMMA 4.2. *For every interval J, every $V\in C^1[0,N]$, and every $\ell\in\mathbb{N}$,*

$$\sum_{\substack{1\le q\le N\\ (q,\ell)=1}} \frac{\varphi(q)}{q} V(q) = C(\ell)\int_0^N V(u)\,du + O_\ell((\|V\|_\infty + T_0^N V)\log N),$$

with

$$C(\ell) = \frac{1}{\zeta(2)}\prod_{\substack{p\in\mathcal{P}\\ p|\ell}}\left(1+\frac{1}{p}\right)^{-1}.$$

Changing b to $q-b$ in Lemma 4.1, we obtain

COROLLARY 4.3. *Suppose q is an odd positive integer. For every interval J, every $g\in C^1(J)$, and every integer x,*

$$\sum_{\substack{a\in J,\, b\in[1,q/2]\\ ab\equiv x \text{ or } -x\,(\mathrm{mod}\,q)\\ (a,q)=1}} g(a) = \frac{\varphi(q)}{q}\int_J g(u)\,du + O((\|g\|_\infty + T_J g)\sigma_0(q)).$$

Since $P'Q-PQ'=\pm1$, P',Q even and Q' odd entail P odd, we infer (with $Q=2q$, $P'=2p'$, $\bar{x}$ the multiplicative inverse of x (mod Q'))

$$\begin{aligned}
(4.1)\qquad \mathcal{L}_I^{\pm}(R) &= \sum_{\substack{Q'\ge R/x_2\\ Q'\equiv1\,(\mathrm{mod}\,2)}} \sum_{\substack{q\in[1,\min\{x_3Q',y_1R\}/2]\\ p'\in[1,Q'/2]\\ p'q\equiv\pm\bar{4}\,(\mathrm{mod}\,Q')}} f_{x_4}^{\pm}(2q,Q')\\
&= \sum_{\substack{Q'\ge R/x_2\\ Q'\equiv1\,(\mathrm{mod}\,2)}} \left(\frac{\varphi(Q')}{Q'}\int_0^{\min\{x_3Q',x_1R\}/2} f_{x_4}^{\pm}(2q,Q')\,dq + O_\varepsilon(Q'^{-2+\varepsilon})\right)\\
&= \frac{1}{2}\sum_{\substack{Q'\ge R/x_2\\ Q'\equiv1\,(\mathrm{mod}\,2)}} \frac{\varphi(Q')}{Q'}\int_0^{\min\{x_3Q',x_1R\}} f_{x_4}^{\pm}(u,Q')\,du + O_\varepsilon(R^{-1+\varepsilon})\\
&= \frac{C(2)F_\pm}{2} + O_\varepsilon(R^{-1+\varepsilon}) = \frac{F_\pm}{3\zeta(2)} + O_\varepsilon(R^{-1+\varepsilon}).
\end{aligned}$$

On the other hand, we have that $P'Q-PQ'=\pm1$ and Q' even entail that both Q and P' are odd, and the condition that P is even is equivalent to $P'Q\equiv\pm1$ (mod $2Q'$). Since in this case $\varphi(2Q')=2\varphi(Q')$, we infer

$$\mathcal{L}_J^{\pm}(R) = \sum_{\substack{Q'\geq R/x_2 \\ Q'\equiv 0 \,(\mathrm{mod}\, 2)}} \sum_{\substack{Q\in[1,\min\{x_3Q',x_1R\}] \\ P'\in[1,Q'] \\ P'Q\equiv\pm 1 \,(\mathrm{mod}\, 2Q')}} f_{x_4}^{\pm}(Q,Q')$$

$$= \sum_{\substack{Q'\geq R/x_2 \\ Q'\equiv 0 \,(\mathrm{mod}\, 2)}} \bigg(\frac{\varphi(2Q')}{2Q'} \int_0^{\min\{x_3Q',x_1R\}} f_{x_4}^{\pm}(u,Q')\,du + O_\varepsilon(Q'^{-2+\varepsilon})\bigg)$$

$$= \bigg(\frac{1}{\zeta(2)} - C(2)\bigg)F_{\pm} + O_\varepsilon(R^{-1+\varepsilon}) = \frac{F_{\pm}}{3\zeta(2)} + O_\varepsilon(R^{-1+\varepsilon}),$$

leading to

$$\mathcal{L}^{E,\pm}(R) = \mathcal{L}_I^{\pm}(R) + \mathcal{L}_J^{\pm}(R) = \frac{2F_{\pm}}{3\zeta(2)} + O_\varepsilon(R^{-1+\varepsilon}),$$

and concluding the proof of (1.3).

The corresponding estimates for $\mathcal{L}_B^{\pm}(R)$ and $\mathcal{L}_A^{\pm}(R)$ are useful for the OCF situation. To estimate $\mathcal{L}_B^{\pm}(R)$, note that $P'Q - PQ' = \pm 1$ and Q' even entail that both P' and Q are odd, $\varphi(2Q') = 2\varphi(Q')$, and thus

$$\mathcal{L}_B^{\pm}(R) = \sum_{\substack{Q'\geq R/x_2 \\ Q'\equiv 0 \,(\mathrm{mod}\, 2)}} \sum_{\substack{Q\in[1,\min\{x_3Q',x_1R\}] \\ P'\in[1,Q'],\, P'Q\equiv\pm 1 \,(\mathrm{mod}\, Q') \\ \frac{P'Q\mp 1}{Q'}\equiv 1 \,(\mathrm{mod}\, 2)}} f_{x_4}^{\pm}(Q,Q') \tag{4.2}$$

$$= \sum_{\substack{Q'\geq R/x_2 \\ Q'\equiv 0 \,(\mathrm{mod}\, 2)}} \Bigg(\sum_{\substack{Q\in[1,\min\{x_3Q',x_1R\}] \\ P'\in[1,Q'],\, P'Q\equiv\pm 1 \,(\mathrm{mod}\, Q')}} f_{x_4}^{\pm}(Q,Q') - \sum_{\substack{Q\in[1,\min\{x_3Q',x_1R\}] \\ P'\in[1,Q'],\; P'Q\equiv\pm 1 \,(\mathrm{mod}\, 2Q')}} f_{x_4}^{\pm}(Q,Q')\Bigg)$$

$$= \sum_{\substack{Q'\geq R/x_2 \\ Q'\equiv 0 \,(\mathrm{mod}\, 2)}} \Bigg(\bigg(\frac{2\varphi(Q')}{Q'} - \frac{\varphi(2Q')}{2Q'}\bigg) \int_0^{\min\{x_3Q',x_1R\}} f_{x_4}^{\pm}(u,Q')\,du + O_\varepsilon(Q'^{-2+\varepsilon})\Bigg)$$

$$= \sum_{\substack{Q'\geq R/x_2 \\ Q'\equiv 0 \,(\mathrm{mod}\, 2)}} \bigg(\frac{\varphi(Q')}{Q'} \int_0^{\min\{x_3Q',x_1R\}} f_{x_4}^{\pm}(u,Q')\,du + O_\varepsilon(Q'^{-2+\varepsilon})\bigg)$$

$$= \bigg(\frac{1}{\zeta(2)} - C(2)\bigg)F_{\pm} + O_\varepsilon(R^{-1+\varepsilon}) = \frac{F_{\pm}}{3\zeta(2)} + O_\varepsilon(R^{-1+\varepsilon}).$$

Finally, $P'Q - PQ' = \pm 1$ and P even entail that both P' and Q are odd, and so

$$\mathcal{L}_A^{\pm}(R) = \sum_{\substack{Q'\geq R/x_2 \\ Q'\equiv 1 \,(\mathrm{mod}\, 2)}} \sum_{\substack{Q\in[1,\min\{x_3Q',x_1R\}] \\ P'\in[1,Q'],\, P'Q\equiv\pm 1 \,(\mathrm{mod}\, 2Q')}} f_{x_4}^{\pm}(Q,Q') \tag{4.3}$$

$$= \sum_{\substack{Q'\geq R/x_2\\ Q'\equiv 1\,(\mathrm{mod}\,2)}} \left(\frac{\varphi(2Q')}{2Q'} \int\limits_0^{\min\{x_3Q',x_1R\}} f^{\pm}_{x_4}(u,Q')\,du + O_\varepsilon(Q'^{-2+\varepsilon})\right)$$
$$= \frac{C(2)}{2}\,F_\pm + O_\varepsilon(R^{-1+\varepsilon}) = \frac{F_\pm}{3\zeta(2)} + O_\varepsilon(R^{-1+\varepsilon}).$$

4.2. The OCF case. This requires more caution as the sequence of denominators of successive convergents is not increasing in general. We wish to characterize those matrices $M\in\mathcal{R}_O$ for which $P/Q, P'/Q'$ are successive convergents of $x\in\Omega$ and $Q=q_{n_R}\leq R<Q'=q_{n_R+1}$. A priori, Lemma 3.7 shows that for each $R>1$ there is at least one and at most two pairs (Q,Q') of denominators of successive convergents of x with $Q\leq R<Q'$. Moreover, if there are two such pairs, then they must be of the form (q_{n_R},q_{n_R+1}) or (q_{n_R+2},q_{n_R+3}). We wish to precisely distinguish n_R from n_R+2. Because all predecessors of Q_0 in the sequence of denominators of OCF convergents are $<Q$ by Lemma 3.7, equality $(Q,Q')=(q_{n_R},q_{n_R+1})$ occurs exactly when

$$Q\leq R<Q' \quad \text{and} \quad R>Q_0.$$

Note that if $\lambda=Q'/Q\in\mathcal{S}_1\cup\mathcal{S}_2$, then necessarily $Q>Q_0$. Furthermore, if $\lambda\in\mathcal{S}_3$, then $Q<Q_0$. The contribution of those pairs (Q,Q') with $\lambda\in\mathcal{S}_3$ and $Q_0=Q(1+\{\lambda\})>R$ should be subtracted, and so we can write

$$\mathcal{L}^{O,+}(R)=\mathcal{L}^+_I(R)+\mathcal{L}^+_A(R)+\mathcal{L}^+_B(R)-\mathcal{D}_1(R),$$
$$\mathcal{L}^{O,-}(R)=\mathcal{L}^-_I(R)+\mathcal{L}^-_A(R)+\mathcal{L}^-_B(R)-\mathcal{D}_2(R)-\mathcal{D}_3(R),$$

with

$$\mathcal{D}_1(R)=\sum_{\substack{M\in\mathcal{R}_O,\,Q'>R/x_2\\ Q\leq\min\{x_3Q',x_1R\}\\ \lambda=Q'/Q\in\mathcal{S}_3,\,Q(1+\{\lambda\})>R}} f^+_{x_4}(Q,Q')=\sum_{\ell\geq 1}\sum_{\substack{M\in\mathcal{R}_O,\,Q'>R/x_2\\ Q\leq\min\{x_3Q',x_1R\}\\ 2\ell Q\leq Q'<(2\ell+g)Q\\ Q'>R+(2\ell-1)Q}} \frac{x_4}{Q'(Q'+x_4Q)},$$

$$\mathcal{D}_2(R)=\sum_{\substack{M\in\mathcal{R}_O,\,Q'>R/x_2\\ Q\leq\min\{x_3Q',x_1R\}\\ \lambda=Q'/Q\in[2,2+g),\,Q'>R+Q}} \frac{x_4}{Q'(Q'-x_4Q)},$$

$$\mathcal{D}_3(R)=\sum_{\substack{M\in\mathcal{R}_O,\,Q'>R/x_2\\ Q\leq\min\{x_3Q',x_1R\}\\ \lambda=Q'/Q\in\mathcal{S}_3,\,\lambda>G^2\\ Q(1+\{\lambda\})>R}} f^-_{x_4}(Q,Q')=\sum_{\ell\geq 2}\sum_{\substack{M\in\mathcal{R}_O,\,Q'>R/x_2\\ Q\leq\min\{x_3Q',x_1R\}\\ 2\ell Q\leq Q'<(2\ell+g)Q\\ Q'>R+(2\ell-1)Q}} \frac{x_4}{Q'(Q'-x_4Q)}.$$

Clearly $\mathcal{D}_2(R)=0$ when $\min\{x_1x_2,x_3\}\le g^2$. When $\min\{x_1x_2,x_3\}>g^2$, the method employed in (4.1)–(4.3) leads, with D_2 as in (1.6), to

$$\mathcal{D}_2(R)=\frac{D_2(x_1,x_2,x_3,x_4)}{\zeta(2)}+O_\varepsilon(R^{-1+\varepsilon}).$$

The estimation of $\mathcal{D}_1(R)$ is slightly more involved because ℓ can take infinitely many values. Note that $\mathcal{D}_1(R)=0$ unless $\min\{x_1x_2,x_3\}>1/(2\ell+g)$. For each $\ell\in\mathbb{N}$ consider the integral

$$I_\ell^+(R):=\iint\limits_{\substack{v\ge R/x_2,\,u\le\min\{x_3v,x_1R\}\\ 2\ell u\le v\le(2\ell+g)u\\ v>R+(2\ell-1)u}}\frac{x_4\,du\,dv}{v(v+x_4u)}.$$

The change of variables $(v,u)=(Ry,Rx)$ shows that $I_\ell^+(R)$ does not depend on R and is given by (1.7). Note also that

$$I_\ell^+(R)\le\int_0^{x_1}dx\int_{2\ell x}^{(2\ell+1)x}\frac{dy}{y^2}\ll\frac{1}{\ell^2}. \tag{4.4}$$

A trivial estimate yields

$$\begin{aligned}\sum_{\substack{\ell\ge R^{1/2}\\ R/x_2\le Q'\le(2\ell+1)R}}\ \sum_{Q'/(2\ell+1)\le Q\le Q'/(2\ell)}\frac{1}{Q'(Q'+x_4Q)}&\\ \le\sum_{\substack{\ell\ge R^{1/2}\\ 1\le Q'\le(2\ell+1)R}}\ \sum_{Q'/(2\ell+1)\le Q\le Q'/(2\ell)}&\frac{1}{Q^2\ell^2}\\ \ll\sum_{\ell\ge R^{1/2}}\frac{1}{\ell^2}\sum_{Q\in[1,2R]}\ \sum_{Q'\in[2\ell Q,(2\ell+1)Q]}&\frac{1}{Q^2}\ll\frac{\log R}{R^{1/2}},\end{aligned}$$

and thus in the definition of $\mathcal{D}_1(R)$ we may take $\ell\in[1,R^{1/2}]$ inserting an error term $\ll R^{-1/2}\log R$. Employing Lemma 4.1, we can express the resulting main term as

$$\sum_{\ell\le R^{1/2}}\ \sum_{\substack{Q'\ge R/x_2\\ Q'<(2\ell+g)x_1R}}\Biggl(\frac{\varphi(Q')}{Q'}\int_{Q'/(2\ell+g)}^{\min\{\frac{Q'}{2\ell},x_3Q',x_1R,\frac{Q'-R}{2\ell-1}\}}\frac{x_4\,du}{Q'(Q'+x_4u)}+O(Q'^{-2+\varepsilon})\Biggr)$$

$$=\Biggl(\sum_{\ell\le R^{1/2}}\ \sum_{\substack{Q'\ge R/x_2\\ Q'<(2\ell+g)x_1R}}\frac{\varphi(Q')}{Q'}\int_{Q'/(2\ell+g)}^{\min\{\frac{Q'}{2\ell},x_3Q',x_1R,\frac{Q'-R}{2\ell-1}\}}\frac{x_4\,du}{Q'(Q'+x_4u)}\Biggr)+O_\varepsilon(R^{-1/2+\varepsilon}).$$

By Lemma 4.2, the main term above becomes

$$\sum_{\ell\le R^{1/2}}\biggl(\frac{I_\ell^+}{\zeta(2)}+O\biggl(\frac{\log R}{R^{1/2}}\biggr)\biggr),$$

and so

$$\mathcal{D}_1(R)=\frac{1}{\zeta(2)}\sum_{\ell\le R^{1/2}} I_\ell^+ + O_\varepsilon(R^{-1/2+\varepsilon}). \tag{4.5}$$

From (4.5) and (4.4) we eventually infer

$$\mathcal{D}_1(R)=\frac{1}{\zeta(2)}\sum_{\ell\ge 1} I_\ell^+ + O_\varepsilon(R^{-1/2+\varepsilon}).$$

The sum $\mathcal{D}_3(R)$ is similarly estimated as in formulas (1.6) and (1.7).

5. Joint distribution for Nakada's α-expansions. We illustrate how explicit renewal type results can be obtained in the case of Nakada's α-expansions NCF_α, $\alpha\in[1/2,1]$. Such continued fractions, defined in [10], have been studied in [10, 6]. Here the unit interval is replaced by $\Omega_\alpha=[\alpha-1,\alpha)$ and the Gauss shift by the map $T_\alpha:\Omega_\alpha\to\Omega_\alpha$ defined for $x\ne 0$ by ([3])

$$T_\alpha(x)=\biggl|\frac{1}{x}\biggr|-\biggl[\biggl|\frac{1}{x}\biggr|+1-\alpha\biggr].$$

A construction of the natural extension $\overline{T}_\alpha$ on a space $\underline{\Omega}_\alpha\subset\mathbb{R}^2$, together with an explicit invariant Borel probability measure μ_α on $\underline{\Omega}_\alpha$, was found by Nakada [10]. He also proved that $(\underline{\Omega}_\alpha,\overline{T}_\alpha,\mu_\alpha)$ is a Kolmogorov automorphism. With $g=1/G=1-g^2$ the set $\underline{\Omega}_\alpha$ is given for $g<\alpha\le 1$ by

$$[\alpha-1,(1-\alpha)/\alpha]\times[0,1/2)\cup((1-\alpha)/\alpha,\alpha)\times[0,1]\cup[\alpha-1,0)\times\{1/2\},$$

and for $1/2\le\alpha\le g$ by

$$\begin{aligned}[]
[\alpha-1,(1-2\alpha)/\alpha]\times[0,g^2)\cup((1-2\alpha)/\alpha,(2\alpha-1)/(1-\alpha)]\times[0,1/2)&\\
\cup((2\alpha-1)/(1-\alpha),\alpha)\times[0,g)\cup[-g^2,(1-2\alpha)/\alpha]\times\{g^2\}&\\
\cup((1-2\alpha)/\alpha,0)\times\{1/2\}.&
\end{aligned}$$

Kraaikamp's thoughtful analysis (see especially Theorem (5.3) and Definitions (5.7) and (5.8) of [6]) also provides characterizations of pairs of successive convergents for such continued fractions if $\alpha\in[1/2,1]$.

PROPOSITION 5.1. *For each $x\in\Omega_\alpha\setminus\mathbb{Q}$ the following are equivalent:*

(i) *$P/Q, P'/Q'$ are successive convergents in* $\mathrm{NCF}_\alpha(x)$ *with $Q,Q'>0$.*
(ii) *$M=\left(\begin{smallmatrix}P & P'\\ Q & Q'\end{smallmatrix}\right)\in\mathrm{GL}_2(\mathbb{Z})$ and $(E_M(x),1/\lambda_M)\in\underline{\Omega}_\alpha$.*

([3]) Here we use the notation from Sections 5 and 6 of [6].

This dynamical system was studied by Kraaikamp [6] in the more general setting of S-expansions, and the above proposition can be likewise generalized if we replace $\mathrm{NCF}_\alpha(x)$ with $\mathrm{CF}_S(x)$, the S-expansion of x, and replace $\underline{\Omega}_\alpha$ with $\underline{\Omega}_S$, the space of the natural extension associated to S.

We wish to estimate the Lebesgue measure $\mathcal{L}^{(\alpha),\pm}_{x_1,x_2,x_3,x_4}(R)$ of the set of numbers $x \in \Omega_\alpha \setminus \mathbb{Q}$ for which there exist successive convergents P/Q, P'/Q' in $\mathrm{NCF}_\alpha(x)$ that satisfy (1.1) and (1.2). We shall require that x_1, x_2, x_3 are in the set $(0,1]$ if $g < \alpha \le 1$, in $(0,1/2]$ if $\alpha = g$, and in $(0,g]$ if $1/2 \le \alpha < g$; moreover, we require $x_4 \in (0,\alpha]$ when we look at $\mathcal{L}^+$ and $x_4 \in (0, 1-\alpha]$ when we look at $\mathcal{L}^-$. The set $\underline{\Omega}_\alpha$ is a union of rectangles and horizontal line segments, but we may ignore the line segments for large R: in particular, the inequality $Q' \ge R/x_2$ shows that the pair $(Q',Q) = (2,1)$ makes no contribution to $\mathcal{L}^\pm$ for $R > 2$, so the situation $\lambda_M^{-1} = 1/2$ can be ignored, and λ_M is always rational, so the situation $\lambda_M^{-1} = g^2$ can also be ignored. Therefore, the cases that appear in $\mathcal{L}^{(\alpha),\pm}_{x_1,x_2,x_3,x_4}(R)$ for $R > 2$ are exactly:

$$\text{For } g < \alpha \le 1: \quad \begin{cases} \lambda_M = Q'/Q > 2 \text{ and } \alpha - 1 \le E_M(x) < \alpha, & \text{or} \\ 1 \le \lambda_M < 2 \text{ and } \frac{1-\alpha}{\alpha} < E_M(x) < \alpha. \end{cases}$$

$$\text{For } 1/2 \le \alpha \le g: \quad \begin{cases} \lambda_M > G^2 \text{ and } \alpha - 1 \le E_M(x) < \alpha, & \text{or} \\ 2 < \lambda_M < G^2 \text{ and } \frac{1-2\alpha}{\alpha} < E_M(x) < \alpha, & \text{or} \\ G < \lambda_M < 2 \text{ and } \frac{2\alpha-1}{1-\alpha} < E_M(x) < \alpha. \end{cases}$$

The varying lower bounds on λ_M depending on the value of α are the reason for our case-based restrictions on the values of x_1, x_2, x_3.

Let $\mathcal{L}^+_{x_1,x_2,x_3,x_4}(\alpha; R)$ denote the Lebesgue measure of the set of numbers $x \in [0,1] \setminus \mathbb{Q}$ for which there exists $M = \left(\begin{smallmatrix} P & P' \\ Q & Q' \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$ with $Q, Q' > 0$, $P/Q, P'/Q' \in [\alpha - 1, \alpha)$ and (1.1) together with $0 \le \frac{Q'x - P'}{-Qx + P} \le x_4$ hold. The corresponding set where the latter inequality is replaced by $-x_4 \le \frac{Q'x-P'}{-Qx+P} \le 0$ is denoted by $\mathcal{L}^-_{x_1,x_2,x_3,x_4}(\alpha; R)$. In both cases, x_1, x_2, x_3, x_4 are parameters in $(0,1]$. When $\alpha = 1$, it is clear that $\mathcal{L}^+$ is exactly the joint distribution considered in [16] (the notation there is $N(R)$). However, as

$$\begin{aligned} \mathcal{L}^\pm_{x_1,x_2,x_3,x_4}(\alpha; R) &= \sum_{Q' \ge R/x_2} \sum_{\substack{Q \in (0, \min\{x_3 Q', x_1 R\}] \\ P' \in (\alpha-1)Q' + [0,Q') \\ P'Q \equiv \pm 1 \,(\mathrm{mod}\, Q')}} \frac{x_4}{Q'(Q' \pm x_4 Q)} \\ &= 2 \sum_{Q' \ge R/x_2} \sum_{\substack{Q \in (0, \min\{x_3 Q', x_1 R\}] \\ (Q,Q')=1}} \frac{x_4}{Q'(Q' \pm x_4 Q)} \\ &= \mathcal{L}^\pm_{x_1,x_2,x_3,x_4}(R), \end{aligned}$$

we see that $\mathcal{L}^{\pm}(\alpha; R)$ does not depend on α. As R tends to infinity, $\mathcal{L}^{\pm}$ converges to $2F^{\pm}/\zeta(2)$.

The joint distributions $\mathcal{L}^{(\alpha),\pm}$ and $\mathcal{L}^{\pm}$ can now be directly related as below. For brevity and readability we omit the appearance of x_1, x_2, and R, which are assumed to be the same on the left- and right-hand sides of the equations.

When $g < \alpha \le 1$, we have

$$\mathcal{L}^{(\alpha),+}_{x_3,x_4} = \begin{cases} \mathcal{L}^{+}_{\min\{x_3,1/2\},x_4} & \text{if } 0 \le x_4 \le (1-\alpha)/\alpha, \\ \mathcal{L}^{+}_{x_3,x_4} - \mathcal{L}^{+}_{x_3,(1-\alpha)/\alpha} + \mathcal{L}^{+}_{\min\{x_3,1/2\},(1-\alpha)/\alpha} & \text{if } (1-\alpha)/\alpha \le x_4 < \alpha, \end{cases}$$

$$\mathcal{L}^{(\alpha),-}_{x_3,x_4} = \mathcal{L}^{-}_{\min\{x_3,1/2\},x_4} \quad \text{if } 0 \le x_4 \le 1-\alpha.$$

When $1/2 \le \alpha \le g$, we have

$$\mathcal{L}^{(\alpha),+}_{x_3,x_4} = \begin{cases} \mathcal{L}^{+}_{\min\{x_3,1/2\},x_4} & \text{if } 0 \le x_4 \le (2\alpha-1)/(1-\alpha), \\ \mathcal{L}^{+}_{x_3,x_4} - \mathcal{L}^{+}_{x_3,(2\alpha-1)/(1-\alpha)} & \\ \quad + \mathcal{L}^{+}_{\min\{x_3,1/2\},(2\alpha-1)/(1-\alpha)} & \text{if } (2\alpha-1)/(1-\alpha) \le x_4 < \alpha. \end{cases}$$

$$\mathcal{L}^{(\alpha),-}_{x_3,x_4} = \begin{cases} \mathcal{L}^{-}_{\min\{x_3,1/2\},x_4} & \text{if } 0 \le x_4 \le (2\alpha-1)/\alpha, \\ \mathcal{L}^{-}_{\min\{x_3,g^2\},x_4} + \mathcal{L}^{-}_{\min\{x_3,1/2\},(2\alpha-1)/\alpha} & \\ \quad - \mathcal{L}^{-}_{\min\{x_3,g^2\},(2\alpha-1)/\alpha} & \text{if } (2\alpha-1)/\alpha \le x_4 \le 1-\alpha. \end{cases}$$

Recall that $x_3 \le g$ in this case.

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