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Convexity and a sum-product type estimate

by

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1. Introduction. Given a finite set $A \subset \mathbb{R}$, the elements of A can be labeled in ascending order, so that $a_1 < \cdots < a_n$. Then A is said to be *convex* if

$$a_i - a_{i-1} < a_{i+1} - a_i$$

for all $2 \leq i \leq n-1$, and it was proved by Elekes, Nathanson and Ruzsa ([ENR]) that $|A \pm A| \geq |A|^{3/2}$, an estimate which stood as the best known for a decade, under various guises. Schoen and Shkredov ([SS2]) recently made significant progress by proving that for any convex set A,

$$|A - A| \gg \frac{|A|^{8/5}}{(\log |A|)^{2/5}} \quad \text{and} \quad |A + A| \gg \frac{|A|^{14/9}}{(\log |A|)^{2/3}}.$$

See [SS2] and the references therein for more details on this problem and its history.

In [ENR], a number of other results were proved connecting convexity with large sumsets. In particular, it was shown that, for any convex or concave function f and any finite set $A \subset \mathbb{R}$,

$$\max\{|A + A|, |f(A) + f(A)|\} \gg |A|^{5/4}, \tag{1.1}$$

$$|A + f(A)| \gg |A|^{5/4}. \tag{1.2}$$

By choosing particularly interesting convex or concave functions f, these results immediately yield interesting corollaries. For example, if we choose $f(x) = \log x$, then (1.1) immediately yields a sum-product estimate. Furthermore, if $f(x) = 1/x$, then (1.2) gives information about another problem posed by Erdős and Szemerédi ([ES]).

In this paper, the methods used by Schoen and Shkredov ([SS2]) are developed further in order to improve on some other results from [ENR]. In particular, the bounds in (1.1) and (1.2) are improved slightly, in the form of the following results.

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Theorem 1.1. *Let f be any continuous, strictly convex or concave function on the reals, and $A, C \subset \mathbb{R}$ be any finite sets such that $|A| \approx |C|$. Then*

$$|f(A)+C|^6|A-A|^5 \gg \frac{|A|^{14}}{(\log |A|)^2}.$$

In particular, choosing $C = f(A)$, this implies that

$$\max\{|f(A)+f(A)|, |A-A|\} \gg \frac{|A|^{14/11}}{(\log |A|)^{2/11}}.$$

Theorem 1.2. *Let f be any continuous, strictly convex or concave function on the reals, and $A, C \subset \mathbb{R}$ be any finite sets such that $|A| \approx |C|$. Then*

$$|f(A)+C|^{10}|A+A|^9 \gg \frac{|A|^{24}}{(\log |A|)^2}.$$

In particular, choosing $C = f(A)$, this implies that

$$\max\{|f(A)+f(A)|, |A+A|\} \gg \frac{|A|^{24/19}}{(\log |A|)^{2/19}}.$$

Theorem 1.3. *Let f be any continuous, strictly convex or concave function on the reals, and $A \subset \mathbb{R}$ be any finite set. Then*

$$|A+f(A)| \gg \frac{|A|^{24/19}}{(\log |A|)^{2/19}}.$$

Applications to sum-product estimates. By choosing $f(x) = \log x$ and applying Theorems 1.1 and 1.2, some interesting sum-product type results can be specified, especially in the case when the product set is small. A *sum-product estimate* is a bound on $\max\{|A+A|, |A\cdot A|\}$, and it is conjectured that at least one of these sets should grow to a near maximal size. Solymosi ([Sol1]) proved that $\max\{|A+A|, |A\cdot A|\} \gg |A|^{4/3}/(\log |A|)^{1/3}$, and this is currently the best known bound. See [Sol1] and the references therein for more details on this problem and its history.

In a similar spirit, one may conjecture that at least one of $|A-A|$ and $|A\cdot A|$ must be large, and indeed this is somewhat true. In an earlier paper of Solymosi ([Sol2]) on sum-product estimates, it was proved that

$$\max\{|A+A|, |A\cdot A|\} \gg \frac{|A|^{14/11}}{(\log |A|)^{3/11}}.$$

It is easy to change the proof slightly to obtain the same result with $|A+A|$ replaced by $|A-A|$, however, in Solymosi's subsequent paper on sum-product estimates, this substitution was not possible. So, $\max\{|A-A|, |A\cdot A|\} \gg |A|^{14/11}/(\log |A|)^{3/11}$ represents the current best known bound of this type. Applying Theorem 1.1 with $f(x) = \log x$, and noting that $|f(A)+f(A)| = |A\cdot A|$, we get the following very marginal improvement.

COROLLARY 1.4. *We have*

$$|A \cdot A|^6 |A - A|^5 \gg \frac{|A|^{14}}{(\log |A|)^2}. \tag{1.3}$$

In particular, this implies that

$$\max\{|A \cdot A|, |A - A|\} \gg \frac{|A|^{14/11}}{(\log |A|)^{2/11}}.$$

By applying Theorem 1.2 in the same way, we establish that

$$|A \cdot A|^{10} |A + A|^9 \gg \frac{|A|^{24}}{(\log |A|)^2}. \tag{1.4}$$

In the case when the productset is small, (1.3) and (1.4) show that the sumset and difference set grow non-trivially. This was shown in [L], and here we get a more explicit version of the same result.

2. Notation and preliminaries. Throughout this paper, the symbols $\ll$, $\gg$ and $\approx$ are used to suppress constants. For example, $X \ll Y$ means that there exists some absolute constant C such that $X < CY$, and $X \approx Y$ means that $X \ll Y$ and $Y \ll X$. Also, all logarithms are to base 2.

For sets A and B, let $E(A, B)$ be the additive energy of A and B, defined in the usual way. So, denoting by $\delta_{A,B}(s)$ (and respectively $\sigma_{A,B}(s)$) the number of representations of an element s of $A - B$ (respectively $A + B$), and writing $\delta_A(s) = \delta_{A,A}(s)$, we define

$$E(A, B) = \sum_s \delta_A(s)\delta_B(s) = \sum_s \delta_{A,B}(s)^2 = \sum_s \sigma_{A,B}(s)^2.$$

Given a set $A \subset \mathbb{R}$ and some $s \in \mathbb{R}$, let $A_s := A \cap (A + s)$. A crucial observation is that $|A_s| = \delta_A(s)$. In this paper, following [SS2], the third moment energy $E_3(A)$ will also be studied, where

$$E_3(A) = \sum_s \delta_A(s)^3.$$

In much the same way, we define

$$E_{1.5}(A) = \sum_s \delta_A(s)^{1.5}.$$

Later on, we will need the following lemma, which was proved in [L]. Note that the proof made use of the Katz–Koester transform (see [KK]).

LEMMA 2.1. *Let A, B be any sets. Then*

$$E_{1.5}(A)^2 |B|^2 \le E_3(A)^{2/3} E_3(B)^{1/3} E(A, A + B).$$

3. Some consequences of the Szemerédi–Trotter theorem. The main preliminary result is an upper bound on the number of high multiplicity elements of a sumset, a result which comes from an application of the Szemerédi–Trotter incidence theorem ([ST]).

THEOREM 3.1. *Let $\mathcal{P}$ be a set of points in the plane and $\mathcal{L}$ a set of curves such that any pair of curves intersect at most once. Then*

$$|\{(p,l) \in \mathcal{P} \times \mathcal{L} : p \in l\}| \leq 4(|\mathcal{P}|\,|\mathcal{L}|)^{2/3} + 4|\mathcal{P}| + |\mathcal{L}|.$$

REMARK. While this paper was in the process of being drafted, a very similar result to the following lemma was included in a paper of Schoen and Shkredov ([SS1, Lemma 24]) which was posted on the arXiv. See their paper for an alternative description of this result and proof. A weaker version of this result was also proved in [L].

LEMMA 3.2. *Let f be a continuous, strictly convex or concave function on the reals, and $A, B, C \subset \mathbb{R}$ be finite sets such that $|B|\,|C| \gg |A|^2$. Then for all $\tau \geq 1$,*

$$|\{x : \sigma_{f(A),C}(x) \geq \tau\}| \ll \frac{|A+B|^2|C|^2}{|B|\tau^3}, \tag{3.1}$$

$$|\{y : \sigma_{A,B}(y) \geq \tau\}| \ll \frac{|f(A)+C|^2|B|^2}{|C|\tau^3}. \tag{3.2}$$

Proof. Let $G(f)$ denote the graph of f in the plane. For any $(\alpha,\beta) \in \mathbb{R}^2$, put $L_{\alpha,\beta} = G(f) + (\alpha,\beta)$. Define a set of points $\mathcal{P} = (A+B) \times (f(A)+C)$ and a set of curves $\mathcal{L} = \{L_{b,c} : (b,c) \in B \times C\}$. By convexity or concavity, $|\mathcal{L}| = |B|\,|C|$, and any pair of curves from $\mathcal{L}$ intersect at most once. Let $\mathcal{P}_\tau$ be the set of points of $\mathcal{P}$ belonging to at least τ curves from $\mathcal{L}$. Applying the aforementioned Szemerédi–Trotter theorem to $\mathcal{P}_\tau$ and $\mathcal{L}$, we get

$$\tau|\mathcal{P}_\tau| \leq 4(|\mathcal{P}_\tau|\,|B|\,|C|)^{2/3} + 4|\mathcal{P}_\tau| + |B|\,|C|.$$

Now we claim for any $\tau > 0$ one has

$$|\mathcal{P}_\tau| \ll |B|^2|C|^2/\tau^3. \tag{3.3}$$

The reason is as follows. Firstly, since there is no point of $\mathcal{P}$ belonging to at least $\min\{|B|+1, |C|+1\}$ curves from $\mathcal{L}$, to prove (3.3) we may assume that $\tau \leq \sqrt{|B|\,|C|}$. Secondly, if $\tau < 8$, then (3.3) holds true since

$$|\mathcal{P}_\tau| \leq |\mathcal{P}| = |(A+B) \times (f(A)+C)| \leq |A|^2|B|\,|C| \ll |B|^2|C|^2 \leq 64\frac{|B|^2|C|^2}{\tau^2}.$$

Finally, we may assume that $8 \leq \tau \leq \sqrt{|B|\,|C|}$. In this case we have

$$\tau|\mathcal{P}_\tau|/2 \leq 4(|\mathcal{P}_\tau|\,|B|\,|C|)^{2/3} + |B|\,|C|.$$

Thus

$$|\mathcal{P}_\tau| \ll \max\{|B|^2|C|^2/\tau^3, |B|\,|C|/\tau\} = |B|^2|C|^2/\tau^3.$$

This proves the claim (3.3).

Next, suppose $\sigma_{f(A),C}(x) \geq \tau$. There exist τ distinct elements $\{a_i\}_{i=1}^{\tau}$ from A and τ distinct elements $\{c_i\}_{i=1}^{\tau}$ from C such that $x = f(a_i) + c_i$ for all i. Now we define $B_i := a_i + B$ for all i, and $\mathcal{M}_x(s) := \sum_{i=1}^{\tau} \chi_{B_i}(s)$, where $\chi_{B_i}(\cdot)$ is the characteristic function of B_i. Since

$$(a_i + b, x) = (a_i + b, f(a_i) + c_i) = (a_i, f(a_i)) + (b, c_i) \in L_{b,c_i}$$

for all i and b, we have $(s, x) \in \mathcal{P}_{\mathcal{M}_x(s)}$. Note also

$$\sum_{s\in A+B} \mathcal{M}_x(s) = \sum_{i=1}^{\tau} \sum_{s\in A+B} \chi_{B_i}(s) = \tau|B|.$$

Let $M := \tau|B|/(2|A+B|)$. Then

$$\sum_{s\in A+B\colon \mathcal{M}_x(s)<M} \mathcal{M}_x(s) < |A+B|M = \tau|B|/2,$$

and hence

$$\sum_{s\in A+B\colon \mathcal{M}_x(s)\geq M} \mathcal{M}_x(s) \geq \tau|B|/2.$$

Dyadically decompose this sum, so that

$$\sum_j X_j(x) \gg \tau|B|, \tag{3.4}$$

where

$$X_j(x) := \sum_{s\colon M2^j\leq \mathcal{M}_x(s)<M2^{j+1}} \mathcal{M}_x(s),$$

$$Y_j(x) := |\{s \in A+B : M2^j \leq \mathcal{M}_x(s) < M2^{j+1}\}|.$$

By (3.3),

$$\sum_{x\colon \sigma_{f(A),C}(x)\geq\tau} Y_j(x) \leq |\mathcal{P}_{M2^j}| \ll \frac{|B|^2|C|^2}{M^3 2^{3j}}.$$

Note that $X_j(x) \approx Y_j(x)M2^j$, thus

$$\sum_{x\colon \sigma_{f(A),C}(x)\geq\tau} X_j(x) \ll \frac{|B|^2|C|^2}{M^2 2^{2j}},$$

which followed by first summing all j's, then applying (3.4), gives

$$\tau|B|\,|\{x : \sigma_{f(A),C}(x) \geq \tau\}| \ll |B|^2|C|^2/M^2.$$

Equivalently,

$$|\{x : \sigma_{f(A),C}(x) \geq \tau\}| \ll \frac{|A+B|^2|C|^2}{|B|\tau^3}.$$

This finishes the proof of (3.1).

In the same way one can prove (3.2). We only sketch the proof and leave the details to the interested readers. Suppose $\sigma_{A,B}(y) \geq \tau$. There exist τ distinct elements $\{a_i\}_{i=1}^{\tau}$ from A and τ distinct elements $\{b_i\}_{i=1}^{\tau}$ from B such that $y = a_i + b_i$. Then we define $C_i := f(a_i) + C$ and $\mathcal{M}_y(s) := \sum_{i=1}^{\tau} \chi_{C_i}(s)$, and as before, $(y,s) \in \mathcal{P}_{\mathcal{M}_y(s)}$. In precisely the same way as in the proof of (3.1), one can prove that

$$\sum_{s\in f(A)+C:\, \mathcal{M}_y(s)\geq M} \mathcal{M}_y(s) \geq \frac{\tau|C|}{2},$$

$$\sum_{y:\, \sigma_{A,B}(y)\geq\tau} Y_j(y) \leq |\mathcal{P}_{M2^j}| \ll \frac{|B|^2|C|^2}{M^3 2^{3j}},$$

$$\sum_{y:\, \sigma_{A,B}(y)\geq\tau} X_j(y) \ll \frac{|B|^2|C|^2}{M^2 2^{2j}},$$

$$\tau|C|\,|\{y : \sigma_{A,B}(y) \geq \tau\}| \ll \frac{|B|^2|C|^2}{M^2},$$

$$|\{y : \sigma_{A,B}(y) \geq \tau\}| \ll \frac{|f(A)+C|^2|B|^2}{|C|\tau^3},$$

where $M := \tau|C|/(2|f(A)+C|)$, $X_j(y) := \sum_{s:\, M2^j\leq\mathcal{M}_y(s)<M2^{j+1}} \mathcal{M}_y(s)$, $Y_j(y) := |\{s \in f(A)+C : M2^j \leq \mathcal{M}_y(s) < M2^{j+1}\}|$. This finishes the whole proof. ■

Corollary 3.3. *Let f be a continuous, strictly convex or concave function on the reals, and $A, C, F \subset \mathbb{R}$ be finite sets such that $|A| \approx |C| \ll |F|$. Then*

$$E(A,A) \ll E_{1.5}(A)^{2/3}|f(A)+C|^{2/3}|A|^{1/3}, \tag{3.5}$$

$$E(A,F) \ll |f(A)+C|\,|F|^{3/2}, \tag{3.6}$$

$$E_3(A) \ll |f(A)+C|^2|A|\log|A|, \tag{3.7}$$

$$E(f(A),f(A)) \ll E_{1.5}(f(A))^{2/3}|A+C|^{2/3}|A|^{1/3}, \tag{3.8}$$

$$E(f(A),F) \ll |A+C|\,|F|^{3/2}, \tag{3.9}$$

$$E_3(f(A)) \ll |A+C|^2|A|\log|A|. \tag{3.10}$$

Proof. Let $\triangle > 0$ be an arbitrary real number. First decomposing $E(A)$, then applying Lemma 3.2 with $B = -A$, gives

$$\begin{aligned} E(A,A) &= \sum_{s:\, \delta_A(s)<\triangle} \delta_A(s)^2 + \sum_{j=0}^{\lfloor \log|A|\rfloor} \sum_{s:\, 2^j\triangle \le \delta_A(s) < 2^{j+1}\triangle} \delta_A(s)^2 \\ &\ll \sqrt{\triangle}\, E_{1.5}(A) + \sum_{j=0}^{\lfloor \log|A|\rfloor} \frac{|f(A)+C|^2|A|}{2^{3j}\triangle^{3j}} \cdot 2^{2j}\triangle^{2j} \\ &\ll \sqrt{\triangle}\, E_{1.5}(A) + \frac{|f(A)+C|^2|A|}{\triangle}. \end{aligned}$$

Choosing an optimal value of $\triangle$ to balance the two terms completes the proof of (3.5).

Similarly, applying Lemma 3.2 with $B = -F$ gives

$$\begin{aligned} E(A,F) &= \sum_{s:\, \delta_{A,F}(s)<\triangle} \delta_{A,F}(s)^2 + \sum_{j=0}^{\lfloor \log|A|\rfloor} \sum_{s:\, 2^j\triangle \le \delta_{A,F}(s) < 2^{j+1}\triangle} \delta_{A,F}(s)^2 \\ &\ll \triangle E_1(A,F) + \sum_{j=0}^{\lfloor \log|A|\rfloor} \frac{|f(A)+C|^2|F|^2}{|C|2^{3j}\triangle^{3j}} \cdot 2^{2j}\triangle^{2j} \\ &\ll \triangle|A|\,|F| + \frac{|f(A)+C|^2|F|^2}{|C|\triangle}. \end{aligned}$$

Choosing an optimal value of $\triangle$ to balance the two terms completes the proof of (3.6).

Once again applying Lemma 3.2 with $B = -A$ gives

$$\begin{aligned} E_3(A) &= \sum_{j=0}^{\lfloor \log|A|\rfloor} \sum_{s:\, 2^j \le \delta_A(s) < 2^{j+1}} \delta_A(s)^3 \\ &\ll \sum_{j=0}^{\lfloor \log|A|\rfloor} |f(A)+C|^2|A| = |f(A)+C|^2|A|\log|A|, \end{aligned}$$

which proves (3.7); and (3.8)–(3.10) can be established in the same way. ∎

4. Proofs of the main results

4.1. Proof of Theorem 1.1. First, apply Hölder's inequality to bound $E_{1.5}(A)$ from below:

$$|A|^6 = \Big(\sum_{s\in A-A} \delta_A(s)\Big)^3 \le \Big(\sum_{s\in A-A} \delta_A(s)^{1.5}\Big)^2 |A-A| = E_{1.5}(A)^2|A-A|.$$

Using this bound and Lemma 2.1 with $B = -A$ gives

$$\frac{|A|^8}{|A-A|} \leq E_{1.5}(A)^2|A|^2 \leq E_3(A)E(A, A-A).$$

Finally, apply (3.7), and (3.6) with $F = A - A$, to conclude that

$$\frac{|A|^8}{|A-A|} \ll |f(A)+C|^3|A-A|^{3/2}|A|\log|A|,$$

and hence

$$|f(A)+C|^6|A-A|^5 \gg \frac{|A|^{14}}{(\log|A|)^2},$$

as required.

4.2. Proof of Theorem 1.2. Using the standard Cauchy–Schwarz bound on the additive energy, and then (3.5), we see that

$$\frac{|A|^{12}}{|A+A|^3} \leq E(A,A)^3 \ll E_{1.5}(A)^2|f(A)+C|^2|A|$$
$$= \frac{|f(A)+C|^2}{|A|}E_{1.5}(A)^2|A|^2.$$

Next, apply Lemma 2.1 with $B = A$ to get

$$\frac{|A|^{12}}{|A+A|^3} \ll \frac{|f(A)+C|^2}{|A|}E_3(A)E(A, A+A),$$

and then apply (3.7), and (3.6) with $F = A + A$, to get

$$\frac{|A|^{12}}{|A+A|^3} \ll \frac{|f(A)+C|^2}{|A|}|f(A)+C|^3|A+A|^{3/2}|A|\log|A|,$$

which, after rearranging, gives

$$|f(A)+C|^{10}|A+A|^9 \gg \frac{|A|^{24}}{(\log|A|)^2}.$$

4.3. Proof of Theorem 1.3. Observe that the Cauchy–Schwarz inequality applied twice tells us that

$$\frac{|A|^{24}}{|A+f(A)|^6} \leq E(A,f(A))^6 \leq E(A,A)^3E(f(A),f(A))^3,$$

so that after applying (3.5) and (3.8), with either $C = A$ or $C = f(A)$,

$$\begin{aligned}\frac{|A|^{26}}{|A+f(A)|^6} &\leq |A|^2E_{1.5}(A)^2|A+f(A)|^2|A|E_{1.5}(f(A))^2|A+f(A)|^2|A| \\ &= (E_{1.5}(A)^2|f(A)|^2)(E_{1.5}(f(A))^2|A|^2)|A+f(A)|^4 \\ &\leq E_3(A)E_3(f(A))E(A, A+f(A)) \\ &\quad \times E(f(A), A+f(A))|A+f(A)|^4,\end{aligned}$$

where the last inequality is a consequence of two applications of Lemma 2.1. Next apply (3.7) and (3.10), again with either $C = A$ or $C = f(A)$, to get

$$\frac{|A|^{26}}{|A+f(A)|^6} \le |A+f(A)|^8|A|^2(\log|A|)^2 E(A, A+f(A))E(f(A), A+f(A)).$$

Finally, apply (3.6) and (3.9), still with either $C = A$ or $C = f(A)$, to obtain

$$\frac{|A|^{26}}{|A+f(A)|^6} \le |A+f(A)|^{13}|A|^2(\log|A|)^2.$$

Then, after rearranging, we get

$$|A+f(A)| \gg \frac{|A|^{24/19}}{(\log|A|)^{2/19}}.$$

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