ACTA ARITHMETICA
152.2 (2012)

Complete asymptotic expansions for certain multiple q-integrals and q-differentials of Thomae–Jackson type

by

Masanori Katsurada (Yokohama)

1. Introduction. We suppose for a moment that q is a real parameter with $0 < q < 1$. Let $\varphi(u)$ be a function integrable on the interval $[0, x]$. A q-analogue of the ordinary integral $\int_0^x \varphi(u)\,du$, in the form

$$\int_0^x \varphi(u)\,d_q u = (1-q)x\sum_{n=0}^{\infty}\varphi(q^n x)q^n, \tag{1.1}$$

was first introduced by Thomae [Th] in 1869 and extensively studied by Jackson [Ja] during 1910–1951 (see also [GR, p. 23, Chap. 1, 1.11]). One natural motivation to formulate (1.1) is that the limiting relation

$$\lim_{q\to 1^-}\int_0^x \varphi(u)\,d_q u = \int_0^x \varphi(u)\,du \tag{1.2}$$

holds for all $\varphi(u)$ continuous on $[0, x]$. A q-analogue, on the other hand, of the ordinary differentiation is defined by

$$\partial_{q,z}\psi(z) = \frac{\psi(z)-\psi(qz)}{(1-q)z} \tag{1.3}$$

(cf. [GR, p. 27, 1.12]), and again satisfies the limiting relation

$$\lim_{q\to 1^-}\partial_{q,z}\psi(z) = \psi'(z) = \partial_z\psi(z), \tag{1.4}$$

say, for all $\psi(z)$ complex differentiable at z.

Throughout this paper, q is a complex parameter with $0 < |q| < 1$, and the substitution $q = e^{-t}$ will be made if necessary, transforming the half-plane $\operatorname{Re} t > 0$ to the unit disk $|q| < 1$. A complex domain $D \subset \mathbb{C}$ is called

2010 *Mathematics Subject Classification*: Primary 11P82; Secondary 11M35.
Key words and phrases: multiple q-integral, multiple q-differential, Mellin transform, asymptotic expansion, Lerch zeta-function.

DOI: 10.4064/aa152-2-1

star-shaped if $0 \in D$ and for any $z \in D$ the line segment $\overline{0, z}$ is included in D. We suppose throughout the paper that $f(z)$ is a function holomorphic in a star-shaped domain D, and ρ_f denotes the distance between 0 and the singularity of $f(z)$ closest to 0.

Let x and y be real numbers with $x > 0$ and $y \geq 0$. Then we define the q-integral and q-differential operators $\mathcal{I}^x_{q,z}$ and $\mathcal{D}^y_{q,z}$ by

$$\mathcal{I}^x_{q,z} f(z) = \int_0^1 u^{x-1} f(uz)\, d_q u = z^{-x} \int_0^z w^{x-1} f(w)\, d_q w \tag{1.5}$$

and

$$\mathcal{D}^y_{q,z} f(z) = \frac{f(z) - q^y f(qz)}{1-q} = z^{-y}(z\partial_{q,z})\{z^y f(z)\} \tag{1.6}$$

for any z in $|z| < \rho_f$, where the latter equalities follow from (1.1) and (1.3) respectively.

Remark. If the base q is restricted to the range $0 < q < 1$, then the domain of z in which the definitions in (1.5) and (1.6) are valid is extended to the whole D by its star-shapedness.

Proposition 1. *The operator relations*

$$\mathcal{I}^x_{q,z}\mathcal{D}^x_{q,z} = 1 \quad \textit{and} \quad \mathcal{D}^x_{q,z}\mathcal{I}^x_{q,z} = 1$$

hold for any $x > 0$, *where* 1 *denotes the identity operation.*

Proof. Let $\varphi(z)$ be a function holomorphic on D. In view of the latter equalities in (1.5) and (1.6), the first assertion reduces to the identity $\int_0^z \partial_{q,w}\varphi(w)\, d_q w = \varphi(z) - \varphi(0)$, while the second to $\partial_{q,z}\int_0^z \varphi(w)\, d_q w = \varphi(z)$; both of these identities are direct consequences of (1.1) and (1.3). ■

It is the principal aim of this paper to extend (1.2) and (1.4); this leads us to show that complete asymptotic expansions as $t \to 0$ through the sector $|\arg t| < \pi/2$ exist for the multiple q-integrals $(\mathcal{I}^x_{q,z})^r f(q^y z)$ (Theorem 1) and the multiple q-differentials $(\mathcal{D}^x_{q,z})^r f(q^y z)$ (Theorem 2) with any integer $r \geq 1$, under fairly generic conditions. A full extension of the domain of z in which Theorems 1 and 2 are valid is possible when $0 < q < 1$ (Theorem 3). Several applications of our main formulae (2.4) and (2.9) will further be given to the generalized Lerch zeta-function (Theorems 4–6), q-factorials (Corollary 4.1), and q-analogues of the exponential function (Corollary 4.2), binomial function (Corollary 4.3), and poly-logarithmic function (Corollaries 4.4 and 5.1). As for the methods used, a Mellin transform technique is crucial in the proofs of Theorems 1 and 2, which reveals that multiple q-integrals and multiple q-differentials are linked to the particular case of Barnes' (multiple) zeta-functions in (4.1) below, through (inverse) Mellin transforms.

The generic function in (4.2) is inserted in a class of zeta-functions such as $\Phi_f(s,x,z)$ in (4.9) defined by Hadamard's convolution with a (generic) function $f(z)$; this is a key to incorporate (4.2) in the Mellin transform formulae (4.8) and (6.7), which play crucial rôles in establishing the asymptotic expansions in (2.4) and (2.9).

It seems that the q-integrals and their asymptotic aspects first appeared in the 1961 paper of Agarwal [Ag]. He defined a q-analogue of McRobert's E-function by means of a certain q-integral, and derived several of its analytic properties including the asymptotic expansions when the variable z tends to ∞. Asymptotic analysis from the point of view of q-analogues has recently been developed by Fitouhi–Brahim–Bettaibi [FBB], who established (for instance) a q-analogue of Watson's lemma on q-Laplace integrals to derive various asymptotic expansions (in small and large variable values) of certain q-special functions including q-error, q-Bessel and q-gamma functions; this direction was further pursued by Bettaibi–Kamel [BeKa] who investigated asymptotic aspects of q-Mellin, q-Laplace, q-Fourier and q-Hankel transforms. Note that all the studies above were carried out with a base q fixed in the range $0 < q < 1$.

Let s be a complex variable, u and v real parameters, and write $e(s) = e^{2\pi i s}$. As for asymptotic aspects of q-series when $q \to 1$, we have established recently in [Ka3, Theorem 0] complete asymptotic expansions of the generalized Lambert series

$$S_s(x,y;u,v;q) = e(ux) \sum_{n=0}^{\infty}{}' (y+n)^{-s} \frac{e(v(y+n))q^{x(y+n)}}{1-e(u)q^{y+n}} \tag{1.7}$$

as $t \to 0$ through the sector $|\arg t| < \pi/2$ by means of a Mellin transform technique, where the term with $n = 0$ is to be omitted if $y = 0$. The customary notation

$$(z;q)_\infty = \prod_{m=0}^{\infty} (1-q^m z) \quad \text{and} \quad (z;q)_n = (z;q)_\infty/(zq^n;q)_\infty \tag{1.8}$$

for any integer n will be used. [Ka3, Theorem 0] in particular implies a complete asymptotic expansion of $\log(q^\alpha;q)_\infty$ as $q \to 1^-$, and it further allows us to treat the q-series

$$F(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2}, \qquad G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n}, \qquad H(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n}.$$

These are typical examples of theta series (in the transformed Eulerian form) whose asymptotic behaviour near the singularities at $q^k = 1$ $(k = 1, 2, \ldots)$ was first considered by Ramanujan in his last letter to Hardy (see [Wa]).

Ramanujan showed

$$
\begin{aligned}
F(q) &= \left(\frac{t}{2\pi}\right)^{1/2} \exp\left(\frac{\pi^2}{6t} - \frac{t}{24}\right) + o(1),\\
G(q) &= \left(\frac{2}{5-\sqrt{5}}\right)^{1/2} \exp\left(\frac{\pi^2}{15t} - \frac{t}{60}\right) + o(1),\\
H(q) &= \left(\frac{2}{5+\sqrt{5}}\right)^{1/2} \exp\left(\frac{\pi^2}{15t} + \frac{11t}{60}\right) + o(1),
\end{aligned}
\tag{1.9}
$$

as $t \to 0^+$, and similar asymptotic formulae for certain other q-series; improvements upon (1.9) to complete forms were deduced as a corollary of [Ka3, Theorem 0]. Zagier [Za] and McIntosh [Mc3] more recently made further progress in this direction. Next let B_k $(k = 0, 1, \ldots)$ denote the Bernoulli numbers (cf. [Er1, p. 35, 1.13(1)]). Then [Ka3, Theorem 0] also yields Ramanujan's famous formula for specific values of the Riemann zeta-function $\zeta(s)$ at odd integers (cf. [Be1, Theorem 2.4], [Be2, Chap. 14, Entry 21(i)]), which asserts that, for any integer $k \neq 0$,

(1.10)

$$
\begin{aligned}
\alpha^{-k}\biggl\{\frac{1}{2}\zeta(2k+1) + \sum_{m=1}^{\infty} \frac{m^{-2k-1}}{e^{2m\alpha}-1}\biggr\} + 2^{2k} \sum_{j=0}^{k+1} \frac{B_{2k+2-2j}B_{2j}}{(2k+2-2j)!(2j)!}\alpha^{k+1-j}(-\beta)^j \\
= (-\beta)^{-k}\biggl\{\frac{1}{2}\zeta(2k+1) + \sum_{m=1}^{\infty} \frac{m^{-2k-1}}{e^{2m\beta}-1}\biggr\},
\end{aligned}
$$

where α and β are positive numbers satisfying $\alpha\beta = \pi^2$ and the finite sum on the left side is to be regarded as null if $k < -1$. The use of a Mellin transform technique to treat (1.7) in fact clarifies that the excluded case $k = 0$ of (1.10) re-emerges (in a sense) in asymptotic formulae for $F(q)$, $G(q)$ and $H(q)$ in (1.9), and in their improvements (see [Ka3, Corollary 1.4]). One of the major features of the results in [Ka3] is that various specific values of zeta-functions appear in the coefficients of the asymptotic series therein. The Mellin transform technique, applied in the present paper (see Sections 4–6 below), also clarifies that the same phenomena as above occur in our main formulae (2.4) and (2.9), whose coefficients may be regarded as specific values of the zeta-functions $\Phi_f(s, x, z)$ in (4.9) and $\zeta_r(s, y)$ in (4.1). One can further observe the same phenomena e.g. in the papers of Berndt–Sohn [BS], Coogan–Ono [CO], and Berndt–Yee [BY]. It is to be remarked that the applications of Theorems 1 and 2 cover the classes of q-series which could not be treated in our previous study [Ka3].

The paper is organized as follows. Our main results (Theorems 1–3 and their corollaries) are stated in the next section, while Section 3 is devoted to the applications of our main formulae. Theorems 1 and 2 will be shown in Sections 4–6, while in Section 7 the proofs of Corollaries 1.1, 2.1 and Theorem 3 are given. The final section is devoted to the derivation of Theorems 4–6 and their corollaries. All the results in the present paper have been announced in [Ka4].

2. Statement of results. Let r be any integer, and w a complex variable. To describe our results we introduce the sequences of functions $A_{f,k}(x,z)$ and the generalized Bernoulli polynomials $B_k^{(r)}(y)$ of rank r (due to Nörlund [Nö]) defined for $k = 0, 1, \ldots$ by the respective Taylor series expansions

$$e^{xw} f(e^w z) = \sum_{k=0}^{\infty} \frac{A_{f,k}(x,z)}{k!} w^k \tag{2.1}$$

and

$$e^{yw} \left(\frac{w}{e^w - 1} \right)^r = \sum_{k=0}^{\infty} \frac{B_k^{(r)}(y)}{k!} w^k \tag{2.2}$$

near $w = 0$. Note that $B_k^{(1)}(y) = B_k(y)$ is the usual Bernoulli polynomial, and so $B_k(0) = B_k$ is the usual Bernoulli number. We write $B_k^{(r)}(0) = B_k^{(r)}$. We will also use Euler's differential operator $\vartheta_z = z\partial_z$.

We now state our first main result.

Theorem 1. *Let x and y be real parameters with $x > 0$ and $y \geq 0$, $q = e^{-t}$, and $r \geq 1$ a fixed integer. Further let $(\mathcal{I}_{q,z}^x)^r f(z)$ denote the r-fold iteration of* (1.5) *applied to any function $f(z)$ holomorphic in a star-shaped domain D, and define the coefficients $A_{f,-j}(x,z)$ $(j = 1, 2, \ldots)$ by*

$$A_{f,-j}(x,z) = \int_0^1 u_j^{x-1} \int_0^1 u_{j-1}^{x-1} \cdots \int_0^1 u_1^{x-1} f(u_1 \cdots u_j z)\, du_1 \cdots du_j. \tag{2.3}$$

Then for any integer $K \geq 0$ the formula

$$\begin{aligned} \frac{q^{xy}}{(1-q)^r} (\mathcal{I}_{q,z}^x)^r f(q^y z) &= \sum_{j=1}^{r} \frac{(-1)^{r-j} A_{f,-j}(x,z) B_{r-j}^{(r)}(y)}{(r-j)!} t^{-j} \\ &\quad + \sum_{k=0}^{K-1} \frac{(-1)^{r+k} A_{f,k}(x,z) B_{r+k}^{(r)}(y)}{(r+k)!} t^k + R_{f,K}^{(r)}(x,y;q,z) \end{aligned} \tag{2.4}$$

holds in the sector $|\arg t| < \pi/2$ and on the disk $|z| < \rho_f$. Here $R_{f,K}^{(r)}$ is the *remainder term expressed by the Mellin transform formula* (5.5) *below, and*

$$R^{(r)}_{f,K}(x,y;q,z)=O(|t|^K) \tag{2.5}$$

as $t\to 0$ *through* $|\arg t|\le\pi/2-\delta$ *with any small* $\delta>0$, *where the implied* O*-constant depends at most on* r, x, y, z, K *and* δ. *In particular if* $0\le y\le r$ *and* $K\ge 1$, *we have the representation*

$$\begin{aligned} R^{(r)}_{f,K}(x,y;q,z)&=(-t)^K\sum_{l=0}^{r-1}\frac{(-1)^{r-1-l}B^{(r)}_{r-1-l}(y)}{l!(r-1-l)!}\sideset{}{'}\sum_{n=-\infty}^{\infty}\frac{e(ny)}{(2\pi in)^{K+l}}\\ &\quad\times\left(\frac{\partial}{\partial u}\right)^l u^{K+l}\int_0^1\xi^{xtu+2\pi in-1}(x+\vartheta_z)^K f(\xi^{tu}z)\,d\xi\bigg|_{u=1}, \end{aligned} \tag{2.6}$$

where the primed summation symbol indicates the omission of the term with $n=0$.

Remark 1. The presence of the negative order terms (with $j=1,\dots,r$) on the right side of (2.4) is reasonable, since the factor $(1-q)^{-r}$ on the left side is asymptotically t^{-r} as $t\to 0$.

Remark 2. The n-sum on the right side of (2.6) converges absolutely for all $K\ge 1$, since the ξ-integral (differentiated with respect to u) is of order $O(|n|^{-1})$ as $n\to\pm\infty$ by partial integration.

Remark 3. The explicit formula (2.6) will be used to extend the domain of z where (2.4) with (2.5) is valid (see Theorem 3).

From the point of view of applications it is necessary to obtain the asymptotic expansions for $(\mathcal{I}^x_{q,z})^r f(z)$ both with and without the associated q-multiples (see (3.5), (3.11) and (3.12) below). The case $y=0$ of Theorem 1 in fact yields, in view of the latter equality in (1.5), the following corollary.

Corollary 1.1. *Let* r *and* x *be as in Theorem* 1. *Then for any integer* $K\ge 0$ *the asymptotic formula*

$$\begin{aligned} &\int_0^z w_r^{-1}\int_0^{w_{r-1}} w_{r-1}^{-1}\cdots w_2^{-1}\int_0^{w_2} w_1^{x-1}f(w_1)\,d_qw_1\cdots d_qw_r\\ &\qquad=\sum_{k=0}^{K-1}\frac{(-1)^kC^{(r)}_{f,k}(x,z)}{k!}t^k+O(|t|^K) \end{aligned} \tag{2.7}$$

holds as $t\to 0$ *through* $|\arg t|\le\pi/2-\delta$ *for any small* $\delta>0$, *on the disk* $|z|<\rho_f$ *with* $|\arg z|<\pi$, *where the implied* O*-constant depends at most on*

x, z, K and δ. Here the coefficients $C^{(r)}_{f,k}$ $(k=0,1,\ldots)$ are given by

$$\begin{aligned}(2.8)\qquad C^{(r)}_{f,k}(x,z) = &\sum_{j=\max(1,r-k)}^{r}\binom{k}{r-j}B^{(-r)}_{k-r+j}B^{(r)}_{r-j}\\ &\times\int_0^z w_j^{-1}\int_0^{w_j} w_{j-1}^{-1}\cdots w_2^{-1}\int_0^{w_2} w_1^{x-1}f(w_1)\,dw_1\cdots dw_j\\ &+\sum_{j=0}^{k-r}\binom{k}{r+j}B^{(-r)}_{k-r-j}B^{(r)}_{r+j}\vartheta_z^j\{z^x f(z)\},\end{aligned}$$

which reduces if $r=1$ to

$$C^{(1)}_{f,k}(x,z) = \frac{1}{k+1}\left[\int_0^z w^{x-1}f(w)\,dw + \sum_{j=0}^{k-1}\binom{k+1}{j+1}B_{j+1}\vartheta_z^j\{z^x f(z)\}\right],$$

where the empty sums are to be regarded as null.

The case $K=1$ of Corollary 1.1 readily implies the following corollary.

COROLLARY 1.2. *Under the same assumptions as in Corollary* 1.1,

$$\begin{aligned}\lim_{\substack{q\to 1\\ |q|<1}}&\int_0^z w_r^{-1}\int_0^{w_{r-1}} w_{r-1}^{-1}\cdots w_2^{-1}\int_0^{w_2} w_1^{x-1}f(w_1)\,d_qw_1\cdots d_qw_r\\ &= C^{(r)}_{f,0}(x,z) = \int_0^z w_r^{-1}\int_0^{w_{r-1}} w_{r-1}^{-1}\cdots w_2^{-1}\int_0^{w_2} w_1^{x-1}f(w_1)\,dw_1\cdots dw_r.\end{aligned}$$

We proceed to state our second main result. Throughout the following, $\Gamma(s)$ denotes the gamma function, and $(s)_n=\Gamma(s+n)/\Gamma(s)$ for any integer n is the shifted factorial of s.

THEOREM 2. *Let $x,y\ge 0$ be real parameters, $q=e^{-t}$, and $r\ge 1$ a fixed integer. Further let $(\mathcal{D}^x_{q,z})^r f(z)$ denote the r-fold iteration of* (1.6) *applied to any function $f(z)$ holomorphic in a star-shaped domain D. Then for any integer $K\ge 0$ the formula*

$$\begin{aligned}(2.9)\qquad q^{xy}&\left(\frac{1-q}{t}\right)^r(\mathcal{D}^x_{q,z})^r f(q^y z)\\ &= \sum_{k=0}^{K-1}\frac{(-1)^k A_{f,r+k}(x,z)B^{(-r)}_k(y)}{k!}t^k + R^{(-r)}_{f,K}(x,y;q,z)\end{aligned}$$

holds in the sector $|\arg t|<\pi/2$ and on the disk $|z|<\rho_f$. Here the remainder term $R^{(-r)}_{f,K}$ is expressed by the Mellin transform formula (6.10) *below, and*

$$(2.10)\qquad R^{(-r)}_{f,K}(x,y;q,z) = O(|t|^K)$$

as $t \to 0$ through $|\arg t| \le \pi/2 - \delta$ with any small $\delta > 0$, where the implied O-constant depends at most on r, x, y, z, K and δ. Furthermore, for any real $x, y \ge 0$, and any integer $K \ge 0$,

(2.11)

$$R_{f,K}^{(-r)}(x,y;q,z) = \frac{(-1)^{r+K}t^K}{\Gamma(r+K)} \sum_{n=0}^{r} \frac{(-r)_n}{n!}(y+n)^{r+K} \int_0^1 (1-\xi)^{r+K-1} q^{x(y+n)\xi} \times (x+\vartheta_z)^{r+K} f(q^{(y+n)\xi}z)\, d\xi.$$

In view of the latter equality in (1.6), the case $y = 0$ of Theorem 2 in fact yields the following corollary.

Corollary 2.1. *Let r and x be as in Theorem 2. Then for any integer $K \ge 0$ the asymptotic formula*

$$(z\partial_{q,z})^r\{z^x f(z)\} = \sum_{k=0}^{K-1} \frac{(-1)^k C_{f,k}^{(-r)}(x,z)}{k!} t^k + O(|t|^K) \tag{2.12}$$

holds as $t \to 0$ through $|\arg t| \le \pi/2 - \delta$ for any small $\delta > 0$, on the disk $|z| < \rho_f$ with $|\arg z| < \pi$, where the implied O-constant depends at most on r, x, z, K and δ. Here the coefficients $C_{f,k}^{(-r)}$ $(k = 0, 1, \ldots)$ are given by

$$C_{f,k}^{(-r)}(x,z) = \sum_{j=0}^{k} \binom{k}{j} B_{k-j}^{(r)} B_j^{(-r)} \vartheta_z^{r+j}\{z^x f(z)\}, \tag{2.13}$$

which reduces if $r = 1$ to

$$C_{f,k}^{(-1)}(x,z) = \frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j+1} B_{k-j} \vartheta_z^{1+j}\{z^x f(z)\}.$$

The case $K = 1$ of Corollary 2.1 readily implies the following corollary.

Corollary 2.2. *Under the same assumptions as in Corollary 2.1,*

$$\lim_{\substack{q\to 1\\ |q|<1}} (z\partial_{q,z})^r f(z) = C_{f,0}^{(-r)}(x,z) = (z\partial_z)^r\{z^x f(z)\}.$$

We lastly proceed to state the full extension of the domain of z in Theorems 1 and 2 under the restriction that $0 < q < 1$ (see Remark just below (1.6)).

Theorem 3. *Set $q = e^{-t}$ with any real $t > 0$, and let $f(z)$ be any function holomorphic in a star-shaped domain D.*

(i) *Let x and y be real with $x > 0$ and $0 \le y \le r$. Then the asymptotic expansion (2.4) with the estimate (2.5) when $t \to 0^+$, as well as the explicit formula (2.6), remain valid throughout the domain D.*

(ii) *Let $x, y \geq 0$ be real. Then the asymptotic expansion* (2.9) *with the estimate* (2.10) *when $t \to 0^+$, as well as the explicit formula* (2.11), *remain valid throughout the domain D.*

(iii) *The asymptotic expansion* (2.7) *with* (2.8) *when $t \to 0^+$ for $x > 0$, and also* (2.12) *with* (2.13) *when $t \to 0^+$ for $x \geq 0$, remain valid throughout the domain D.*

3. Applications of Theorems 1 and 2. We suppose for simplicity that $0 < q < 1$ throughout this section. Let $[s]_q = (1-q^s)/(1-q)$ be a q-analogue of s, and let $[s]_{q;n} = \prod_{m=0}^{n-1}[s+m]_q$ and $[1]_{q;n} = [n]_q!$ for $n = 0, 1, \ldots$ denote q-analogues of the shifted factorial of s and the factorial of n respectively (cf. [GR, p. 7, Chap. 1]), where the empty products are regarded to be 1. Note that $\lim_{q\to 1^-}[s]_q = s$ readily implies that

$$\lim_{q\to 1^-}[s]_{q;n} = (s)_n \quad \text{and} \quad \lim_{q\to 1^-}[n]_q! = n!. \tag{3.1}$$

We define the *generalized Lerch zeta-function* $\varPhi(s,x,z)$ by

$$\varPhi(s,x,z) = \sum_{m=0}^{\infty}(x+m)^{-s}z^m \tag{3.2}$$

for any complex s if $|z| < 1$, and for $\operatorname{Re} s > 1$ if $|z| = 1$ (cf. [Er1, p. 27, Chap. I, 1.11(1)]); this reduces to the ordinary Lerch zeta-function $\phi(s,x,\lambda)$ when $z = e(\lambda)$ with any $\lambda \in \mathbb{R}$, and further to the Hurwitz zeta-function $\zeta(s,x)$ when $z = 1$, while $\zeta_\lambda(s) = e(\lambda)\phi(s,1,\lambda)$ is the exponential zeta-function, and so $\zeta(s) = \zeta(s,1) = \zeta_\lambda(s)$ for $\lambda \in \mathbb{Z}$ is the Riemann zeta-function. The contour integral expression (5.2) below in fact shows that $\varPhi(s,x,z)$ continues to a holomorphic function of $(s,z) \in \mathbb{C} \times D$, where

$$D = \{z \in \mathbb{C} : |\arg(1-z)| < \pi\} = \mathbb{C} \setminus [1,\infty) \tag{3.3}$$

is a complex cut-plane. Note here that D is a star-shaped domain. We can therefore apply Theorem 3(i) (on (2.4) with (2.5)) to $f(z) = \varPhi(s,x,z)$, and obtain the following theorem.

THEOREM 4. *Let x and y be real with $x > 0$ and $0 \leq y \leq r$, and s any complex number. Then for any integer $K \geq 0$ the asymptotic expansion*

$$\begin{aligned} \frac{q^{xy}}{(1-q)^r}(\mathcal{I}^x_{q,z})^r\varPhi(s,x,q^y z) &= \sum_{j=1}^{r}\frac{(-1)^{r-j}\varPhi(s+j,x,z)B^{(r)}_{r-j}(y)}{(r-j)!}t^{-j} \\ &\quad + \sum_{k=0}^{K-1}\frac{(-1)^{r+k}\varPhi(s-k,x,z)B^{(r)}_{r+k}(y)}{(r+k)!}t^k + O(t^K) \end{aligned} \tag{3.4}$$

holds as $t \to 0^+$, in $|\arg(1-z)| < \pi$, where the implied O-constant depends at most on r, s, x, y, z and K.

Let $\mathrm{Li}_l(z)$ for any $l \in \mathbb{Z}$ be the poly-logarithmic function defined by $\mathrm{Li}_l(z) = z\Phi(l, 1, z)$ for any $z \in D$. It is seen from (1.1), (1.5), (1.8) and the relation $\log(1-z) = -z\Phi(1, 1, z)$, by (3.2), that

$$\log(q^y z; q)_\infty = -\frac{q^y z}{1-q} \mathcal{I}^1_{q,z} \Phi(1, 1, q^y z) \tag{3.5}$$

for any real $y \geq 0$ and in $|\arg(1-z)| < \pi$. Then the case $(r, s, x) = (1, 1, 1)$ of Theorem 4 yields the following corollary.

Corollary 4.1. *Let y be real with $0 \leq y \leq 1$. Then for any integer $K \geq 0$ the asymptotic expansion*

$$\log(q^y z; q)_\infty = -\mathrm{Li}_2(z) t^{-1} - \sum_{k=0}^{K-1} \frac{(-1)^{k+1} \mathrm{Li}_{1-k}(z) B_{k+1}(y)}{(k+1)!} t^k + O(t^K) \tag{3.6}$$

holds as $t \to 0^+$, in $|\arg(1-z)| < \pi$, where the implied O-constant depends at most on y, z and K.

Remark. The assertion (3.6) was first established by McIntosh [Mc1], [Mc2] in a more general setting.

We next present applications to q-analogues of the exponential and binomial functions defined respectively by

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} \qquad \left(|z| < \frac{1}{1-q}\right),$$
$$f_q(y; z) = \sum_{n=0}^{\infty} \frac{[y]_{q;n}}{[n]_q!} z^n \qquad (|z| < 1),$$

from which together with (3.1) the limiting relations $\lim_{q\to 1^-} e_q(z) = e^z$ and $\lim_{q\to 1^-} f_q(y; z) = (1-z)^{-y}$ follow. It is known that the q-binomial theorem (cf. [GR, p. 8, Chap. 1, 1.3]) asserts that

$$e_q(z) = \frac{1}{((1-q)z; q)_\infty} \quad \text{and} \quad f_q(y; z) = \frac{(q^y z; q)_\infty}{(z; q)_\infty} \tag{3.7}$$

for any $y \geq 0$; these further provide the meromorphic continuations of $e_q(z)$ and $f_q(y; z)$ respectively over the whole z-plane. Corollary 4.1 can therefore be applied to the right sides above, yielding the following corollaries.

Corollary 4.2. *For any integer $K \geq 0$ the asymptotic expansion*

$$\log e_q(z) = z + \sum_{k=1}^{K-1} \alpha_k(z) t^k + O(t^K) \tag{3.8}$$

holds as $t \to 0^+$, *in* $|\arg(1-z)| < \pi$, *and this further implies that*

$$e_q(z) = e^z \Big\{ 1 + \sum_{k=1}^{K-1} \beta_k(z) t^k + O(t^K) \Big\}$$

as $t \to 0^+$, *where the coefficients* $\alpha_k(z)$ *and* $\beta_k(z)$ *are given by*

$$\alpha_k(z) = \sum_{j=0}^{k} \frac{(-1)^{k-j} B_{k-j}}{(k-j)!} \sum_{h=0}^{j} (1+h)^{k-j-2} \frac{B_{j-h}^{(-h-1)} z^{1+h}}{(j-h)!}, \tag{3.9}$$

$$\beta_k(z) = \sum_{\substack{\sum_{j=1}^k j l_j = k \\ l_j \geq 0\,(j=1,\dots,k)}} \prod_{j=1}^{k} \frac{\alpha_j(z)^{l_j}}{l_j!}$$

for $k = 0, 1, \dots$, *and the implied* O*-constants depend on* z *and* K.

COROLLARY 4.3. *Let* y *be real with* $0 \leq y \leq 1$. *Then for any integer* $K \geq 0$ *the asymptotic expansion*

$$\log f_q(y;z) = \sum_{k=0}^{K-1} \frac{(-1)^{k+1} \operatorname{Li}_{1-k}(z)}{(k+1)!} \{B_{k+1} - B_{k+1}(y)\} t^k + O(t^K)$$

holds as $t \to 0^+$, *in* $|\arg(1-z)| < \pi$, *and this further implies that*

$$f_q(y;z) = (1-z)^{-y} \Big\{ 1 + \sum_{k=1}^{K-1} \gamma_k(y,z) t^k + O(t^K) \Big\}$$

as $t \to 0^+$, *where the coefficients* $\gamma_k(y,z)$ *are given by*

$$\gamma_k(y,z) = (-1)^k \sum_{\substack{\sum_{j=1}^k j l_j = k \\ l_j \geq 0\,(j=1,\dots,k)}} \prod_{j=1}^{k} \frac{1}{l_j!} \left[\frac{\operatorname{Li}_{1-j}(z)}{(j+1)!} \{B_{j+1}(y) - B_{j+1}\} \right]^{l_j}$$

for $k = 0, 1, \dots$. *Here the implied* O*-constants depend at most on* y, z *and* K.

We thirdly present applications to a q-analogue $\operatorname{Li}_{q,l}(z)$ of the polylogarithmic function for any $l \in \mathbb{Z}$, defined by

$$\operatorname{Li}_{q,l}(z) = \sum_{m=0}^{\infty} \frac{z^{1+m}}{[1+m]_q^l} \qquad (|z| < 1), \tag{3.10}$$

which by (3.1) satisfies $\lim_{q \to 1^-} \operatorname{Li}_{q,l}(z) = \operatorname{Li}_l(z)$. We can in fact show

$$\operatorname{Li}_{q,r}(z) = z (\mathcal{I}_{q,z}^1)^r \Phi(0,1,z) \tag{3.11}$$

for any integer $r \geq 0$; this further provides the meromorphic continuation of $\operatorname{Li}_{q,r}(z)$ for all $z \in D$. Corollary 1.1 can therefore be applied upon taking $f(z) = \Phi(0,1,z)$ to yield the following corollary.

Corollary 4.4. *Let $r\in\mathbb{Z}$ be fixed with $r\geq 1$. Then for any integer $K\geq 0$ the asymptotic expansion*

$$\mathrm{Li}_{q,r}(z)=\sum_{k=0}^{K-1}\frac{(-1)^k C_{f,k}^{(r)}(1,z)}{k!}t^k+O(t^K)$$

holds as $t\to 0^+$, in $|\arg(1-z)|<\pi$, where the coefficients $C_{f,k}^{(r)}$ are given by

$$\begin{aligned}C_{f,k}^{(r)}(1,z)=&\sum_{j=\max(1,r-k)}^{r}\binom{k}{r-j}B_{k-r+j}^{(-r)}B_{r-j}^{(r)}\,\mathrm{Li}_j(z)\\&+\sum_{j=0}^{k-r}\binom{k}{r+j}B_{k-r-j}^{(-r)}B_{r+j}^{(r)}\,\mathrm{Li}_{-j}(z)\end{aligned}$$

for $k=0,1,\ldots$. Here the implied O-constant depends at most on r, z and K.

We fourthly discuss applications of Theorem 2; it first yields, upon taking $f(z)=\varPhi(s,x,z)$, the following theorem.

Theorem 5. *Let $x\geq 0$ and $y\geq 0$ be real, and s any complex number. Then for any integer $K\geq 0$ the asymptotic expansion*

$$\begin{aligned}q^{xy}\Bigl(\frac{1-q}{t}\Bigr)^r(\mathcal{D}_{q,z}^x)^r\varPhi(s,x,q^y z)=&\sum_{k=0}^{K-1}\frac{(-1)^k\varPhi(s-r-k,x,z)B_k^{(-r)}(y)}{k!}t^k\\&+O(t^K)\end{aligned}$$

holds as $t\to 0^+$, in $|\arg(1-z)|<\pi$, where the implied O-constant depends at most on r, s, x, y, z and K.

We can in fact show

$$\mathrm{Li}_{q,-r}(z)=z(\mathcal{D}_{q,z}^1)^r\varPhi(0,1,z)\tag{3.12}$$

for any integer $r\geq 0$. Corollary 2.1 can therefore be applied by taking $f(z)=\varPhi(0,1,z)$ to yield the following corollary.

Corollary 5.1. *Let $r\in\mathbb{Z}$ be fixed with $r\geq 1$. Then for any integer $K\geq 0$ the asymptotic expansion*

$$\mathrm{Li}_{q,-r}(z)=\sum_{k=0}^{K-1}\frac{(-1)^k C_{f,k}^{(-r)}(1,z)}{k!}t^k+O(t^K)$$

holds as $t\to 0^+$, in $|\arg(1-z)|<\pi$, where the coefficients $C_{f,k}^{(-r)}$ are given by

$$C_{f,k}^{(-r)}(1,z)=\sum_{j=0}^{k}\binom{k}{j}B_{k-j}^{(r)}B_j^{(-r)}\,\mathrm{Li}_{-r-j}(z)$$

for $k=0,1,\ldots$. Here the implied O-constant depends at most on r, z and K.

We finally present applications of Theorems 1 and 2 to the ordinary Lerch zeta-function $\phi(s,x,\lambda)=\Phi(s,x,e(\lambda))$ with any $\lambda\in\mathbb{R}$. Let $a>0$ be any fixed real number. Then the domain of the parameter z in $\phi(s,a+z,\lambda)$ can be extended, by the same argument as in [Ka1], to the whole sector $|\arg(a+z)|<\pi$, which is again star-shaped; Theorems 1 and 2 can therefore be applied with $f(z)=\phi(s,a+z,\lambda)$ to yield the following theorem.

THEOREM 6. *Set $q=e^{-t}$ with any $t>0$.*

(i) *Let $x>0$ and $0\le y\le r$. Then $q^{xy}(1-q)^{-r}(\mathcal{I}^x_{q,z})^r\phi(s,a+q^yz,\lambda)$ has the asymptotic expansion* (2.4) *with* (2.5) *when $t\to 0^+$, and for its remainder term the explicit formula* (2.6) *holds, both in the sector $|\arg(a+z)|<\pi$, where the coefficients $A_{f,k}(x,z)$ $(k\in\mathbb{Z})$ are given by* (2.1) *and* (2.3).

(ii) *Let $x\ge 0$ and $y\ge 0$. Then $q^{xy}\{(1-q)/t\}^r(\mathcal{D}^x_{q,z})^r\phi(s,a+q^yz,\lambda)$ has the asymptotic expansion* (2.9) *with* (2.10) *when $t\to 0^+$, and for its remainder term the explicit formula* (2.11) *holds, both in the sector $|\arg(a+z)|<\pi$, where the coefficients $A_{f,r+k}(x,z)$ $(k\ge 0)$ are given by* (2.1).

REMARK. The q-integral $\mathcal{I}^1_{q,z}\zeta(s,1\pm z)$ has recently been studied in an extensive manner by Kurokawa–Mimachi–Wakayama [KMW], in connection with various evaluations of infinite series involving the values of zeta-functions; a glimpse of this aspect can be seen in formula (3.16) below.

It is in fact possible to express the coefficients of the asymptotic series above in terms of the generalized hypergeometric functions defined by

$$(3.13)\qquad {}_{m+1}F_m\left(\begin{matrix}\alpha_1,\dots,\alpha_{m+1}\\ \beta_1,\dots,\beta_m\end{matrix};z\right)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_{m+1})_n}{(\beta_1)_n\cdots(\beta_m)_n n!}z^n\qquad(|z|<1)$$

for $m=0,1,\dots$ (cf. [Sl, p. 40, Chap. 2, 2.1.1(2.1.1.1)]). We can then prove the following corollary.

COROLLARY 6.1. *The following expressions hold for the coefficients in* (2.4) *and* (2.9) *if $f(z)=\phi(s,a+z,\lambda)$:*

(i) *In the region $\sigma>1$ and in the sector $|\arg(a+z)|<\pi$,*

$$(3.14)\qquad A_{f,-j}(x,z)=x^{-j}\sum_{l=0}^{\infty}e(\lambda l)(a+l)^{-s}{}_{j+1}F_j\left(\begin{matrix}s,x,\dots,x\\ x+1,\dots,x+1\end{matrix};-\frac{z}{a+l}\right)$$

for $j=1,2,\dots,$ and

$$(3.15)\qquad A_{f,k}(x,z)=x^{k}\sum_{l=0}^{\infty}e(\lambda l)(a+l)^{-s}{}_{k+1}F_k\left(\begin{matrix}s,x+1,\dots,x+1\\ x,\dots,x\end{matrix};-\frac{z}{a+l}\right)$$

for $k=0,1,\dots$.

(ii) *For any $s\in\mathbb{C}$ and any z in the disk $|z|<a$,*

$$A_{f,k}(x,z)=\sum_{n=0}^{\infty}\frac{(-1)^n(s)_n}{n!(x+n)^k}\phi(s+n,a,\lambda)z^n \qquad (k\in\mathbb{Z}). \tag{3.16}$$

4. A Mellin transform formula for $(\mathcal{I}^x_{q,z})^r f(q^y z)$. The aim of this section is to deduce the Mellin transform formula (4.8) below, which plays a crucial rôle in establishing the asymptotic expansion (2.4) with the estimate (2.5).

Let at first $y>0$. We will use the particular case of Barnes' (multiple) zeta-function $\zeta_r(s,y)$, defined by

$$\zeta_r(s,y)=\sum_{n=0}^{\infty}\frac{(r)_n}{n!}(y+n)^{-s} \qquad (\operatorname{Re} s>r), \tag{4.1}$$

and its meromorphic continuation over the whole s-plane (see [Ka2, Theorem 3]). Suppose temporarily that $|z|<\rho_f$, where the power series expansion

$$f(z)=\sum_{m=0}^{\infty}f_m z^m \tag{4.2}$$

is valid. The proof begins with the observation that (1.1) and (1.5) yield

$$\begin{aligned}
&\frac{q^{xy}}{(1-q)^r}(\mathcal{I}^x_{q,z})^r f(q^y z) \qquad (4.3)\\
&=q^{xy}\sum_{n_1,\dots,n_r=0}^{\infty} f(q^{y+n_1+\dots+n_r}z)(q^{n_1})^{x-1}q^{n_1}\cdots(q^{n_r})^{x-1}q^{n_r}\\
&=\sum_{n_1,\dots,n_r=0}^{\infty} f(q^{y+n_1+\dots+n_r}z)q^{x(y+n_1+\dots+n_r)}=\sum_{n=0}^{\infty}\frac{(r)_n}{n!}f(q^{y+n}z)q^{x(y+n)},
\end{aligned}$$

upon noting that the number of r-tuples $(n_1,\dots,n_r)$ of non-negative integers with $\sum_{j=1}^r n_j=n$ is equal to $\binom{n+r-1}{r-1}=(r)_n/n!$. The last expression in (4.3) is modified by applying Cauchy's formula into

$$\sum_{n=0}^{\infty}\frac{(r)_n}{n!}\frac{1}{2\pi i}\oint_{|w|=\rho}\frac{q^{x(y+n)}f(w)}{w-q^{y+n}z}\,dw=\frac{1}{2\pi i}\oint_{|w|=\rho}f(w)K_r\left(x,y;q,\frac{z}{w}\right)\frac{dw}{w}, \tag{4.4}$$

say, where the radius ρ satisfies $|z|<|w|=\rho<\rho_f$, and K_r is given by

$$K_r(x,y;q,Z)=\sum_{n=0}^{\infty}\frac{(r)_n}{n!}\frac{q^{x(y+n)}}{1-q^{y+n}Z} \tag{4.5}$$

for $|Z|<1$. Then the right side of (4.5) can be transformed by applying the expression

$$\frac{q^{xy}}{1-q^y Z} = \frac{1}{2\pi i}\int_{(c)} \Gamma(s)\Phi(s,x,Z)(yt)^{-s}\,ds, \tag{4.6}$$

where c is a constant satisfying $c > r$, providing for the location of the (possible) poles (see (4.8) and (5.3)), and (c) denotes the vertical straight line from $c-i\infty$ to $c+i\infty$; this is obtained from the Mellin inversion formula

$$q^{xy} = \frac{1}{2\pi i}\int_{(c)} \Gamma(s)(xyt)^{-s}\,ds$$

for $|\arg t| < \pi/2$ (cf. [Er2, p. 347, 7.3(1)], [Ka3, (6.2)]), where both sides with $x+m$ instead of x are to be multiplied by Z^m and summed up over $m = 0, 1, \ldots$. The expression (4.6) with y replaced by $y+n$ is substituted into each term on the right side of (4.5) to imply

$$K_r(x,y;q,Z) = \frac{1}{2\pi i}\int_{(c)} \Gamma(s)\Phi(s,x,Z)\zeta_r(s,y)t^{-s}\,ds. \tag{4.7}$$

By further substituting (4.7) into the integrand on the right side of (4.4) and then changing the order of the w- and s-integrals we find that

$$\frac{q^{xy}}{(1-q)^r}(\mathcal{I}^x_{q,z})^r f(q^y z) = \frac{1}{2\pi i}\int_{(c)} \Gamma(s)\Phi_f(s,x,z)\zeta_r(s,y)t^{-s}\,ds, \tag{4.8}$$

where the resulting w-integral is equal to the zeta-function $\Phi_f(s,x,z)$ (associated with $f(z)$) of the form

$$\Phi_f(s,x,z) = \frac{1}{2\pi i}\oint_{|w|=\rho} f(w)\Phi\left(s,x,\frac{z}{w}\right)\frac{dw}{w} = \sum_{m=0}^{\infty} f_m (x+m)^{-s} z^m \tag{4.9}$$

for any complex s and $|z| < \rho_f$ (see (4.2) and (3.2)). Here the interchange of the integrals above is justified by Fubini's theorem, because, from the known vertical order estimate $\zeta_r(s,y) = O(|\mathrm{Im}\, s|^{\nu_r(\mathrm{Re}\, s)})$ as $\mathrm{Im}\, s \to \pm\infty$ (see [Ka2, Lemma 2]), the right side of (4.7) is

$$\ll \int_{(c)} (|\mathrm{Im}\, s|+1)^A e^{-(\pi/2-|\arg t|)|\mathrm{Im}\, s|}|t|^{-c}\,|ds| \ll 1$$

for some constant $A > 0$ depending on r, x, y, c and Z, in the sector $|\arg t| < \pi/2$ and uniformly on the circle $|Z| = R$ with any $R < 1$.

5. Derivation of the asymptotic expansion for $(\mathcal{I}^x_{q,z})^r f(q^y z)$. The aim of this section is to prove Theorem 1. For this we first prepare the following.

Lemma 1. *Let $\Phi_f(s,x,z)$ be defined by (4.9), where $f(z)$ is holomorphic in a star-shaped domain D. Then $\Phi_f(s,x,z)$ continues to a holomorphic* *function of* (s, z) *over* $\mathbb{C} \times D$; *its values at non-positive integers are*

$$\Phi_f(-k, x, z) = A_{f,k}(x, z) \qquad (k = 0, 1, \ldots). \tag{5.1}$$

Proof. Multiplying by f_m both sides of

$$(x+m)^{-s} z^m = \Gamma(s)^{-1} \int_0^\infty u^{s-1} e^{-xu} (e^{-u} z)^m \, du$$

for $\operatorname{Re} s > 0$, and then summing up over $m = 0, 1, \ldots$ we see from (4.9) that

$$\Phi_f(s, x, z) = \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} e^{-xu} f(e^{-u} z) \, du$$

for $\operatorname{Re} s > 0$, provided that $|z| < \rho_f$; this is further transformed by a standard argument into

$$\Phi_f(s, x, z) = \frac{1}{\Gamma(s)\{e(s) - 1\}} \int_{\mathcal{C}} w^{s-1} e^{-xw} f(e^{-w} z) \, dw. \tag{5.2}$$

Here $\mathcal{C}$ is a contour which starts from infinity, proceeds along the real axis to a small positive δ, rounds the origin counter-clockwise, and returns to infinity along the real axis, where $\arg w$ varies from 0 to 2π round $\mathcal{C}$. Note that $e^{-w} z \in D$ for all $w \in \mathcal{C}$ and $z \in D$ if δ is chosen sufficiently small. Formula (5.2) therefore provides the analytic continuation of $\Phi_f(s, x, z)$ for all $(s, z) \in \mathbb{C} \times D$.

The evaluation of $\Phi_f(s, x, z)$ at non-positive integers proceeds from (5.2) as follows:

$$\begin{aligned}\Phi_f(-k, x, z) &= \frac{(-1)^k k!}{2\pi i} \oint_{|w|=\delta} w^{-k-1} e^{-xw} f(e^{-w} z) \, dw \\ &= (-1)^k \left(\frac{\partial}{\partial w}\right)^k e^{-xw} f(e^{-w} z) \Big|_{w=0} = A_{f,k}(x, z)\end{aligned}$$

for $k = 0, 1, \ldots,$ where the last equality follows from (2.1). ∎

We have shown in [Ka2, Theorem 3] the following properties of $\zeta_r(s, y)$ defined by (4.1).

LEMMA 2. *The zeta-function* $\zeta_r(s, y)$ *continues to a meromorphic function over the whole* s*-plane*; *its only singularities are the simple poles at* $s = j$ $(j = 1, \ldots, r)$ *with the residues*

$$\operatorname{Res}_{s=j} \zeta_r(s, y) = \frac{(-1)^{r-j} B_{r-j}^{(r)}(y)}{(j-1)!(r-j)!} \qquad (j = 1, \ldots, r), \tag{5.3}$$

while its values at non-positive integers are

$$\zeta_r(-k, y) = \frac{(-1)^r k! B_{r+k}^{(r)}(y)}{(r+k)!} \qquad (k = 0, 1, \ldots). \tag{5.4}$$

We are now ready to prove the assertion (2.4). Let $K \geq 0$ be any integer, and c_K a constant satisfying $-K < c_K < -K+1$. Then the path of integration in (4.8) can be moved to the left, from (c) to (c_K), upon passing over the poles of the integrand at $s = j$ $(j = 1, \ldots, r)$ and $s = -k$ $(k = 0, 1, \ldots, K-1)$, since the integrand for $\operatorname{Re} s \leq c$ is of order $O\{|\operatorname{Im} s|^B e^{-(\pi/2-|\arg t|)|\operatorname{Im} s|}\}$ as $\operatorname{Im} s \to \pm\infty$, with some constant $B > 0$ depending on r, x, y, z and $\operatorname{Re} s$. Collecting the residues of these poles, which are computed by using (5.1), (5.3) and (5.4), we obtain (2.4) with

$$(5.5) \qquad R^{(r)}_{f,K}(x, y; q, z) = \frac{1}{2\pi i} \int_{(c_K)} \Gamma(s)\Phi_f(s, x, z)\zeta_r(s, y)t^{-s}\, ds.$$

The error estimate (2.5) is deduced by further moving the path in (5.5) from (c_K) to (c_{K+1}); this leads to

$$(5.6) \qquad R^{(r)}_{f,K}(x, y; q, z)$$
$$= \frac{(-1)^{r+K} A_{f,K}(x, z) B^{(r)}_{r+K}(y)}{(r+K)!} t^K + \frac{1}{2\pi i} \int_{(c_{K+1})} \Gamma(s)\Phi_f(s, x, z)\zeta_r(s, y)t^{-s}\, ds$$
$$\ll |t|^K + |t|^{-c_{K+1}} \ll |t|^K$$

as $t \to 0$, since $-K-1 < c_{K+1} < -K$. The remaining case $y = 0$ of (2.5) can be shown by using the following lemma.

LEMMA 3. *There exist polynomials $p_{r,j}(y)$ in y with rational coefficients such that*

$$\zeta_r(s, y) = \sum_{j=0}^{r-1} p_{r,j}(y)\zeta(s-j, y)$$

for all real $y > 0$.

Proof. We have

$$\frac{(r)_n}{n!} = \frac{(n+1)_{r-1}}{(r-1)!} = \frac{(y+n+1-y)_{r-1}}{(r-1)!} = \sum_{j=0}^{r-1} p_{r,j}(y)(y+n)^j$$

for some $p_{r,j}(y) \in \mathbb{Q}[y]$; this with (4.1) readily implies the assertion. ■

We now let $y \to 0^+$ in (2.4). It is straightforward to see that the limiting operations are possible for all the terms in (2.4) except $R^{(r)}_{f,K}(x, y; q, z)$. On the other hand, from Lemma 3 and $\zeta(s, y) = y^{-s} + \zeta(s, 1+y)$ for any $y > 0$ it follows that $\zeta_r(s, y)$ converges (as $y \to 0^+$) to $\sum_{j=0}^{r-1} p_{r,j}(0)\zeta(s-j)$ uniformly on the line $\operatorname{Re} s = c_{K+1}$ (< 0); this further implies from (5.6) that

$\lim_{y\to+0} R^{(r)}_{f,K}(x,y;q,z)$ exists and equals

$$\frac{(-1)^{r+K}A_{f,K}(x,z)B^{(r)}_{r+K}}{(r+K)!}t^K+\sum_{j=0}^{r-1}\frac{p_{r,j}(0)}{2\pi i}\int_{(c_{K+1})}\Gamma(s)\Phi_f(s,x,z)\zeta(s-j)t^{-s}\,ds,$$

which is again $\ll t^K+t^{-c_{K+1}}\ll t^K$. The case $y=0$ of (2.5) thus remains valid.

We next proceed to deduce the explicit formula (2.6). To this end we first substitute the series representation in (4.9) into the right side of (5.5), and then integrate term-by-term, to obtain

$$R^{(r)}_{f,K}(x,y;q,z)=\sum_{m=0}^{\infty}f_m z^m I_K((x+m)t), \tag{5.7}$$

where

$$I_K(T)=\frac{1}{2\pi i}\int_{(c_K)}\Gamma(s)\zeta_r(s,y)T^{-s}\,ds \tag{5.8}$$

for $|\arg T|<\pi/2$. We have shown in [Ka2, Theorem 4] the functional equation

$$\zeta_r(s,y)=\sum_{l=0}^{r-1}\frac{(-1)^{r-l-1}B^{(r)}_{r-l-1}(y)}{l!(r-l-1)!}\frac{\Gamma(1-s+l)}{(2\pi)^{1-s+l}}\{e^{-\pi i(1-s+l)/2}\zeta_y(1-s+l)$$
$$+e^{\pi i(1-s+l)/2}\zeta_{-y}(1-s+l)\}$$

for $0<y\le r$; this is substituted into the integrand in (5.8) to give, upon term-by-term integration,

(5.9)
$$I_K(T)=\sum_{l=0}^{r-1}\frac{(-1)^{r-l-1}B^{(r)}_{r-l-1}(y)}{l!(r-l-1)!}T^{-l-1}\Big\{\sum_{n=1}^{\infty}e(ny)\varphi_{K,l}(T/2\pi ne^{\pi i/2})$$
$$+\sum_{n=1}^{\infty}e(-ny)\varphi_{K,l}(T/2\pi ne^{-\pi i/2})\Big\}$$

with $\arg(e^{\pm\pi i/2})=\pm\pi/2$, where

$$\varphi_{K,l}(Z)=\frac{1}{2\pi i}\int_{(c_K)}\Gamma(s)\Gamma(1+l-s)Z^{1+l-s}\,ds \tag{5.10}$$

for $|\arg Z|<\pi$. We shall prove the following.

LEMMA 4. *For any $K\ge1$ and $l\ge0$ we have the equality*

$$\varphi_{K,l}(Z)=(-1)^K Z^{K+l+1}\left(\frac{\partial}{\partial u}\right)^l\frac{u^{K+l}}{1+uZ}\bigg|_{u=1}. \tag{5.11}$$

Proof. It follows from (5.10) that

$$\varphi_{K,l}(Z) = \frac{(-1)^K}{2\pi i} \int_{(d)} \Gamma(w)\Gamma(1-w)(1+K-w)_l Z^{1+K+l-w}\, dw, \tag{5.12}$$

by changing the variable $s = w - K$, setting $c_K = d - K$ (with $0 < d < 1$), and noting that $\Gamma(s+l) = (s)_l \Gamma(s)$. The expression $(1+K-w)_l = (\partial/\partial u)^l u^{K+l-w}|_{u=1}$ is then inserted into the integrand in (5.12), and the order of differentiation and integration is interchanged. We then evaluate the resulting w-integral by the Mellin inversion formula for $(1+Z)^{-1}$ (cf. [Er2, p. 346, 7.2(18)]) to find the assertion (5.11). ■

The expression (5.11) is substituted into each term of the infinite series in (5.9) to show that the two infinite sums in the curly brackets on the right side of (5.9) sum up to

$$(-T)^K \left(\frac{\partial}{\partial u}\right)^l u^{K+l} \sum_{n=-\infty}^{\infty}{}' \frac{e(ny)}{(2\pi i n)^{K+l}(2\pi i n + Tu)}\bigg|_{u=1}. \tag{5.13}$$

We further substitute (5.9), incorporating (5.13) with $(2\pi i n + Tu)^{-1} = \int_0^1 \xi^{2\pi i n + Tu - 1}\, d\xi$, into each term on the right side of (5.7) to obtain the assertion (2.6) by changing the order of the m- and n-sums, upon noting (6.1) below. Theorem 1 is thus proved.

6. Derivation of the asymptotic expansion for $(\mathcal{D}^x_{q,z})^r f(q^y z)$. The aim of this section is to prove Theorem 2. We first deduce the Mellin transform formula (6.7) below, which plays a crucial rôle in establishing the asymptotic expansion (2.9) with the estimate (2.10). We set $f_x^{\langle k\rangle}(z) = (x+\vartheta_z)^k f(z)$ $(k = 0, 1, \ldots)$, which from (4.2) and (8.2) below gives

$$f_x^{\langle k\rangle}(z) = \sum_{m=0}^{\infty} f_m (x+m)^k z^m \tag{6.1}$$

in particular for $|z| < \rho_f$, and first show the following relation.

LEMMA 5. *For any integer $r \ge 1$, any real $x, y \ge 0$, and any $z \in D$ we have the formula*

$$\begin{aligned} q^{xy}\left(\frac{1-q}{t}\right)^r &(\mathcal{D}^x_{q,z})^r f(q^y z) \\ &= \int_0^1 \cdots \int_0^1 f_x^{\langle r\rangle}(q^{y+\tau_1+\cdots+\tau_r} z) q^{x(y+\tau_1+\cdots+\tau_r)}\, d\tau_1 \cdots d\tau_r. \end{aligned} \tag{6.2}$$

Proof. The case $r = 1$ of (6.2) can be verified by partial integration from (1.6) and the fact $\vartheta_z f(q^{y+\tau} z) = (\partial/\partial\tau) f(q^{y+\tau} z)/(-t)$; the general case follows by induction on r. ■

The multiple integral on the right side is further modified by Cauchy's formula into

$$\int_0^1\cdots\int_0^1\frac{1}{2\pi i}\oint_{|w|=\rho}\frac{f_x^{\langle r\rangle}(w)q^{x(y+\tau_1+\cdots+\tau_r)}}{w-q^{y+\tau_1+\cdots+\tau_r}z}\,dw\,d\tau_1\cdots d\tau_r = \frac{1}{2\pi i}\oint_{|w|=\rho}f_x^{\langle r\rangle}(w)K_{-r}\left(x,y;q,\frac{z}{w}\right)\frac{dw}{w}, \tag{6.3}$$

say, where the radius ρ again satisfies $|z|<|w|=\rho<\rho_f$, and K_{-r} is given by

$$K_{-r}(x,y;q,Z)=\int_0^1\cdots\int_0^1\frac{q^{x(y+\tau_1+\cdots+\tau_r)}}{1-q^{y+\tau_1+\cdots+\tau_r}Z}\,d\tau_1\cdots d\tau_r \tag{6.4}$$

for $|Z|<1$. Suppose temporarily that $c>r$, providing for the location of the (possible) poles (see (6.7) and (6.8) below). Then the right side of (6.4) is further transformed, similarly to the derivation of (4.7), into

$$\int_0^1\cdots\int_0^1\frac{1}{2\pi i}\int_{(c)}\Gamma(s)\Phi(s,x,Z)\{(y+\tau_1+\cdots+\tau_r)t\}^{-s}\,ds\,d\tau_1\cdots d\tau_r = \frac{1}{2\pi i}\int_{(c)}\Gamma(s-r)\Phi(s,x,Z)\zeta_{-r}(s-r,y)t^{-s}\,ds, \tag{6.5}$$

where the change of the order of integration (using Fubini's theorem) is justified by absolute convergence, and the resulting inner iterated τ_j-integrals ($j=1,\ldots,r$) are evaluated by the following relation.

LEMMA 6. *For any integer $r\geq 1$, and any complex s except $s=j$ ($j=1,\ldots,r$), we have the relation*

$$\int_0^1\cdots\int_0^1(y+\tau_1+\cdots+\tau_r)^{-s}\,d\tau_1\cdots d\tau_r=\frac{\Gamma(s-r)}{\Gamma(s)}\zeta_{-r}(s-r,y), \tag{6.6}$$

where $\zeta_{-r}(s,y)$ is defined by replacing r with $-r$ in (4.1).

Proof. Suppose temporarily that $\operatorname{Re}s>0$. Then integrating both sides of

$$(y+\tau_1+\cdots+\tau_r)^{-s}=\frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}e^{-(y+\tau_1+\cdots+\tau_r)u}\,du$$

with respect to τ_j ($j=1,\ldots,r$), we see that the left side of (6.6) equals

$$\frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}e^{-yu}\left(\frac{1-e^{-u}}{u}\right)^r du,$$

which can be evaluated by (6.9) below, and the assertion hence follows by analytic continuation. ∎

We now substitute the last expression of (6.5) (see (6.4)) into the integrand on the right side of (6.3), change the order of integration (by Fubini's theorem), and note that $\Phi_{f_x^{\langle r\rangle}}(s,x,z)=\Phi_f(s-r,x,z)$ (see (4.9) and (6.1)), to infer from (6.2) that

$$(6.7)\qquad q^{xy}\left(\frac{1-q}{t}\right)^r(\mathcal{D}^x_{q,z})^r f(q^y z) = \frac{1}{2\pi i}\int_{(c)}\Gamma(s-r)\Phi_f(s-r,x,z)\zeta_{-r}(s-r,y)t^{-s}\,ds,$$

which is a key to deduce (2.9). For this we first prepare the following.

Lemma 7. *Let $r\ge 1$ be any integer. Then the zeta-function $\zeta_{-r}(s,y)$ is entire over the s-plane; its values at non-positive integers are*

$$(6.8)\qquad \zeta_{-r}(-k,y)=\begin{cases}0 & \text{if } k=0,1,\ldots,r-1,\\ \dfrac{(-1)^r k! B^{(-r)}_{k-r}(y)}{(k-r)!} & \text{if } k=r,r+1,\ldots.\end{cases}$$

Proof. We can transform the defining series in (4.1), similarly to the proof of [Ka2, Theorem 3], into

$$(6.9)\qquad \zeta_{-r}(s,y)=\frac{1}{\Gamma(s)\{e(s)-1)\}}\int_{\mathcal{C}} w^{s-1}e^{-yw}(1-e^{-w})^r\,dw,$$

where $\mathcal{C}$ is the same contour as in the proof of Lemma 1; this implies

$$\zeta_{-r}(-k,y)=\frac{(-1)^k k!}{2\pi i}\oint_{|w|=\delta} e^{-yw}w^{r-k-1}\left(\frac{w}{1-e^{-w}}\right)^{-r}dw \qquad (k=0,1,\ldots),$$

which readily yields (6.8) in view of (2.2). ∎

Let $K\ge 0$ be any integer, and c_K a constant satisfying $-K<c_K<-K+1$. We can then move the path of integration in (6.7) to the left, from (c) to (c_K), where the residues of the relevant (possible) poles at $s=h$ ($h=1,\ldots,r-1$) and $s=-k$ ($k=0,1,\ldots,K-1$) are computed by using (5.1) and (6.8); this yields (2.9) with

$$(6.10)\qquad R^{(-r)}_{f,K}(x,y;q,z)=\frac{1}{2\pi i}\int_{(c_K)}\Gamma(s-r)\Phi_f(s-r,x,z)\zeta_{-r}(s-r,y)t^{-s}\,ds,$$

which further leads to the bound in (2.10), similarly to (5.6), by moving the path of integration in (6.10) from (c_K) to (c_{K+1}). The case $y=0$ of (2.10) again remains valid by an argument using Lemma 3, similar to that for $R^{(r)}_{f,K}(x,y;q,z)$.

We next proceed to deduce the explicit formula (2.11). To this end we substitute the series representation in (4.9) into the integrand in (6.10), and integrate term-by-term, to find

$$R_{f,K}^{(-r)}(x,y;q,z)=\sum_{m=0}^{\infty}f_m(x+m)^r z^m J_K((x+m)t), \tag{6.11}$$

where

$$\begin{aligned} J_K(T)&=\frac{1}{2\pi i}\int\limits_{(c_K)}\Gamma(s-r)\zeta_{-r}(s-r,y)T^{-s}\,ds \\ &=\sum_{n=0}^{r}\frac{(-r)_n}{n!}(y+n)^r\psi_K((y+n)T) \end{aligned} \tag{6.12}$$

with

$$\psi_K(Z)=\frac{1}{2\pi i}\int\limits_{(c_K)}\Gamma(s-r)Z^{-s}\,ds \tag{6.13}$$

for $|\arg Z|<\pi/2$. Here the latter equality in (6.12) is obtained by substituting the series representation in (4.1) with (r,s) replaced by $(-r,s-r)$, where the series terminates at $n=r$ since $(-r)_n=0$ for $n>r$. We shall prove the following.

Lemma 8. *For any $K\geq 1$ we have the equality*

$$\psi_K(Z)=\frac{(-1)^{r+K}Z^K}{\Gamma(r+K)}\int_0^1(1-\xi)^{r+K-1}e^{-\xi Z}\,d\xi. \tag{6.14}$$

Proof. To remove the poles at $s=r-k$ $(k=0,1,\ldots,r+K-1)$ of the integrand in (6.13), we substitute the relation

$$\begin{aligned} \Gamma(s-r)&=\frac{(-1)^{r+K}\Gamma(s+K)}{(-s+r)(-s+r-1)\cdots(-s-K+1)} \\ &=\frac{(-1)^{r+K}\Gamma(s+K)}{\Gamma(r+K)}\int_0^1\xi^{-s-K}(1-\xi)^{r+K-1}\,d\xi, \end{aligned}$$

which is valid on the line $\operatorname{Re}s=c_K$ $(<-K+1\leq 0)$. The order of the s- and ξ-integrals is interchanged to yield (6.14), since the resulting s-integral is evaluated as

$$\frac{1}{2\pi i}\int\limits_{(c_K)}\Gamma(s+K)(\xi Z)^{-s}\,ds=\frac{1}{2\pi i}\int\limits_{(d)}\Gamma(w)(\xi Z)^{K-w}\,dw=(\xi Z)^K e^{-\xi Z},$$

where $d=c_K+K$ (>0), by the Mellin inversion formula for e^{-Z} (cf. [Er2, p. 312, 6.3(1)]). ∎

We therefore substitute the last expression in (6.12), incorporating (6.14), into each term on the right side of (6.11), and then interchange the order of the m- and n-sums to obtain the assertion (2.11) in view of (6.1). Theorem 2 is thus proved.

7. Proofs of Corollaries 1.1, 2.1 and Theorem 3. Before starting the proofs we prepare the following lemma.

LEMMA 9. *For any integers $j \geq 1$ and $k \geq 0$, and all $z \in D$,*

$$A_{f,-j}(x,z) = z^{-x} \int_0^z w_j^{-1} \int_0^{w_j} w_{j-1}^{-1} \cdots w_2^{-1} \int_0^{w_2} w_1^{x-1} f(w_1)\, dw_1 \cdots dw_j \tag{7.1}$$

and

$$A_{f,k}(x,z) = z^{-x} \vartheta_z^k \{z^x f(z)\} = (x + \vartheta_z)^k f(z). \tag{7.2}$$

Proof. Changing the variable $u_i = w_i/z$ $(i = 1, \ldots, j)$ in (2.3), we readily obtain (7.1). Next the definition (2.1) gives

$$A_{f,k}(x,z) = \partial_w^k e^{xw} f(e^w z)|_{w=0} = \vartheta_\tau^k \tau^x f(\tau z)|_{\tau=1} = z^{-x} \vartheta_\tau^k (\tau z)^x f(\tau z)|_{\tau=1}$$

for any integer $k \geq 0$ and all $z \in D$, where the substitution $e^w = \tau$ and the operator change $\partial_w = (\partial\tau/\partial w)\partial_\tau = \vartheta_\tau$ are made. The former equality in (7.2) hence follows from the fact that $\vartheta_w \varphi(wz) = wz\varphi'(wz) = \vartheta_{wz}\varphi(wz) = \vartheta_z \varphi(wz)$ for any differentiable function $\varphi(z)$, while the latter from the repeated use of the operator identity $\vartheta_z z^x = z^x(x + \vartheta_z)$. ∎

Proof of Corollary 1.1. It follows from the case $(y, w) = (0, -t)$ of (2.2) with $-r$ instead of r that for any integer $H \geq 0$ the asymptotic expansion

$$\left(\frac{1-q}{t}\right)^r = \sum_{h=0}^{H-1} \frac{(-1)^h B_h^{(-r)}}{h!} t^h + O(|t|^H) \tag{7.3}$$

holds as $t \to 0$ through the sector $|\arg t| \leq \pi - \delta$ with any small $\delta > 0$. We hence multiply the asymptotic series above (times t^r) by the right side of (2.4) with $y = 0$; the resulting coefficient of t^k $(k = 0, 1, \ldots)$, coming from the product with the first j-sum, is

$$\frac{(-1)^k}{k!} \sum_{l=0}^{\min(k,r-1)} \binom{k}{l} B_{k-l}^{(-r)} B_l^{(r)} A_{f,l-r}(x,z), \tag{7.4}$$

while that of t^{r+m} $(m = 0, 1, \ldots)$, from the product with the second k-sum, is

$$\frac{(-1)^{r+m}}{(r+m)!} \sum_{j=0}^{m} \binom{r+m}{r+j} B_{m-j}^{(-r)} B_{r+j}^{(r)} A_{f,j}(x,z). \tag{7.5}$$

Rewriting the summation indices as $l = r - j$ in (7.4) and $m = k - r$ in (7.5) respectively, and adding the resulting expressions, we obtain the assertion (2.7) with (2.8) in view of (7.1) and (7.2), upon noting that the left side of (2.7) is equal to $z^x(\mathcal{I}^x_{q,z})^r f(z)$. The particular case $r=1$ of (2.8) is derived by incorporating

$$B^{(-1)}_k = \frac{1}{k+1} \quad \text{and} \quad \binom{k}{h}\frac{1}{h+1} = \frac{1}{k+1}\binom{k+1}{h+1} \tag{7.6}$$

for any integers h and k with $0 \le h \le k$. ∎

Proof of Corollary 2.1. We multiply the asymptotic series in (7.3) with $-r$ instead of r by the right side of (2.9) with $y=0$; the resulting coefficient of t^k $(k=0,1,\ldots)$ is equal to

$$\frac{(-1)^k}{k!}\sum_{j=0}^{k}\binom{k}{j}B^{(r)}_{k-j}B^{(-r)}_j A_{f,r+j}(x,z),$$

which readily implies the assertion (2.12) with (2.13) in view of (7.2), upon noting that the left side of (2.12) is equal to $z^x(\mathcal{D}^x_{q,z})^r f(z)$. The particular case $r=1$ of (2.13) is derived by incorporating (7.6). ∎

Proof of Theorem 3. Consider first (i). We observe that the left side in (2.4) is holomorphic for all $z\in D$ by (4.3), and that the same is true for the first and second sums on the right side in view of (2.1) and (2.3). Further, one more path move from (c_{K+1}) to (c_{K+2}) in (5.6) implies that

$$R^{(r)}_{f,K}(x,y;q,z) = \sum_{k=K}^{K+1}\frac{(-1)^{r+k}A_{f,k}(x,z)B^{(r)}_{r+k}(y)}{(r+k)!}t^k + R^{(r)}_{f,K+2}(x,y;q,z)$$

at first for $|z|<\rho_f$. Hence if the ξ-integral (differentiated with respect to u) in (2.6) with $K+2$ instead of K is $\ll 1$ for any $z\in D$ with the implied $\ll$-constant independent of t, then (2.4)–(2.6) remain valid throughout D; this can be seen as follows. The operator identities

$$\partial^l_u u^{K+l} = \sum_{j=0}^{l} P_{K,l,j}(u)\partial^j_u \quad \text{and} \quad (x+\vartheta_z)^K = \sum_{k=0}^{K} Q_{K,k}(x,z)\partial^k_z \tag{7.7}$$

hold with some $P_{K,l,j}(u)\in\mathbb{Z}[u]$ and $Q_{K,k}(x,z)\in\mathbb{Z}[x,z]$, and the right sides above are applied to the ξ-integral in (2.6), showing that the resulting sum is a collection of terms of the form $\int_0^1 \xi^{axtu}(\log\xi)^b f^{(c)}(\xi^{tu}z)\,d\xi$ with the coefficients being some polynomials in t, u and x; the operations in (7.7) therefore do not decrease the order in t.

Consider next (ii). We observe from (6.2), (2.1) and (2.11) that (2.9) holds for all $z\in D$ by analytic continuation. Moreover the ξ-integral on the right side of (2.11) is $\ll 1$ for all $z\in D$ with the implied $\ll$-constant independent of t, since the latter operation in (7.7), applied to the ξ-integral, again does not decrease the order in t; this therefore shows that (2.9)–(2.11) remain valid throughout D.

The arguments for cases (i) and (ii) above imply the assertion of (iii). ∎

8. Proofs of Theorems 4–6 and their corollaries

Proofs of Theorem 4 and Corollaries 4.1–4.4. Theorem 4 readily follows from Theorem 3(i) (on (2.4) with (2.5)) by setting $f(z) = \Phi(s,x,z)$, and noting that

$$\int_0^1 u_j^{x-1} \cdots \int_0^1 u_1^{x-1}(u_1 \cdots u_j z)^m \, du_1 \cdots du_j = (x+m)^{-j} z^m, \tag{8.1}$$

$$(x+\vartheta_z)^k z^m = (x+m)^k z^m \tag{8.2}$$

for any integers $j \geq 1$ and $k, m \geq 0$. Corollary 4.1 is then from (3.5) just the particular case $(r,s,x) = (1,1,1)$ of Theorem 4. Next a straightforward application of (3.6) with $y = 0$ upon replacing z by $(1-q)z$ at first gives for any integer $K \geq 0$ that

$$\log e_q(z) = \sum_{k=-1}^{K-1} \frac{(-1)^{k+1} \operatorname{Li}_{1-k}((1-q)z) B_{k+1}}{(k+1)!} t^k + O(t^K) \tag{8.3}$$

as $t \to 0^+$. We substitute the asymptotic series in (7.3) (times t^r) into each term of the power series expansion of $\operatorname{Li}_l((1-q)z)$; this shows for any integer $J \geq 0$ that

$$\operatorname{Li}_l((1-q)z) = \sum_{j=0}^{J-1} \left\{ \sum_{h=0}^{j} (1+h)^{-l} \frac{B_{j-h}^{(-h-1)} z^{1+h}}{(j-h)!} \right\} t^{j+1} + O(t^{J+1}),$$

which is further substituted into each term of the k-sum in (8.3) to yield the assertion (3.8) with (3.9). To exponentiate the asymptotic series in (3.8) we appeal to the following lemma, which is obtained by manipulation of formal power series.

LEMMA 10. *Let $\varphi(t) = \sum_{j=0}^{\infty} c_j t^j \in \mathbb{C}[[t]]$, and define $\psi(t) = e^{\varphi(t)}$, where e^T is defined by $\sum_{k=0}^{\infty} T^k/k! \in \mathbb{C}[[T]]$. Then the formal power series expansion of $\psi(t)$ is given by $\psi(t) = \sum_{k=0}^{\infty} d_k t^k$ with*

$$d_k = e^{c_0} \sum_{\substack{\sum_{j=1}^k j l_j = k \\ l_j \geq 0 \, (j=1,\ldots,k)}} \prod_{j=1}^{k} \frac{c_j^{l_j}}{l_j!} \quad (k = 0,1,\ldots).$$

Lemma 10 applies to (3.8) with (3.9), yielding the remaining assertions of Corollary 4.2. Corollary 4.3 can be shown similarly, by using (3.6), the latter equality in (3.7) and Lemma 10. Lastly, Theorem 3(iii) (on (2.7) with (2.8)) yields Corollary 4.4 by taking $f(z) = \Phi(0,1,z)$, in view of (3.11). ■

Proofs of Theorem 5 and Corollary 5.1. Theorem 5 is deduced from Theorem 3(ii) (on (2.9) with (2.10)) by taking $f(z) = \Phi(s,x,z)$, in view of (8.2). Corollary 5.1 then readily follows from Theorem 3(iii) (on (2.12) with (2.13)), in view of (3.12). ■

Proofs of Theorem 6 and Corollary 6.1. Theorem 6 follows by direct application of Theorem 3(i)&(ii) to $f(z) = \phi(s,a+z,\lambda)$. We next prove Corollary 6.1. The series representations

$$\phi(s,a+z,\lambda) = \sum_{l=0}^{\infty} e(\lambda l)(a+l+z)^{-s} = \sum_{l=0}^{\infty} e(\lambda l)(a+l)^{-s}\left(1+\frac{z}{a+l}\right)^{-s} \tag{8.4}$$

hold for $\operatorname{Re} s > 1$ from (3.2); the last expression is substituted into the integrand in (2.3), and then the order of the sum and the integrals is inverted to imply (3.14), where the resulting multiple $(u_1,\ldots,u_j)$-integral is evaluated by Euler's formula for the generalized hypergeometric function (cf. [Sl, p. 108, Chap. 4, 4.1(4.1.3)]). The last expression in (8.4) is again substituted into the rightmost side of (7.2); the term-by-term operation in each term gives

$$(x+\vartheta_z)^k\left(1+\frac{z}{a+l}\right)^{-s} = \sum_{n=0}^{\infty} \frac{(s)_n(x+n)^k}{n!}\left(-\frac{z}{a+l}\right)^n \qquad (|z|<a),$$

which is evaluated by (3.13) to yield (3.15). Formulae (3.14) and (3.15) both hold in the sector $|\arg(a+z)| < \pi$ by analytic continuation of generalized hypergeometric functions.

The remaining (3.16) can be derived by substituting the series representation in (3.13) (with the respective sets of parameters) into (3.14) and (3.15), and by changing the order of the n- and l-sums, where each resulting term is continued to a meromorphic function over the whole s-plane by the contour integral expression in (5.2). ■

Acknowledgments. The author would like to thank the referee, whose suggestions on refinements made an earlier manuscript into the present form. The author is also indebted to Professor Masaaki Amou and Professor Ryotaro Okazaki for their valuable comments on the present research.

The present investigation was initiated during the author's academic stay at Mathematisches Institut, Westfälische Wilhelms-Universität Münster. He would like to express his sincere gratitude to Professor Christopher Deninger and the institution for their warm hospitality and constant support.

The author was also supported by Grant-in-Aid for Scientific Research (No. 16540038), The Ministry of Education, Culture, Sports, Science & Technology in Japan.

References

[Ag] R. P. Agarwal, *A basic analogue of MacRobert's E-function*, Proc. Glasgow Math. Soc. 5 (1961), 4–7.

[Be1] B. C. Berndt, *Modular transformations and generalizations of several formulae of Ramanujan*, Rocky Mountain J. Math. 7 (1977), 147–189.

[Be2] —, *Ramanujan's Notebooks*, Part II, Springer, New York, 1989.

[BS] B. C. Berndt and J. Sohn, *Asymptotic formulas for two continued fractions in Ramanujan's Lost Notebook*, J. London Math. Soc. 65 (2002), 271–284.

[BY] B. C. Berndt and A. J. Yee, *On the generalized Rogers–Ramanujan continued fraction*, Ramanujan J. 7 (2003), 321–331.

[BeKa] N. Bettaibi and J. Kamel, *Asymptotic expansions for a class of q-integral transforms*, Appl. Math. Comput. 198 (2008), 433–444.

[CO] G. H. Coogan and K. Ono, *A q-series identity and the arithmetic of Hurwitz zeta functions*, Proc. Amer. Math. Soc. 131 (2003), 719–724.

[Er1] A. Erdélyi (ed.), W. Magnus, F. Oberhettinger and F. G. Tricomi (The Bateman Manuscript Project), *Higher Transcendental Functions*, Vol. I, McGraw-Hill, New York, 1953.

[Er2] —, *Tables of Integral Transforms*, Vol. I, McGraw-Hill, New York, 1954.

[FBB] A. Fitouhi, K. Brahim and N. Bettaibi, *Asymptotic approximations in quantum calculus*, J. Nonlinear Math. Phys. 12 (2005), 586–606.

[GR] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, 2nd ed., Cambridge Univ. Press, Cambridge, 2004.

[Ja] F. H. Jackson, *On q-definite integrals*, Quart. J. Pure Appl. Math. 41 (1910), 193–203.

[Ka1] M. Katsurada, *Power series and asymptotic series associated with the Lerch zeta-function*, Proc. Japan Acad. Ser. A 74 (1998), 167–170.

[Ka2] —, *An asymptotic formula of Ramanujan for a certain theta-type series*, Acta Arith. 97 (2001), 157–172.

[Ka3] —, *Asymptotic expansions of certain q-series and a formula of Ramanujan for specific values of the Riemann zeta-function*, ibid. 107 (2003), 269–298.

[Ka4] —, *Asymptotic expansions for certain multiple q-integrals and q-differentials of Thomae–Jackson type*, in: Diophantine Analysis and Related Fields 2010, T. Komatsu (ed.), A.I.P. Press, New York, 2010, 100–113.

[KMW] N. Kurokawa, K. Mimachi and M. Wakayama, *Jackson's integral of the Hurwitz zeta function*, Rend. Circ. Mat. Palermo (2) 56 (2007), 43–56.

[Mc1] R. J. McIntosh, *Some asymptotic formula for q-hypergeometric series*, J. London Math. Soc. 51 (1995), 157–172.

[Mc2] —, *Some asymptotic formulae for q-shifted factorials*, Ramanujan J. 3 (1999), 205–214.

[Mc3] —, *Unclosed asymptotic expansions and mock theta functions*, ibid. 19 (2009), 183–189.

[Nö] N. E. Nörlund, *Vorlesungen über Differenzenrechnung*, Chelsea, New York, 1954.

[Sl] L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge Univ. Press, Cambridge, 1966.

[Th] J. Thomae, *Beiträge zur Theorie der durch die Heinesche Reihe:* $1+\frac{1-q^a}{1-q}\cdot\frac{1-q^b}{1-q^c}\cdot x+\frac{1-q^a}{1-q}\cdot\frac{1-q^{a+1}}{1-q^2}\cdot\frac{1-q^b}{1-q^c}\cdot\frac{1-q^{b+1}}{1-q^{c+1}}\cdot x^2+\cdots$ *darstellbaren Functionen*, J. Reine Angew. Math. 70 (1869), 258–281.

[Wa] G. N. Watson, *The final problem: an account of the mock theta functions*, J. London Math. Soc. 11 (1936), 55–80.

[Za] D. B. Zagier, *The dilogarithm function*, in: Frontiers in Number Theory, Physics and Geometry II, P. Cartier et al. (eds.), Springer, Berlin, 2007, 3–65.

Masanori Katsurada
Department of Mathematics
Hiyoshi Campus
Keio University
4-1-1 Hiyoshi, Kouhoku-ku
Yokohama 223-8521, Japan
E-mail: katsurad@hc.cc.keio.ac.jp
katsurad@z3.keio.jp

Received on 29.6.2010
and in revised form on 27.2.2011 (6430)