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Cantor series constructions of sets of normal numbers

by

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1. Introduction

Definition 1.1. Let b and k be positive integers. A *block of length k in base b* is an ordered k-tuple of integers in $\{0,1,\dots,b-1\}$. A *block of length k* is a block of length k in some base b. A *block* is a block of length k in base b for some integers k and b. Given a block B, $|B|$ will represent the length of B.

Definition 1.2. Given an integer $b \geq 2$, the *b-ary expansion* of a real x in $[0,1)$ is the (unique) expansion of the form

$$x = \sum_{n=1}^{\infty} \frac{E_n}{b^n} = 0.E_1E_2\dots \tag{1.1}$$

such that E_n is in $\{0,1,\dots,b-1\}$ for all n with $E_n \neq b-1$ infinitely often.

Denote by $N_n^b(B,x)$ the number of times a block B occurs with its starting position no greater than n in the b-ary expansion of x.

Definition 1.3. A real number x in $[0,1)$ is *normal in base b* if for all k and blocks B in base b of length k, one has

$$\lim_{n\to\infty} \frac{N_n^b(B,x)}{n} = b^{-k}. \tag{1.2}$$

A number x is *simply normal in base b* if (1.2) holds for $k=1$.

Borel introduced normal numbers in 1909 and proved that almost all (in the sense of Lebesgue measure) real numbers in $[0,1)$ are normal in all bases [3]. The best known example of a number that is normal in base 10 is due to Champernowne [5]. The number $H_{10} = 0.1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12\dots$, formed by concatenating the digits of every natural number written in increasing order in base 10, is normal in base 10. Any H_b, formed similarly to H_{10} but in base b, is known to be normal in base b. Since then, many examples have

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been given of numbers that are normal in at least one base. One can find a more thorough literature review in [6] and [10].

The Q-Cantor series expansion, first studied by Georg Cantor in [4], is a natural generalization of the b-ary expansion.

DEFINITION 1.4. $Q = (q_n)_{n=1}^{\infty}$ is a *basic sequence* if each q_n is an integer greater than or equal to 2.

DEFINITION 1.5. Given a basic sequence Q, the *Q-Cantor series expansion* of a real x in $[0, 1)$ is the (unique (1)) expansion of the form

$$x = \sum_{n=1}^{\infty} \frac{E_n}{q_1 \cdots q_n} \tag{1.3}$$

such that E_n is in $\{0, 1, \ldots, q_n - 1\}$ for all n with $E_n \neq q_n - 1$ infinitely often. We abbreviate (1.3) with the notation $x = 0.E_1E_2E_3\ldots$ with respect to Q.

Clearly, the b-ary expansion is a special case of (1.3) where $q_n = b$ for all n. If one thinks of a b-ary expansion as representing an outcome of repeatedly rolling a fair b-sided die, then a Q-Cantor series expansion may be thought of as representing an outcome of rolling a fair q_1-sided die, followed by a fair q_2-sided die and so on. For example, if $q_n = n + 1$ for all n, then the Q-Cantor series expansion of $e - 2$ is

$$e - 2 = \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots .$$

If $q_n = 10$ for all n, then the Q-Cantor series expansion of $1/4$ is

$$\frac{1}{4} = \frac{2}{10} + \frac{5}{10^2} + \frac{0}{10^3} + \frac{0}{10^4} + \cdots .$$

For a given basic sequence Q, let $N_n^Q(B, x)$ denote the number of times a block B occurs starting at a position no greater than n in the Q-Cantor series expansion of x. Additionally, define

$$Q_n^{(k)} = \sum_{j=1}^{n} \frac{1}{q_j q_{j+1} \cdots q_{j+k-1}}.$$

A. Rényi [15] defined a real number x to be normal with respect to Q if for all blocks B of length 1,

$$\lim_{n \to \infty} \frac{N_n^Q(B, x)}{Q_n^{(1)}} = 1. \tag{1.4}$$

If $q_n = b$ for all n, then (1.4) is equivalent to *simple normality in base b*, but

(1) Uniqueness can be proven in the same way as for the b-ary expansion.

not equivalent to *normality in base b*. Thus, we want to generalize normality in a way that is equivalent to normality in base b when all $q_n = b$.

Definition 1.6. A real number x is *Q-normal of order k* if for all blocks B of length k,

$$\lim_{n\to\infty} \frac{N_n^Q(B,x)}{Q_n^{(k)}} = 1.$$

We say that x is *Q-normal* if it is Q-normal of order k for all k. Additionally, x is *simply Q-normal* if it is Q-normal of order 1.

We make the following definitions:

Definition 1.7. A basic sequence Q is *k-divergent* if $\lim_{n\to\infty} Q_n^{(k)} = \infty$; *fully divergent* if it is k-divergent for all k; and *k-convergent* if it is not k-divergent.

Definition 1.8. A basic sequence Q is *infinite in limit* if $q_n \to \infty$.

For Q that are infinite in limit, it has been shown that the set of all x in $[0,1)$ that are Q-normal of order k has full Lebesgue measure if and only if Q is k-divergent [15]. Therefore if Q is infinite in limit, then the set of all x in $[0,1)$ that are Q-normal has full Lebesgue measure if and only if Q is fully divergent.

Definition 1.9. Let x be a number in $[0,1)$ and let Q be a basic sequence. Then $T_{Q,n}(x)$ is defined as $q_1 \cdots q_n x \pmod 1$.

Definition 1.10. A number x in $[0,1)$ is *Q-distribution normal* if the sequence $(T_{Q,n}(x))_{n=0}^{\infty}$ is uniformly distributed in $[0,1)$.

Remark 1.11. For every basic sequence Q, the set of Q-distribution normal numbers has full Lebesgue measure.

Note that in base b, where $q_n = b$ for all n, the notions of Q-normality and Q-distribution normality are equivalent. This equivalence is fundamental in the study of normality in base b. It is surprising that this equivalence breaks down in the more general context of Q-Cantor series for general Q. Examples are given in [2] of numbers that satisfy one notion of normality and not others.

In general, it is more difficult to give explicit constructions of normal numbers (for various notions of normality) than it is to give typicality results. An explicit construction of a basic sequence Q and a real number x such that x is Q-normal and Q-distribution normal is given in [2] and [12]. In this paper, we will construct a set of Q-distribution normal numbers for any Q that is infinite in limit. None of these numbers will be Q-normal. Additionally, this set of Q-distribution normal numbers will be perfect and nowhere dense.

We recall the following standard definition that will be useful in studying distribution normality:

Definition 1.12. For a finite sequence $z = (z_1, \ldots, z_n)$, we define the *star discrepancy* $D_n^*(z) = D_n^*(z_1, \ldots, z_n)$ as

$$\sup_{0<\gamma\le 1} \left| \frac{A([0,\gamma), z)}{n} - \gamma \right|.$$

Given an infinite sequence $w = (w_1, w_2, \ldots)$, we furthermore define $D_n^*(w) = D_n^*(w_1, \ldots, w_n)$. For convenience, set $D^*(z_1, \ldots, z_n) = D_n^*(z_1, \ldots, z_n)$.

The star discrepancy will be useful to us due to the following theorem:

Theorem 1.13. *The sequence $w = (w_1, w_2, \ldots)$ is uniformly distributed mod 1 if and only if $\lim_{n\to\infty} D_n^*(w) = 0$.*

Remark 1.14. For any sequence w, $1/n \le D_n^*(w) \le 1$.

The following theorem ([2]) was proven by N. Korobov in [9] and will be of central importance in this paper:

Theorem 1.15. *Given a basic sequence Q and a real number x with Q-Cantor series expansion $x = \sum_{n=1}^{\infty} \frac{E_n}{q_1 \cdots q_n}$, if Q is infinite in limit, then x is Q-distribution normal if and only if $(E_n/q_n)_{n=1}^{\infty}$ is uniformly distributed mod 1.*

We note the following theorem of J. Galambos [8]:

Theorem 1.16. *Let Q be a 1-divergent basic sequence. Let E_k be the digits of the Q-Cantor series expansion of x and put $\theta_k = \theta_k(x) = E_k/q_k$. Then, for almost all x in $[0,1)$,*

$$D_n^*(\theta) \ge \frac{1}{2n} \sum_{k=1}^{n} \frac{1}{q_k}$$

for sufficiently large n.

A discrepancy estimate, valid for certain Q, will be given for the Q-distribution normal numbers that we will construct. We will make use of the following definition from [10]:

Definition 1.17. For $0 \le \delta < 1$ and $\epsilon > 0$, a finite sequence $x_1 < \cdots < x_N$ in $[0,1)$ is called an *almost arithmetic progression*-(δ, ϵ) if there exists an η, $0 < \eta \le \epsilon$, such that the following conditions are satisfied:

(1.5) $$0 \le x_1 \le \eta + \delta\eta;$$

(1.6) $$\eta - \delta\eta \le x_{n+1} - x_n \le \eta + \delta\eta \quad \text{for } 1 \le n \le N-1;$$

(1.7) $$1 - \eta - \delta\eta \le x_N < 1.$$

([2]) T. Šalát proved a stronger result in [16], but we will not need it in this paper.

Almost arithmetic progressions were introduced by P. O'Neil in [14]. He proved that a sequence $(x_n)_n$ of real numbers in $[0,1)$ is uniformly distributed mod 1 if and only if the following holds: for any three positive real numbers δ, ϵ, and ϵ', there exists a positive integer N such that for all $n > N$, the initial segment $x_1, \ldots, x_n$ can be decomposed into an almost arithmetic progression-(δ, ϵ) with at most N_0 elements left over, where $N_0 < \epsilon' N$.

In [1], R. Adler, M. Keane, and M. Smorodinsky showed that the real number whose continued fraction expansion is given by the concatenation of the digits of the continued fraction expansion of the rational numbers

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots \tag{1.8}$$

is normal with respect to the continued fraction expansion. For every Q that is infinite in limit, we use Definition 1.17 to construct a set Θ_Q of Q-distribution normal numbers that are defined similarly to the concatenation of the numbers in (1.8). We prove the following results on Θ_Q:

1. If $x \in \Theta_Q$, then x is Q-distribution normal and not simply Q-normal (Theorem 2.20 and Proposition 2.21).
2. Θ_Q is perfect and nowhere dense (Theorems 3.8 and 3.10).
3. If $x \in \Theta_Q$, $x = 0.E_1E_2\ldots$ with respect to Q, and $X = (E_n/q_n)_{n=1}^{\infty}$, then for certain basic sequences Q, there exists a constant γ_Q such that for all $\psi > 1$,
$$D_n^*(X) < \psi \cdot \gamma_Q \cdot n^{-1/2}$$
for large enough n (Theorem 3.3). For many basic sequences, we can determine the constant γ_Q. In particular, $\gamma_Q = \sqrt{8}$ if $q_n \geq 5n$ for all n.
4. The Hausdorff dimension of Θ_Q is evaluated or approximated for several classes of basic sequences (Theorems 3.17, 3.21, 3.22, and 3.24). Given any $\alpha \in [0,1]$, we provide an example of a basic sequence Q such that Θ_Q has α as its Hausdorff dimension (Theorem 3.26).

2. The construction.

For the rest of this section, we fix a basic sequence Q that is infinite in limit.

2.1. Notation and conventions.

For the rest of this paper, let $\tau(n) = 1 + \cdots + n = n(n+1)/2$ be the nth triangular number. Given a basic sequence Q, we will construct a sequence $l_1, l_2, \ldots$ of positive integers. The following definition will be needed:

Definition 2.1. For each positive integer j, we define

$$\nu_j = \min\{N : q_m \geq 2j^2 \text{ for all } m \geq N\}.$$

We now recursively define the sequence $l_1, l_2, \ldots$:

Definition 2.2. We set $l_1 = \max(\nu_2 - 1, 1)$. Given $l_1, \ldots, l_{i-1}$, we define l_i to be the smallest positive integer such that

$$l_1 + 2l_2 + 3l_3 + \cdots + il_i \geq \nu_{i+1} - 1.$$

Thus, we have

$$l_i = \max(\min\{k : l_1 + 2l_2 + \cdots + (i-1)l_{i-1} + ik \geq \nu_{i+1} - 1\}, 1).$$

Additionally, for any non-negative integer i, we set

$$L_i = \sum_{j=1}^{i} jl_j = l_1 + 2l_2 + \cdots + il_i.$$

Lemma 2.3. *Suppose that a, c, and q are positive integers and $q \geq 2a^2$. Then there exist at least two integers F such that*

$$\frac{F}{q} \in \left[\frac{c}{a} - \frac{1}{2a^2}, \frac{c}{a} + \frac{1}{2a^2}\right]. \tag{2.1}$$

Proof. We assume, for contradiction, that there are fewer than two solutions to (2.1). Thus, there exists an integer F such that

$$\frac{F}{q} < \frac{c}{a} - \frac{1}{2a^2} \quad \text{and} \quad \frac{c}{a} + \frac{1}{2a^2} < \frac{F+2}{q},$$

so

$$\left[\frac{c}{a} - \frac{1}{2a^2}, \frac{c}{a} + \frac{1}{2a^2}\right] \not\subseteq \left[\frac{F}{q}, \frac{F+2}{q}\right]. \tag{2.2}$$

By (2.2), we conclude that

$$\left(\frac{c}{a} + \frac{1}{2a^2}\right) - \left(\frac{c}{a} - \frac{1}{2a^2}\right) < \frac{F+2}{q} - \frac{F}{q},$$

so

$$\frac{1}{a^2} < \frac{2}{q}. \tag{2.3}$$

Cross-multiplying (2.3) gives $q < 2a^2$, which contradicts $q \geq 2a^2$. ∎

Definition 2.4. Let $S_Q = \{(a,b,c) \in \mathbb{N}^3 : b \leq l_a,\ c \leq a\}$ and define $\phi_Q : S_Q \to \mathbb{N}$ by $\phi_Q(a,b,c) = L_{a-1} + (b-1)a + c$.

Lemma 2.5. *The function ϕ_Q is a bijection from S_Q to $\mathbb{N}$.*

Proof. Starting at $n = 1$, put l_1 boxes of length 1, followed by l_2 boxes of length 2, l_3 boxes of length 3, and so on. Then the position of the component c of the bth box of length a is at

$$1l_1 + 2l_2 + \cdots + (a-1)l_{a-1} + (b-1)a + c = \phi_Q(a,b,c),$$

so ϕ_Q is a bijection from S_Q to $\mathbb{N}$. ∎

Definition 2.6. The sequence $F = (F_{(a,b,c)})_{(a,b,c)\in S_Q}$ is a *Q-special sequence* if $F_{(a,b,1)} = 0$ for $(a,b,1) \in S_Q$ and

$$\frac{F_{(a,b,c)}}{q_{\phi_Q(a,b,c)}} \in \left[\frac{c-1}{a} - \frac{1}{2a^2}, \frac{c-1}{a} + \frac{1}{2a^2}\right]$$

for $(a,b,c) \in S_Q$ with $c > 1$. Let Γ_Q denote the set of all Q-special sequences.

Given a Q-special sequence F, Lemma 2.5 allows us to define $E_F = (E_{F,n})_{n=1}^{\infty}$ as follows:

Definition 2.7. Suppose that F is a Q-special sequence. For any positive integer n, we define $E_{F,n} = F_{\phi_Q^{-1}(n)}$ and let $E_F = (E_{F,n})_{n=1}^{\infty}$.

Given finite sequences $w_1, w_2, \ldots$, we let $w_1 w_2 w_3 \ldots$ denote their concatenation.

Definition 2.8. If F is a Q-special sequence and $(a,b,1) \in S_Q$, then we define

$$y_{F,a,b} = \left(\frac{F_{(a,b,c)}}{q_{\phi_Q(a,b,c)}}\right)_{c=1}^{a}$$

and let $D^*_{F,a,b} = D^*(y_{F,a,b})$. We also set

$$y_F = y_{F,1,1} y_{F,1,2} \cdots y_{F,1,l_1} y_{F,2,1} y_{F,2,2} \cdots y_{F,2,l_2} y_{F,3,1} y_{F,3,2} \cdots y_{F,3,l_3} y_{F,4,1} \cdots.$$

Definition 2.9. If F is a Q-special sequence, define

$$x_F = \sum_{n=1}^{\infty} \frac{E_{F,n}}{q_1 \cdots q_n}.$$

We also let $\Theta_Q = \{x_F : F \in \Gamma_Q\}$.

Remark 2.10. By construction, $y_F = (E_{F,n}/q_n)_{n=1}^{\infty}$, so by Theorem 1.15, x_F is Q-distribution normal if and only if y_F is uniformly distributed mod 1.

2.2. Basic lemmas. We will use the following theorem from [13]:

Theorem 2.11. *Let $x_1 < \cdots < x_N$ be an almost arithmetic progression-(δ,ϵ) and let η be the positive real number corresponding to the sequence according to Definition* 1.17. *Then*

$$D_N^* \le \frac{1}{N} + \frac{\delta}{1+\sqrt{1-\delta^2}} \quad \textit{for } \delta > 0 \quad \textit{and} \quad D_N^* \le \min\left(\eta, \frac{1}{N}\right) \quad \textit{for } \delta = 0.$$

Corollary 2.12. *Let $x_1 < \cdots < x_N$ be an almost arithmetic progression-(δ,ϵ) and let η be the positive real number corresponding to the sequence according to Definition* 1.17. *Then $D_N^* \le 1/N + \delta$.*

Lemma 2.13. *If F is a Q-special sequence, then the sequence $y_{F,a,b}$ is an almost arithmetic progression-$(1/a, 1/a)$ and $D^*_{F,a,b} \le 2/a$.*

Proof. The case $a = 1$ is trivial, so suppose that $a > 1$. To show that $y_{F,a,b}$ is an almost arithmetic progression-$(1/a, 1/a)$, we first note that $F_{(a,b,1)} = 0$,

$$0 \leq \frac{F_{(a,b,1)}}{q_{\phi_Q(a,b,1)}} \leq \frac{1}{a} + \frac{1}{a^2},$$

so (1.5) holds.

Next, suppose that $2 \leq c \leq a-1$. By construction,

$$\frac{F_{(a,b,c)}}{q_{\phi_Q(a,b,c)}} \in \left[\frac{c-1}{a} - \frac{1}{2a^2}, \frac{c-1}{a} + \frac{1}{2a^2}\right], \quad \frac{F_{(a,b,c+1)}}{q_{\phi_Q(a,b,c+1)}} \in \left[\frac{c}{a} - \frac{1}{2a^2}, \frac{c}{a} + \frac{1}{2a^2}\right]$$

so

$$\frac{F_{(a,b,c+1)}}{q_{\phi_Q(a,b,c+1)}} - \frac{F_{(a,b,c)}}{q_{\phi_Q(a,b,c)}} \leq \left(\frac{c}{a} + \frac{1}{2a^2}\right) - \left(\frac{c-1}{a} - \frac{1}{2a^2}\right), \tag{2.4}$$

$$\frac{F_{(a,b,c+1)}}{q_{\phi_Q(a,b,c+1)}} - \frac{F_{(a,b,c)}}{q_{\phi_Q(a,b,c)}} \geq \left(\frac{c}{a} - \frac{1}{2a^2}\right) - \left(\frac{c-1}{a} + \frac{1}{2a^2}\right). \tag{2.5}$$

Combining (2.4) and (2.5), we see that

$$\frac{1}{a} - \frac{1}{a^2} \leq \frac{F_{(a,b,c+1)}}{q_{\phi_Q(a,b,c+1)}} - \frac{F_{(a,b,c)}}{q_{\phi_Q(a,b,c)}} \leq \frac{1}{a} + \frac{1}{a^2},$$

so (1.6) holds.

Lastly, by construction,

$$\frac{a-1}{a} - \frac{1}{a^2} \leq \frac{F_{(a,b,a)}}{q_{\phi_Q(a,b,a)}} < \frac{a-1}{a} + \frac{1}{a^2},$$

so

$$1 - \frac{1}{a} - \frac{1}{a^2} \leq \frac{F_{(a,b,a)}}{q_{\phi_Q(a,b,a)}} \leq 1 - \frac{1}{a} + \frac{1}{a^2} < 1$$

and we have verified (1.7). Therefore, $y_{F,a,b}$ is an almost arithmetic progression-$(1/a, 1/a)$. By Corollary 2.12, $D^*_{F,a,b} \leq 1/a + 1/a = 2/a$. ∎

Throughout the rest of this paper, for a given n, the symbol $i = i(n)$ means the unique integer satisfying $L_i < n \leq L_{i+1}$. Given a positive integer n, let $m = n - L_i$. Note that m can be written uniquely as $m = \alpha(i+1) + \beta$ with

$$0 \leq \alpha \leq l_{i+1} \quad \text{and} \quad 0 \leq \beta < i+1.$$

We define α and β as the unique integers satisfying these conditions.

The following results from [10] will be needed:

LEMMA 2.14. *If t is a positive integer and for $1 \leq j \leq t$, z_j is a finite sequence in $[0,1)$ with star discrepancy at most ϵ_j, then*

$$D^*(z_1 \cdots z_t) \leq \frac{\sum_{j=1}^t |z_j| \epsilon_j}{\sum_{j=1}^t |z_j|}$$

COROLLARY 2.15. *If t is a positive integer and for $1 \leq j \leq t$, z_j is a finite sequence in $[0,1)$ with star discrepancy at most ϵ_j, then*

$$D^*(l_1 z_1 \cdots l_t z_t) \leq \frac{\sum_{j=1}^{t} l_j |z_j| \epsilon_j}{\sum_{j=1}^{t} l_j |z_j|}.$$

Recall that $D^*(z)$ is bounded above by 1 for all finite sequences z of real numbers in $[0,1)$. By Corollary 2.15,

$$D_n^*(y_F) \leq f_i(\alpha,\beta) := \frac{\sum_{j=1}^{i} l_j \cdot j \cdot \frac{2}{j} + \alpha \cdot (i+1) \cdot \frac{2}{i+1} + \beta}{\sum_{j=1}^{i} j l_j + (i+1)\alpha + \beta}$$
$$= \frac{\sum_{j=1}^{i} 2 l_j + 2\alpha + \beta}{\sum_{j=1}^{i} j l_j + (i+1)\alpha + \beta}.$$

Note that $f_i(\alpha,\beta)$ is a rational function of α and β. We consider the domain of f_i to be $\mathbb{R}_0^+ \times \mathbb{R}_0^+$, where $\mathbb{R}_0^+$ is the set of all non-negative real numbers. Given a Q-special sequence F, we now give an upper bound for $D_n^*(y_F)$. Since $D_n^*(y_F)$ is at most $f_i(\alpha,\beta)$, it is enough to bound $f_i(\alpha,\beta)$ from above on $[0, l_{i+1}] \times [0, i]$. Set

$$\bar{\epsilon}_i = f_i(0, i+1) = \frac{\sum_{j=1}^{i} 2 l_j + i + 1}{\sum_{j=1}^{i} j l_j + i + 1}.$$

The following lemma is proven similarly to Lemma 11 in [2]:

LEMMA 2.16. *If $i > 2$ and*

$$\sum_{j=1}^{i} j l_j > \sum_{j=1}^{i} 2 l_j \tag{2.6}$$

and

$$(w,z) \in \{0,\ldots,l_{i+1}\} \times \{0,\ldots,i\},$$

then

$$f_i(w,z) < f_i(0,i+1) = \bar{\epsilon}_i.$$

We will now prove a series of lemmas to show that $\bar{\epsilon}_i \to 0$. The following was proven by O. Toeplitz in [17]:

THEOREM 2.17. *Let $(\gamma_{n,k} : 1 \leq k \leq n, n \geq 1)$ be an array of real numbers such that:*

(1) $\lim_{n\to\infty} \gamma_{n,k} = 0$ *for each $k \in \mathbb{N}$;*
(2) $\lim_{n\to\infty} \sum_{k=1}^{n} \gamma_{n,k} = 1$;
(3) *there exists $C > 0$ such that $\sum_{k=1}^{n} |\gamma_{n,k}| \leq C$ for all positive integers n.*

Then for any convergent sequence (α_n), *the transformed sequence* (β_n) *given by*

$$\beta_n = \sum_{k=1}^{n} \gamma_{n,k}\alpha_k, \qquad n \ge 1,$$

is also convergent and $\lim_{n\to\infty}\beta_n = \lim_{n\to\infty}\alpha_n$.

We will need the following result that follows from Theorem 2.17:

LEMMA 2.18. *Let* L *be a real number and* $(a_n)_{n=1}^{\infty}$ *and* $(b_n)_{n=1}^{\infty}$ *be two sequences of positive real numbers such that* $\sum_{n=1}^{\infty} b_n = \infty$ *and* $\lim_{n\to\infty} a_n/b_n = L$. *Then*

$$\lim_{n\to\infty} \frac{a_1+\cdots+a_n}{b_1+\cdots+b_n} = L.$$

We may now show that $\bar{\epsilon}_i \to 0$.

LEMMA 2.19. $\lim_{n\to\infty} \bar{\epsilon}_i = 0$.

Proof. We will first show that $\lim_{i\to\infty} \bar{\epsilon}_i \to 0$. The lemma will then follow as $i = i(n)$ satisfies $\lim_{n\to\infty} i(n) = \infty$.

We apply Lemma 2.18 with $a_1 = 2l_1 + 2$, $b_1 = l_1 + 2$ and for $j > 1$, $a_j = 2l_j + 1$ and $b_j = jl_j + 1$. Thus,

$$a_1 + \cdots + a_i = \sum_{j=1}^{i} 2l_j + i + 1 \quad \text{and} \quad b_1 + \cdots + b_i = \sum_{j=1}^{i} jl_j + i + 1.$$

Since $\lim_{i\to\infty} \frac{a_i}{b_i} = \lim_{i\to\infty} \frac{2l_i+1}{il_i+1} = 0$, we see that

$$\lim_{i\to\infty} \bar{\epsilon}_i = \lim_{i\to\infty} \frac{\sum_{j=1}^{i} 2l_j + i + 1}{\sum_{j=1}^{i} jl_j + i + 1} = \lim_{i\to\infty} \frac{a_i}{b_i} = 0. \ \blacksquare$$

2.3. Main theorem

THEOREM 2.20. *Suppose that* F *is a* Q*-special sequence. Then* x_F *is* Q*-distribution normal.*

Proof. Suppose that n is large enough so that $i > 2$ and $(i-2)l_i > l_1$. Then

$$il_i + 2l_2 + l_1 > 2l_i + 2l_2 + 2l_1. \tag{2.7}$$

We also note that

$$jl_j > 2l_j \quad \text{for } j > 2. \tag{2.8}$$

Combining (2.7) and (2.8) gives

$$\sum_{j=1}^{i} jl_j > \sum_{j=1}^{i} 2l_j.$$

By Lemma 2.16, $D_n^*(y_F) < \bar{\epsilon}_{i(n)}$, and by Lemma 2.19, $\bar{\epsilon}_{i(n)} \to 0$, so the sequence y_F is uniformly distributed mod 1. Thus, by Theorem 1.15, x_F is Q-distribution normal. ∎

We will now show that while Theorem 2.20 allows us to construct Q-distribution normal numbers, none of these numbers will be simply Q-normal.

PROPOSITION 2.21. *If F is a Q-special sequence, then x_F is not simply Q-normal.*

Proof. If Q is 1-convergent, then x_F is not simply Q-normal as the digit 0 occurs infinitely often in the Q-Cantor series expansion of x_F.

Next, suppose that Q is 1-divergent. We will show that the digit 1 may only occur finitely often in the Q-Cantor series expansion of x_F. Suppose that $(a, b, 2) \in F$ and $a \geq 2$. Then, by construction, we have

$$\frac{F_{(a,b,2)}}{q_{\phi_Q(a,b,2)}} \in \left[\frac{1}{a} - \frac{1}{2a^2}, \frac{1}{a} + \frac{1}{2a^2}\right]$$

and $q_{\phi_Q(a,b,2)} \geq 2a^2$. Thus, we see that

$$\frac{F_{(a,b,2)}}{q_{\phi_Q(a,b,2)}} \geq \frac{1}{a} - \frac{1}{2a^2}$$

so

$$F_{(a,b,2)} \geq \left(\frac{1}{a} - \frac{1}{2a^2}\right) q_{\phi_Q(a,b,2)} \geq \left(\frac{1}{a} - \frac{1}{2a^2}\right) \cdot 2a^2 = 2a - 1 > 1. \tag{2.9}$$

Thus, by (2.9), $F_{(a,b,2)} > 1$ when $a \geq 2$. Since $F_{(a,b,1)} = 0$ whenever $(a, b, 1) \in S_Q$, there are at most finitely many n such that $E_{F,n} = 1$, so x_F is not simply Q-normal. ∎

3. Other properties of Θ_Q

3.1. Discrepancy results

LEMMA 3.1. *Suppose that Q is a basic sequence such that there exist constants M and t with $\nu_{i+1} - \nu_i \leq Mi$ for all $i > t$. Then $l_i \leq \lceil M + 1 \rceil$ for all $i > t$.*

Proof. Suppose that $i > t$ and $l_i \geq 2$. Then by definition of $(l_i)_i$, we have ([3])

$$\nu_{i+1} - 1 \leq L_i < \nu_{i+1} + i - 1 \quad \text{and} \quad \nu_i - 1 \leq L_{i-1}.$$

Thus,

$$L_i = L_{i-1} + il_i < \nu_{i+1} + i - 1$$

([3]) Note that we cannot conclude that $L_i < \nu_{i+1} + i - 1$ if $l_i = 1$: consider $q_n = 8^n$, where $l_i = 1$, $L_i = i(i + 1)/2$, and $\nu_i = \lceil \log_8(2i^2) \rceil$ for all i.

so

$$l_i < \frac{\nu_{i+1} + i - 1 - L_{i-1}}{i} \le \frac{\nu_{i+1} + i - 1 - (\nu_i - 1)}{i} = 1 + \frac{\nu_{i+1} - \nu_i}{i}$$
$$\le 1 + \frac{Mi}{i} = 1 + M \le \lceil 1 + M \rceil. \blacksquare$$

PROPOSITION 3.2. *Suppose that Q is a basic sequence such that there exist constants M and t where $l_j \le M$ for $j > t$. Then for all Q-special sequences F and real numbers $\psi > 1$, we have*

$$D_n^*(y_F) < \psi\sqrt{2M}(2M+1) \cdot n^{-1/2},$$

for large enough n.

Proof. By Lemma 2.16, for large enough n, we have

$$D_n^*(y_F) < \frac{\sum_{j=1}^{i} 2l_j + i + 1}{\sum_{j=1}^{i} jl_j + i + 1}.$$

Set $\kappa = \sum_{j=1}^{t} jl_j$. Since $l_j \ge 1$ for all j, we see that

$$\begin{aligned}(3.1)\qquad D_n^*(y_F) &< \frac{2\kappa + \sum_{j=1}^{i} 2M + i + 1}{\sum_{j=1}^{i} j \cdot 1 + i + 1} \\ &= \frac{2\kappa + 2Mi + i + 1}{i(i+1)/2 + i + 1} = \frac{(2\kappa+1) + (2M+1)i}{i^2 + 3i + 3/2} \\ &< \frac{2(2\kappa+1) + 2(2M+1)i}{i^2 + 3i/2} = \frac{2(2\kappa+1)/i + 2(2M+1)}{i + 3/2}.\end{aligned}$$

However,

$$\begin{aligned}\frac{i(i+1)}{2} &= \sum_{j=1}^{i} j \cdot 1 \le \sum_{j=1}^{i} jl_j \\ &< n \le \kappa + \sum_{j=1}^{i+1} jl_j \le \kappa + \sum_{j=1}^{i+1} jM = \kappa + \frac{(i+1)(i+2)}{2}M.\end{aligned}$$

Thus, we see that $i \ge p$, where p is the positive solution to $n = \kappa + (p+1)(p+2)M/2$. Therefore,

$$(3.2)\qquad p = \frac{-3 + \sqrt{(8/M)n + (1 - 8\kappa/M)}}{2}.$$

Substituting (3.2) into (3.1), we arrive at the inequality

$$D_n^*(y_F) < \frac{2(\kappa+1)/i + 2(2M+1)}{\frac{-3+\sqrt{(8/M)n+(1-8\kappa/M)}}{2} + \frac{3}{2}} = \frac{4(\kappa+1)/i + 4(2M+1)}{\sqrt{(8/M)n + (1 - 8\kappa/M)}}.$$

Let $\psi > 1$. Then for large enough n,

$$\frac{4(\kappa+1)/i(n)+4(2M+1)}{\sqrt{(8/M)n+(1-8\kappa/M)}} < \psi\frac{4(2M+1)}{\sqrt{(8/M)n}} = \psi\sqrt{2M}\,(2M+1)n^{-1/2},$$

so $D_n^*(y_F) < \psi\sqrt{2M}\,(2M+1)n^{-1/2}$. ∎

THEOREM 3.3. *Suppose that Q is a basic sequence such that there exist constants M and t where $\nu_{i+1}-\nu_i \le Mi$ for $j > t$. Then for all Q-special sequences F and real numbers $\psi > 1$, we have*

$$D_n^*(y_F) < \psi\sqrt{2\lceil M+1\rceil}\,(2\lceil M+1\rceil+1)n^{-1/2}$$

for large enough n.

Proof. This follows directly from Lemma 3.1 and Proposition 3.2. ∎

REMARK 3.4. If $q_m \ge 2n^2$ for $\tau(n-1) < m \le \tau(n)$, then $l_i = 1$ for all i and Theorem 3.3 implies that for all $\psi > 1$ and large enough n, we have

$$D_n^*(y_F) < \psi\sqrt{8}\,n^{-1/2}. \tag{3.3}$$

For example, (3.3) holds if $q_n \ge 5n$ for all n.

3.2. Θ_Q is perfect and nowhere dense. The goal of this subsection will be to show that Θ_Q is a perfect, nowhere dense subset of $[0,1)$. We first remark that the *existence* of a set of normal numbers that is perfect and nowhere dense should not be surprising. However, constructing a specific example of such a set may not lend itself to an obvious solution.

We will now work towards showing that Θ_Q is perfect and nowhere dense. In order to proceed, we define a function, d, from $\Gamma_Q \times \Gamma_Q$ to $\mathbb{R}$:

DEFINITION 3.5. Suppose F_1 and F_2 are Q-special sequences. If $F_1 \ne F_2$, we define $\zeta_{F_1,F_2} = \min\{n : E_{F_1,n} \ne E_{F_2,n}\}$. Define (4) $d : \Gamma_Q \times \Gamma_Q \to \mathbb{R}$ by

$$d(F_1,F_2) = \begin{cases} \dfrac{1}{q_1\cdots q_{\zeta_{F_1,F_2}-1}} & \text{if } F_1 \ne F_2, \\ 0 & \text{if } F_1 = F_2. \end{cases}$$

LEMMA 3.6. *If $F_1, F_2 \in \Gamma_Q$, then $|x_{F_1}-x_{F_2}| \le d(F_1,F_2)$.*

Proof. Let $n = \zeta_{F_1,F_2}$. We write the Q-Cantor series expansions of x_{F_1} and x_{F_2} as follows:

$$x_{F_1} = \frac{E_1}{q_1}+\frac{E_2}{q_1q_2}+\cdots+\frac{E_{n-1}}{q_1\cdots q_{n-1}}+\frac{E_{F_1,n}}{q_1\cdots q_n}+\frac{E_{F_1,n+1}}{q_1\cdots q_{n+1}}+\cdots,$$
$$x_{F_2} = \frac{E_1}{q_1}+\frac{E_2}{q_1q_2}+\cdots+\frac{E_{n-1}}{q_1\cdots q_{n-1}}+\frac{E_{F_2,n}}{q_1\cdots q_n}+\frac{E_{F_2,n+1}}{q_1\cdots q_{n+1}}+\cdots,$$

(4) (Γ_Q, d) is a metric space.

so

$$
\begin{aligned}
&|x_{F_1} - x_{F_2}| \\
&= \left| \left(\frac{E_{F_1,n}}{q_1 \cdots q_{n-1}} - \frac{E_{F_2,n}}{q_1 \cdots q_{n-1}} \right) + \left(\frac{E_{F_1,n+1}}{q_1 \cdots q_{n+1}} - \frac{E_{F_2,n+1}}{q_1 \cdots q_{n+1}} \right) + \cdots \right| \\
&\le \frac{|E_{F_1,n} - E_{F_2,n}|}{q_1 \cdots q_n} + \frac{|E_{F_1,n+1} - E_{F_2,n+1}|}{q_1 \cdots q_{n+1}} + \cdots \le \frac{1}{q_1 \cdots q_{n-1}} = d(F_1, F_2). \blacksquare
\end{aligned}
$$

Lemma 3.7. *If $F \in \Gamma_Q$, then there exists a sequence of Q-special sequences $F_1, F_2, F_3, \ldots$ such that $F \neq F_n$ for all n and* $\lim_{n\to\infty} d(F, F_n) = 0$.

Proof. By Lemma 2.3, we may define a sequence of Q-special sequences as follows. Let n be any positive integer and put $(\alpha, \beta, \gamma) = \phi_Q^{-1}(n)$. We must now consider three cases. First, if $\gamma \neq 1$, then for $m \neq n$, we set $E_{n,m} = E_{F,m}$ and we let $E_{n,n} \neq E_{F,n}$ be any value that satisfies

$$
\frac{E_{n,n}}{q_n} \in \left[\frac{\gamma - 1}{\alpha} - \frac{1}{2\alpha^2}, \frac{\gamma - 1}{\alpha} + \frac{1}{2\alpha^2} \right].
$$

Second, we suppose that $\gamma = 1$ and $\alpha > 1$. Put $(\alpha', \beta', \gamma') = \phi_Q^{-1}(n+1)$. Then for $m \neq n+1$, we set $E_{n,m} = E_{F,m}$ and we let $E_{n,n+1} \neq E_{F,n+1}$ be any value that satisfies

$$
\frac{E_{n,n+1}}{q_{n+1}} \in \left[\frac{\gamma' - 1}{\alpha'} - \frac{1}{2\alpha'^2}, \frac{\gamma' - 1}{\alpha'} + \frac{1}{2\alpha'^2} \right].
$$

Third, we consider the case where $\alpha = \gamma = 1$. Set $t = \phi_Q(2, 1, 2)$ and note that $t > n$. Then for $m \neq t$, put $E_{n,m} = E_{F,m}$ and let $E_{n,t} \neq E_{F,t}$ be any value that satisfies

$$
\frac{E_{n,t}}{q_t} \in \left[\frac{2 - 1}{2} - \frac{1}{2 \cdot 2^2}, \frac{2 - 1}{2} + \frac{1}{2 \cdot 2^2} \right] = \left[\frac{3}{8}, \frac{5}{8} \right].
$$

Now that we have determined $(E_{n,m})_{m=1}^{\infty}$, set $F_n = (E_{n,\phi_Q(a,b,c)})_{(a,b,c)\in S_Q}$. Thus, $F \neq F_n$ for all n, and for large enough m we have

$$
d(F, F_m) \le \max\left(\frac{1}{q_1 \cdots q_m}, \frac{1}{q_1 \cdots q_{m-1}} \right) = \frac{1}{q_1 \cdots q_{m-1}},
$$

so $F_n \to F$. ∎

Theorem 3.8. *The set Θ_Q is perfect.*

Proof. Suppose that $x \in \Theta_Q$ and that $x = x_F$. By Lemma 3.7, there exists a sequence of Q-special sequences $F_1, F_2, \ldots$, none of which are equal to F, with $F_n \to F$. Thus, $x \neq x_{F_n}$ for all n. Let $\epsilon > 0$ and suppose that N is large enough so that for all $n > N$, we have $d(F, F_n) < \epsilon$. Clearly, $|x - x_{F_n}| \le d(F, F_n) < \epsilon$, so $x_{F_n} \to x_F$ and Θ_Q is perfect. ∎

We need the following simple lemma:

LEMMA 3.9. *If $a \geq 1$, then*

$$\frac{a-1}{a} + \frac{1.5}{2a^2} < 1.$$

THEOREM 3.10. *The set Θ_Q is nowhere dense.*

Proof. Let $I \subset [0,1)$ be any interval such that $\Theta_Q \cap I \neq \emptyset$. We will show that there exists an interval $K \subset I$ such that $\Theta_Q \cap K = \emptyset$. Thus, there exists a positive integer n and an interval $J \subset I$ with

$$J = \left[\frac{E_1}{q_1} + \frac{E_2}{q_1 q_2} + \cdots + \frac{E_n}{q_1 \cdots q_n}, \frac{E_1}{q_1} + \frac{E_2}{q_1 q_2} + \cdots + \frac{E_n + 1}{q_1 \cdots q_n}\right)$$

and $E_j \in [0, q_j - 1) \cap \mathbb{Z}$ for $j = 1, \ldots, n$. Put $(a, b, c) = \phi_Q^{-1}(n+1)$. By Lemma 3.9, we may set

$$K = \left[\frac{E_1}{q_1} + \cdots + \frac{E_n}{q_1 \cdots q_n} + \left(\frac{a-1}{a} + \frac{1.5}{2a^2}\right)\frac{1}{q_1 \cdots q_n}, \frac{E_1}{q_1} + \cdots + \frac{E_n + 1}{q_1 \cdots q_n}\right).$$

If $\Theta_Q \cap J = \emptyset$, we are finished, so assume that $\Theta_Q \cap J \neq \emptyset$. Suppose that $F \in \Gamma_Q$ is such that $x_F \in J$ and

$$x = 0.E_1 \ldots E_n E_{n+1} \ldots \quad \text{with respect to } Q.$$

By construction, if $c \neq 1$, we have

$$\frac{E_{n+1}}{q_{n+1}} \in \left[\frac{c-1}{a} - \frac{1}{2a^2}, \frac{c-1}{a} + \frac{1}{2a^2}\right].$$

If $c = 1$, then $E_{n+1} = 0$. Therefore,

$$\begin{aligned} x_F &\leq \frac{E_1}{q_1} + \frac{E_2}{q_1 q_2} + \cdots + \frac{E_n}{q_1 \cdots q_n} + \left(\frac{c-1}{a} + \frac{1}{2a^2}\right)\frac{1}{q_1 \cdots q_n} \\ &< \frac{E_1}{q_1} + \frac{E_2}{q_1 q_2} + \cdots + \frac{E_n}{q_1 \cdots q_n} + \left(\frac{a-1}{a} + \frac{1.5}{2a^2}\right)\frac{1}{q_1 \cdots q_n}, \end{aligned}$$

so $x_F \notin K$. Hence, $K \cap \Theta_Q = \emptyset$ and Θ_Q is nowhere dense. ∎

3.3. Hausdorff dimension of Θ_Q. Given a basic sequence Q and a positive integer n, we will define the functions $a(n)$, $b(n)$, and $c(n)$ by $(a(n), b(n), c(n)) = \phi_Q^{-1}(n)$. Set $\omega_n = \#\{E_{F,n} : F \in \Gamma_Q\}$,

$$A(k) = \begin{cases} 1 & \text{if } 1 \leq k \leq l_1, \\ p & \text{if } l_1 + \cdots + l_{p-1} < k \leq l_1 + \cdots + l_p, \end{cases}$$

and $\gamma(k) = A(1) + \cdots + A(k)$.

Note that $\omega_n = 1$ if and only if $c(n) = 1$. By Lemma 2.3, we are guaranteed that $\omega_n \geq 2$ if $c(n) \neq 1$. Additionally, we can say that

$$\frac{q_n}{a(n)^2} \leq \omega_n \leq \frac{q_n}{a(n)^2} + 1 < \frac{2q_n}{a(n)^2} \tag{3.4}$$

when $\omega_n \neq 1$. If q_n grows quickly enough that $l_1 = l_2 = \cdots = 1$, then $A(k) = k$ and $\gamma(k) = \tau(k)$, so $a(n) = \lfloor (1+\sqrt{8n-7})/2 \rfloor$. Thus, we have

$$\sqrt{n} \leq a(n) < \sqrt{3n}. \tag{3.5}$$

Combining (3.4) and (3.5), we see that

$$\frac{q_n}{3n} \leq \omega_n < \frac{2q_n}{n}. \tag{3.6}$$

Definition (3.6) ([5]). A basic sequence Q *grows nicely* if $n^s = o(q_n)$ for all positive integers s, and

$$\log q_{\tau(k-1)+1} + \log q_{\tau(k)} = o\Big(\log \prod_{n=1}^{\tau(k-1)-1} q_n\Big),$$

$$\log \prod_{n=0}^{k-2} q_{\tau(k)+1} = o\Big(\log \prod_{n=1}^{\tau(k-1)-1} q_n\Big).$$

A basic sequence Q *grows slowly* if there exists a constant M such that $\omega_n \leq M$ for all $n \geq 1$. Lastly, Q *grows quickly* if

$$\log \prod_{n=1}^{\tau(k)-1} q_n = o(\log q_{\tau(k)}).$$

Example 3.12. The basic sequences given by $q_n = n+1$ and by $q_n = \max(2, \lfloor \log n \rfloor)$ grow slowly. If $t \geq 2$, then $q_n = \lfloor t^n \rfloor$ and $q_n = 2^{2^n}$ are examples of nicely growing basic sequences. If we let $q_1 = 2$ and $q_{n+1} = 2^{q_1 \cdots q_n}$, then Q grows quickly.

If $J \subset [0,1)$ is a subset of $[0,1)$, we will denote its Hausdorff dimension by $\dim_{\mathrm{H}} J$. In this section, we will compute the Hausdorff dimension of Θ_Q for a few classes of basic sequences. We will show that $\dim_{\mathrm{H}} \Theta_Q = 0$ when Q grows slowly or quickly. When Q grows nicely, we will have $\dim_{\mathrm{H}} \Theta_Q = 1$.

Definition 3.13. Let J be any non-empty subset of $[0,1)$ and let $C_\delta(J)$ be the smallest number of sets of diameter at most δ which can cover J. Then the *box-counting dimension* of J, if it exists, is defined as

$$\dim_{\mathrm{B}} J = \lim_{\delta \to 0} \frac{\log C_\delta(J)}{-\log \delta}.$$

The *lower box-counting dimension* and *upper box-counting dimension* of J are defined as

$$\underline{\dim}_{\mathrm{B}} J = \liminf_{\delta \to 0} \frac{\log C_\delta(J)}{-\log \delta} \quad \text{and} \quad \overline{\dim}_{\mathrm{B}} J = \limsup_{\delta \to 0} \frac{\log C_\delta(J)}{-\log \delta},$$

respectively.

([5]) A basic sequence may still grow slowly no matter how fast q_n grows when n is restricted to those values for which $\omega_n = 1$.

The following standard result (see [7]) will be used frequently and without mention:

THEOREM 3.14. *Let J be a non-empty subset of* $[0,1]$. *Then*

$$0 \le \dim_{\mathrm{H}} J \le \underline{\dim}_{\mathrm{B}} J \le \overline{\dim}_{\mathrm{B}} J \le 1.$$

We will make use of the following general construction found in [7]. Suppose that $[0,1] = I_0 \supset I_1 \supset I_2 \supset \cdots$ is a decreasing sequence of sets, with each I_k a union of a finite number of disjoint closed intervals (called *kth level basic intervals*). Then we will consider the set $\bigcap_{k=0}^{\infty} I_k$. We will construct a set Θ'_Q that may be written in this form such that $\dim_{\mathrm{H}} \Theta_Q = \dim_{\mathrm{H}} \Theta'_Q$.

Given a block of digits $B = (b_1, \ldots, b_s)$ and a positive integer n, define

$$\mathcal{S}_{Q,B} = \{x = 0.E_1E_2\ldots \text{ with respect to } Q : E_1 = b_1, \ldots, E_t = b_s\}.$$

Let P_n be the set of all possible values of $E_n(x)$ for $x \in \Theta_Q$. Put $J_0 = [0,1)$ and

$$J_k = \bigcup_{B \in \prod_{n=1}^{\gamma(k)} P_n} \mathcal{S}_{Q,B}.$$

Then $J_k \subset J_{k-1}$ for all $k \ge 0$ and $\Theta_Q = \bigcap_{k=0}^{\infty} J_k$, which gives the following:

PROPOSITION 3.15. *Θ_Q can be written in the form $\bigcap_{k=0}^{\infty} J_k$, where each J_k is the union of a finite number of disjoint half-open intervals.*

We now set $I_k = \bar{J}_k$ for all $k \ge 0$ and put $\Theta'_Q = \bigcap_{k=0}^{\infty} I_k$. Since each set J_k consists of only a finite number of intervals, the set $I_k \setminus J_k$ is finite.

LEMMA 3.16. $\dim_{\mathrm{H}} \Theta_Q = \dim_{\mathrm{H}} \Theta'_Q$.

Proof. The lemma follows as $\Theta'_Q \setminus \Theta_Q$ is a countable set. ∎

For $k \ge 1$, we note that, by construction, there are $\omega_1 \cdots \omega_{\gamma(k)-1}$ kth level intervals and they are all of length $(q_1 \cdots q_{\gamma(k)})^{-1}$. Additionally, they are all separated by a distance of at least $(q_1 \cdots q_{\gamma(k)})^{-1}(1 + 2/A(k)^2)$. This gives

$$\underline{\dim}_{\mathrm{B}} \Theta_Q = \liminf_{k\to\infty} \frac{\log(\omega_1 \cdots \omega_{\gamma(k)-1})}{\log(q_1 \cdots q_{\gamma(k)})},$$

$$\overline{\dim}_{\mathrm{B}} \Theta_Q = \limsup_{k\to\infty} \frac{\log(\omega_1 \cdots \omega_{\gamma(k)-1})}{\log(q_1 \cdots q_{\gamma(k)})}, \tag{3.7}$$

$$\dim_{\mathrm{B}} \Theta_Q = \lim_{k\to\infty} \frac{\log(\omega_1 \cdots \omega_{\gamma(k)-1})}{\log(q_1 \cdots q_{\gamma(k)})}.$$

THEOREM 3.17. *Suppose that Q grows slowly. Then*

$$\dim_{\mathrm{H}} \Theta_Q = \dim_{\mathrm{B}} \Theta_Q = 0.$$

Proof. Since Q is infinite in limit, for all $z > M$, there exists a positive integer t such that $q_1 \cdots q_{\gamma(k)} \geq z^{\gamma(k)}$ for all $k > t$. Substituting $\omega_n \leq M$ into (3.7), we see that

$$\overline{\dim}_{\mathrm{B}} \Theta_Q \leq \limsup_{k\to\infty} \frac{\log M^{\gamma(k)-1}}{\log z^{\gamma(k)}} = \frac{\log M}{\log z},$$

so $\overline{\dim}_{\mathrm{B}} \Theta_Q = 0$. ∎

We will use the following result from [7]:

THEOREM 3.18. *Suppose that each $(k-1)$th level interval of I_{k-1} contains at least m_k kth level intervals $(k = 1, 2, \ldots)$ which are separated by gaps of at least ϵ_k, where $0 \leq \epsilon_{k+1} < \epsilon_k$ for each k. Then*

$$\dim_{\mathrm{H}}\Big(\bigcap_{k=0}^{\infty} I_k\Big) \geq \liminf_{k\to\infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log(m_k \epsilon_k)}.$$

LEMMA 3.19.

$$\dim_{\mathrm{H}} \Theta'_Q \geq \liminf_{k\to\infty} \frac{\log(\omega_1 \cdots \omega_{\gamma(k-1)-1})}{\log \frac{q_1 \cdots q_{\gamma(k)}}{\omega_{\gamma(k-1)}\omega_{\gamma(k-1)+1}\cdots\omega_{\gamma(k)-1}}}.$$

Proof. We substitute

$$m_1 \cdots m_{k-1} = \omega_1 \cdots \omega_{\gamma(k-1)-1}, \qquad m_k = \omega_{\gamma(k-1)}\omega_{\gamma(k-1)+1} \cdots \omega_{\gamma(k)-1},$$

and

$$\epsilon_k = (q_1 \cdots q_{\gamma(k)})^{-1}(1 + 2/A(k)^2)$$

into Theorem 3.18. Since $\lim_{k\to\infty} A(k) = \infty$, we see that $1 < 1 + 2/A(k)^2 \leq 3$, so

$$\liminf_{k\to\infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log(m_k \epsilon_k)} = \liminf_{k\to\infty} \frac{\log(\omega_1 \cdots \omega_{\gamma(k-1)-1})}{\log\big(\frac{q_1 \cdots q_{\gamma(k)}}{\omega_{\gamma(k-1)}\omega_{\gamma(k-1)+1}\omega_{\gamma(k)-1}}(1+2/A(k))^{-1}\big)}$$

$$= \liminf_{k\to\infty} \frac{\log(\omega_1 \cdots \omega_{\gamma(k-1)-1})}{\log \frac{q_1 \cdots q_{\gamma(k)}}{\omega_{\gamma(k-1)}\omega_{\gamma(k-1)+1}\cdots\omega_{\gamma(k)-1}}}. \ ∎$$

LEMMA 3.20. *Suppose that $l_i = 1$ for all i. Then*

$$\dim_{\mathrm{H}} \Theta'_Q \geq \liminf_{k\to\infty} \frac{\mathrm{N}}{\mathrm{D}}, \tag{3.8}$$

where

$$\mathrm{N} = \log \prod_{n=1}^{\tau(k-1)-1} q_n - \log 3^{\tau(k-2)-1} - \log(\tau(k-1)-1)! - \log \prod_{n=0}^{k-2} q_{\tau(n)+1}$$

$$\mathrm{D} = \log \prod_{n=1}^{\tau(k-1)-1} q_n + \log q_{\tau(k-1)+1}$$

$$+ \log q_{\tau(k)} + \log 3^{k-1} + \log \frac{(\tau(k)-1)!}{(\tau(k-1)-1)!(\tau(k-1)+1)}.$$

Proof. Since $l_i = 1$ for all i, (3.6) holds and $\gamma(k) = \tau(k)$ for all k. Note that $\omega_n = 1$ if and only if $n = \tau(k) + 1$ for some k. Therefore,

$$\begin{aligned}
&\omega_1\omega_2\cdots\omega_{\gamma(k-1)-1} \\
&\quad \geq \frac{q_1}{3\cdot 1}\frac{q_2}{3\cdot 2}\cdots\frac{q_{\tau(k-1)-1}}{3(\tau(k-1)-1)}\prod_{n=0}^{k-2}\frac{3(\tau(n)+1)}{q_{\tau(n)+1}} \\
&\quad \geq \Big(\prod_{n=1}^{\tau(k-1)-1} q_n\Big)\cdot 3^{-(\tau(k-1)-1)}(\tau(k-1)-1)!^{-1}\cdot 3^{k-1}\prod_{n=0}^{k-2} q_{\tau(n)+1}^{-1} \\
&\quad = \Big(\prod_{n=1}^{\tau(k-1)-1} q_n\Big)\cdot 3^{-(\tau(k-2)-1)}(\tau(k-1)-1)!^{-1}\prod_{n=0}^{k-2} q_{\tau(n)+1}^{-1},
\end{aligned}$$

so

$$\begin{aligned}
&\log(\omega_1\cdots\omega_{\gamma(k-1)-1}) \\
&\quad \geq \log\prod_{n=1}^{\tau(k-1)-1} q_n - \log 3^{\tau(k-2)-1} - \log(\tau(k-1)-1)! - \log\prod_{n=0}^{k-2} q_{\tau(n)+1}.
\end{aligned}$$

Next, since $\omega_{\gamma(k-1)+1} = 1$, we arrive at the estimate

$$\begin{aligned}
&\frac{q_1\cdots q_{\gamma(k)}}{\omega_{\gamma(k-1)}\omega_{\gamma(k-1)+1}\cdots\omega_{\gamma(k)-1}} \\
&\quad \leq \frac{q_1\cdots q_{\tau(k)}}{\Big(\frac{q_{\tau(k-1)}}{3\tau(k-1)}\frac{q_{\tau(k-1)+1}}{3(\tau(k-1)+1)}\cdots\frac{q_{\tau(k)-1}}{3(\tau(k)-1)}\Big)\frac{3(\tau(k-1)+1)}{q_{\tau(k-1)+1}}} \\
&\quad = \Big(\prod_{n=1}^{\tau(k-1)-1} q_n\Big) q_{\tau(k-1)+1}q_{\tau(k)}\cdot 3^{k-1}\frac{(\tau(k)-1)!}{(\tau(k-1)-1)!(\tau(k-1)+1)}.
\end{aligned}$$

Thus, Lemma 3.19 yields the conclusion. ∎

THEOREM 3.21. *Suppose that Q grows nicely. Then* $\dim_{\mathrm{H}} \Theta_Q = 1$.

Proof. We will show that $\dim_{\mathrm{H}} \Theta'_Q = 1$, so that $\dim_{\mathrm{H}} \Theta_Q = 1$ immediately follows. We need only consider the case where $l_i = 1$ for all i. Since Q grows nicely, the dominant term of both the numerator and denominator in (3.8) is $\log\prod_{n=1}^{\tau(k-1)-1} q_n$, so $\dim_{\mathrm{H}} \Theta'_Q = 1$ by Lemma 3.20. ∎

THEOREM 3.22. *Suppose that Q grows quickly. Then* $\dim_{\mathrm{H}} \Theta_Q = \dim_{\mathrm{B}} \Theta_Q = 0$.

Proof. It will be sufficient to consider the case where $l_k = 1$ for all k. We will show that $\overline{\dim}_{\mathrm{B}} \Theta_Q = 0$. Recall that $\omega_n < 2q_n/n$, so $\gamma(k) = \tau(k)$ and

$$\omega_1\cdots\omega_{\gamma(k)-1} < \frac{2q_1}{1}\frac{2q_2}{2}\cdots\frac{2q_{\tau(k)-1}}{\tau(k)-1} = \Big(\prod_{n=1}^{\tau(k)-1} q_n\Big)\cdot 2^{\tau(k)-1}/(\tau(k)-1)!,$$

so

$$\overline{\dim}_{\mathrm{B}} \Theta_Q \le \limsup_{k\to\infty} \frac{\log \prod_{n=1}^{\tau(k)-1} q_n + \log 2^{\tau(k)-1} - \log(\tau(k)-1)!}{\log \prod_{n=1}^{\tau(k)-1} q_n + \log q_{\tau(k)}}. \tag{3.9}$$

However, the dominant terms in the numerator and denominator of (3.9) are $\log \prod_{n=1}^{\tau(k)-1} q_n$ and $\log q_{\tau(k)}$, respectively, so

$$\overline{\dim}_{\mathrm{B}} \Theta_Q \le \limsup_{k\to\infty} \frac{\log \prod_{n=1}^{\tau(k)-1} q_n}{\log q_{\tau(k)}} = 0. \ \blacksquare$$

The Hausdorff dimension of Θ_Q is less certain when q_n grows like a polynomial. The following lemma will be needed:

Lemma 3.23.

$$\log \frac{(\tau(k)-1)!}{(\tau(k-1)-1)!(\tau(k-1)+1)} = o(\log(\tau(k-1)-1)!).$$

Proof. Suppose that $k > 2$. Then

$$\frac{(\tau(k)-1)!}{(\tau(k-1)-1)!(\tau(k-1)+1)} < \frac{(\tau(k)-1)!}{(\tau(k-1)-1)!} < (\tau(k)-1)^k = e^{k\log(\frac{1}{2}(k^2+k-2))} < e^{k\log(k^2)} = e^{2k\log k},$$

so

$$\log \frac{(\tau(k)-1)!}{(\tau(k-1)-1)!(\tau(k-1)+1)} < 2k\log k.$$

By Stirling's formula,

$$\begin{aligned}(\tau(k-1)-1)! &> \sqrt{2\pi}\,(\tau(k-1)-1)^{\tau(k-1)-1/2} e^{-(\tau(k-1)-1)}\\ &= \sqrt{2\pi}\left(\frac{1}{2}(k^2-k-2)\right)^{\frac{1}{2}(k^2-k-1)} e^{-\frac{1}{2}(k^2-k-2)}\\ &= \sqrt{2\pi} e^{\frac{1}{2}(k^2-k-2)\log(\frac{1}{2}(k^2-k-1)) - \frac{1}{2}(k^2-k-2)},\end{aligned}$$

so

$$\log(\tau(k-1)-1)! > \frac{1}{2}(k^2-k-2)\left(\log\left(\frac{1}{2}(k^2-k-1)\right) - 1\right).$$

Since $\lim_{k\to\infty} \frac{2k\log k}{\frac{1}{2}(k^2-k-2)(\log(\frac{1}{2}(k^2-k-1))-1)} = 0$, the lemma follows. $\blacksquare$

Theorem 3.24. *Suppose that there exists reals number $t > 1$ and $\lambda_1, \lambda_2 \ge 1$ such that $\lambda_1 n^t \le q_n \le \lambda_2 n^t$ for all n and $q_m \ge 2p^2$ for $\tau(p-1) < m \le \tau(p)$. Then $\dim_{\mathrm{B}} \Theta_Q = 1$ and*

$$1 - 1/t \le \dim_{\mathrm{H}} \Theta_Q \le 1.$$

Proof. Since $\lambda_1 n^t \le q_n \le \lambda_2 n^t$,

$$\log \lambda_1^{\tau(k-1)-1} + t\log(\tau(k-1)-1)! \le \log \prod_{n=1}^{\tau(k-1)-1} q_n \le \log \lambda_2^{\tau(k-1)-1} + t\log(\tau(k-1)-1)!,$$

so

$$\log \prod_{n=1}^{\tau(k-1)-1} q_n - \log(\tau(k-1)-1)! \ge (t-1)\log(\tau(k-1)-1)! + \log \lambda_1^{\tau(k-1)-1}.$$

Note that $l_i = 1$ for all i, so by Lemmas 3.20 and 3.23,

$$\dim_{\mathrm{H}} \Theta'_Q \ge \liminf_{k\to\infty} \frac{(t-1)\log(\tau(k-1)-1)! + \log \lambda_1^{\tau(k-1)-1}}{t\log(\tau(k-1)-1)! + \log \lambda_2^{\tau(k-1)-1}} = \liminf_{k\to\infty} \frac{(t-1)\log(\tau(k-1)-1)!}{t\log(\tau(k-1)-1)!} = 1 - \frac{1}{t}.$$

A similar computation gives $\underline{\dim}_{\mathrm{B}} \Theta_Q = 1$, so $\dim_{\mathrm{B}} \Theta_Q = 1$. ∎

Let $\alpha \in (0,1)$. We will now work towards constructing a basic sequence Q_α such that $\dim_{\mathrm{H}} \Theta_{Q_\alpha} = \alpha$. Define the basic sequence $Q_\alpha = (q_{\alpha,n})_n$ by

(3.10)
$$q_{\alpha,n} = \begin{cases} \max\left(\left\lfloor\left(\prod_{m=1}^{n-1} q_{\alpha,m}\right)^{(1-\alpha)/\alpha}\right\rfloor, 2n^2\right) & \text{if } n = \tau(k) \text{ for even } k, \\ 2n^2 & \text{for all other values of } n. \end{cases}$$

We will write $V_k = q_{\alpha,\tau(k)}$ and $P_k = \prod_{n=1}^{\tau(k)-1} q_{\alpha,n}$, so for large enough integers k that are even,

$$V_k = \lfloor P_k^{(1-\alpha)/\alpha} \rfloor. \tag{3.11}$$

LEMMA 3.25. *If k is even, then*

$$\frac{1-\alpha}{\alpha}\log P_{k-1} < \log V_k < \frac{1-\alpha}{\alpha}\log P_{k-1} + \frac{4-4\alpha}{\alpha} k\log k.$$

Proof. We have

$$\log V_k \le \log\Big(P_{k-1}\prod_{n=\tau(k-1)}^{\tau(k)-1} 2n^2\Big)^{(1-\alpha)/\alpha} = \frac{1-\alpha}{\alpha}\log P_{k-1} + \frac{1-\alpha}{\alpha}\log \prod_{n=\tau(k-1)}^{\tau(k)-1} 2n^2$$
$$< \frac{1-\alpha}{\alpha}\log P_{k-1} + \frac{1-\alpha}{\alpha}\log(2\tau(k)^2)^k$$
$$< \frac{1-\alpha}{\alpha}\log P_{k-1} + \frac{1-\alpha}{\alpha} k\log k^4 = \frac{1-\alpha}{\alpha}\log P_{k-1} + \frac{4-4\alpha}{\alpha} k\log k.$$

The lower bound follows similarly. ∎

THEOREM 3.26. *If* $\alpha \in (0,1)$, *then* $\dim_{\mathrm{H}} \Theta_{Q_\alpha} = \underline{\dim}_{\mathrm{B}} \Theta_{Q_\alpha} = \alpha$ *and* $\overline{\dim}_{\mathrm{B}} \Theta_{Q_\alpha} = 1$.

Proof. For this basic sequence, $l_i = 1$ for all i, so we may use our usual estimates. Thus, by (3.7) and (3.11),

$$\begin{aligned}\underline{\dim}_{\mathrm{B}} \Theta_{Q_\alpha} &= \liminf_{k\to\infty} \frac{\log(\omega_1 \cdots \omega_{\gamma(k)-1})}{\log(q_{\alpha,1} \cdots q_{\alpha,\gamma(k)})} \le \liminf_{k\to\infty} \frac{\log \prod_{n=1}^{\tau(k)-1} \frac{2q_{\alpha,n}}{n}}{\log \prod_{n=1}^{\tau(k)-1} q_{\alpha,n} + \log q_{\alpha,\tau(n)}} \\ &= \min\left(\lim_{k\to\infty,\, k \text{ even}} \frac{\log P_k}{\log P_k + \frac{1-\alpha}{\alpha}\log P_k}, \lim_{k\to\infty,\, k \text{ odd}} \frac{\log P_k}{\log P_k + \log(2\tau(k)^2)} \right). \\ &= \min\left(\frac{1}{1+\frac{1-\alpha}{\alpha}}, 1 \right) = \alpha.\end{aligned}$$

Following a similar computation, $\overline{\dim}_{\mathrm{B}} \Theta_{Q_\alpha} = 1$. By Lemmas 3.20 and 3.25,

$$\begin{aligned}\dim_{\mathrm{H}} \Theta'_{Q_\alpha} &\ge \liminf_{k\to\infty} \frac{\log P_{k-1}}{\log P_{k-1} + \log V_k} \\ &= \min\left(\lim_{k\to\infty,\, k \text{ even}} \frac{\log P_{k-1}}{\log P_{k-1} + \frac{1-\alpha}{\alpha}\log P_{k-1}}, \right. \\ &\qquad \left. \lim_{k\to\infty,\, k \text{ odd}} \frac{\log P_{k-1}}{\log P_{k-1} + \log(2\tau(k)^2)} \right) = \alpha,\end{aligned}$$

so $\dim_{\mathrm{H}} \Theta_{Q_\alpha} = \underline{\dim}_{\mathrm{B}} \Theta_{Q_\alpha} = \alpha$. ■

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