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The equation $n_1n_2 = n_3n_4$, the gcd-sum function and the mean values of certain character sums

by

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1. Introduction. For any real number $B \geq 2$, let $N(B)$ denote the number of solutions of the equation $n_1n_2 = n_3n_4$ with $1 \leq n_j \leq B$ $(j = 1,2,3,4)$. A. Ayyad, T. Cochrane and Z. Y. Zheng [1] proved

$$N(B) = \frac{12}{\pi^2}B^2\log B + CB^2 + O(B^{19/13}\log^{7/13} B), \tag{1.1}$$

where $C := 24\gamma/\pi^2 - 6/\pi^2 - 2\zeta'(2)/\zeta^2(2) - 2$, and γ is the Euler constant. Suppose $p > 2$ is a large prime number. As an application of (1.1), in [1] it is proved that the asymptotic formula

$$\frac{1}{p-1}\sum_{\chi\neq\chi_0}\Big|\sum_{n\leq B}\chi(n)\Big|^4 = \frac{12}{\pi^2}B^2\log B + \left(C - \frac{B^2}{p}\right)B^2 + O(B^{19/13}\log^{7/13} B)$$

holds for $B \leq \sqrt{p}$.

Let $d(n)$ denote the Dirichlet divisor function. The well-known Dirichlet divisor problem is to study properties of the error term $\Delta(x)$, defined by

$$\Delta(x) := \sum_{n\leq x} d(n) - x\log x - (2\gamma - 1)x \qquad (x > 0).$$

The latest upper bound of $\Delta(x)$ is due to M. N. Huxley [5], who proved that

$$\Delta(x) = O(x^{\frac{131}{416}}\log^{\frac{26947}{8320}} x). \tag{1.2}$$

S. Shi [7] connected the evaluation of $N(B)$ with $\Delta(x)$ and improved (1.1) to

$$N(B) = \frac{12}{\pi^2}B^2\log B + CB^2 + O(B^{\frac{547}{416}+\varepsilon}), \tag{1.3}$$

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which yields correspondingly

$$\frac{1}{p-1}\sum_{\chi\neq\chi_0}\Big|\sum_{n\le B}\chi(n)\Big|^4=\frac{12}{\pi^2}B^2\log B+\Big(C-\frac{B^2}{p}\Big)B^2+O(B^{\frac{547}{416}+\varepsilon}),\quad B\le\sqrt{p}.$$

In her proof, besides the Dirichlet divisor problem, Shi also used estimates of sums involving the fractional part function $\{\cdot\}$. The first aim of this paper is to give a different and much simpler proof of (1.3), which depends only on the estimation of $\Delta(x)$, not on sums involving the function $\{\cdot\}$. The new proof is connected with the so-called gcd-sum function.

Pillai [6] defined the gcd-sum (*Pillai's function*) by the relation

$$g(n):=\sum_{j=1}^{n}(j,n),$$

where (a,b) denotes the greatest common divisor of a and b. This is Sequence A018804 in Sloane's Online Encyclopedia of Integer Sequences. This function was recently studied by several authors. In particular, the asymptotic behaviour of the partial sum $G_\alpha(x):=\sum_{n\le x}g(n)n^{-\alpha}$ was studied in [2–4], where α is any fixed real number. In [8] Y. Tanigawa and W. G. Zhai proved the unified asymptotic formula

$$G_\alpha(x)=N_\alpha(x)+C(\alpha)+O(x^{\theta+1-\alpha+\varepsilon}),\tag{1.4}$$

where $\theta:=\inf\{a\mid\Delta(x)\ll x^a\}$,

$$N_\alpha(x)=\begin{cases}\dfrac{x^{2-\alpha}\log x}{(2-\alpha)\zeta(2)}+\frac{x^{2-\alpha}}{(2-\alpha)\zeta(2)}\Big(2\gamma-\dfrac{1}{2-\alpha}-\dfrac{\zeta'(2)}{\zeta(2)}\Big) & \text{if } \alpha\neq 2,\\[2mm] \dfrac{\log^2 x}{2\zeta(2)}-\Big(\dfrac{\zeta'(2)}{\zeta^2(2)}+\dfrac{2\gamma}{\zeta(2)}\Big)\log x & \text{if } \alpha=2,\end{cases}\tag{1.5}$$

$$C(\alpha)=\begin{cases}0 & \text{if } \alpha\le 1,\\ c(\alpha-1)/\zeta(\alpha) & \text{if } 1<\alpha<2,\\ \dfrac{c(1)}{\zeta(2)}+\dfrac{2\zeta'^2(2)-\zeta(2)\zeta''(2)}{2\zeta^3(2)}-\dfrac{2\gamma\zeta'(2)}{\zeta^2(2)} & \text{if } \alpha=2,\\ \dfrac{\zeta^2(\alpha-1)}{\zeta(\alpha)} & \text{if } \alpha>2,\end{cases}$$

$$c(\beta)=\begin{cases}\beta\displaystyle\int_0^\infty\Delta(x)x^{-\beta-1}\,dx & \text{if } 0<\beta<1,\\ 2\gamma-1+\displaystyle\int_1^\infty\Delta(x)x^{-2}\,dx & \text{if } \beta=1.\end{cases}$$

Define the error term $E_\alpha(x)$ by

$$E_\alpha(x):=G_\alpha(x)-N_\alpha(x)-C(\alpha).$$

In [8] Y. Tanigawa and W. G. Zhai proved the asymptotic formula

$$\int_1^T x^{2\alpha-2}E_\alpha^2(x)\,dx = \frac{D_0}{6\pi^2}T^{3/2} + O(T^{5/4+\varepsilon}), \tag{1.6}$$

where

$$D_0 := \sum_{n=1}^{\infty} h_0^2(n)n^{-3/2}, \qquad h_0(n) := \sum_{n=ml} \mu(m)d(l)m^{-1/2}. \tag{1.7}$$

The evaluation of $N(B)$ is closely related to the mean value of $g(n)$. Actually we will show in Section 2 that

$$N(B) = 4G_0(B) - 2B^2 + O(B). \tag{1.8}$$

Define

$$E(B) := N(B) - \frac{12}{\pi^2}B^2\log B - CB^2. \tag{1.9}$$

The formula (1.8) implies that

$$E(B) = 4E_0(B) + O(B).$$

So as a corollary of (1.6) we have the following theorem.

THEOREM 1.1. *The asymptotic formula*

$$\int_2^T E^2(B)\,dB = \frac{8D_0}{7\pi^2}T^{7/2} + O(T^{13/4+\varepsilon})$$

holds, where D_0 is defined by (1.7).

Furthermore, define

$$M_1(B) = \sum_{n_1\le B}\sum_{\substack{n_2\le B\\ n_1n_2=n_3n_4}}\sum_{n_3\le B}\sum_{n_4\le B} n_1n_2n_3n_4,$$

$$M_2(B) = \sum_{n_1\le B}\sum_{\substack{n_2\le B\\ n_1n_2=n_3n_4}}\sum_{n_3\le B}\sum_{n_4\le B} (-1)^{n_1+n_2+n_3+n_4}.$$

We shall prove the following theorems in Sections 3 and 5 respectively.

THEOREM 1.2. *For any real number $B \ge 2$, we have*

$$M_1(B) = \frac{4}{9\pi^2}B^6\log B + C_1B^6 + O(B^{\frac{2211}{416}+\varepsilon}),$$

where

$$C_1 = \frac{4}{9\pi^2}\left(2\gamma - \frac{1}{6} - \frac{\zeta'(2)}{\zeta(2)} + \zeta(3)\right).$$

Let $F_1(B) = M_1(B) - \frac{4}{9\pi^2}B^6\log B - C_1B^6$. *Then*

$$\int_2^T F_1^2(B)\,dB = \frac{8D_0}{1863\pi^2}T^{23/2} + O(T^{45/4+\varepsilon}).$$

THEOREM 1.3. *For any real number* $B \ge 2$, *we have*

$$M_2(B) = \frac{4}{\pi^2}B^2\log B + C_2B^2 + O(B^{\frac{547}{416}+\varepsilon}),$$

where

$$C_2 = \frac{4}{\pi^2}\biggl(\frac{2\log 2}{3} + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)}\biggr).$$

Let $F_2(B) = M_2(B) - \frac{4}{\pi^2}B^2\log B - C_2B^2$. *Then*

$$\int_2^T F_2^2(B)\,dB = \frac{2D_*}{7\pi^2}T^{7/2} + O(T^{13/4+\varepsilon}),$$

where

$$D_* := \sum_{n=1}^{\infty} h_*^2(n)n^{-3/2}, \qquad h_*(n) := \sum_{n=kl}\frac{\mu^*(k)d(l)}{k^{1/2}},$$

$$\mu^*(k) := \begin{cases} \mu(k) & \text{if } 2 \nmid k,\\ 2\mu(k) & \text{if } 2 \mid k.\end{cases}$$

REMARK 1.1. S. Shi did not study the mean value of $E(B)$, and no results on the integral $\int_2^T E^2(B)\,dB$ are known.

From Theorems 1.1–1.3 we have

COROLLARY 1.1.

$$E(B) = \Omega(B^{5/4}), \qquad F_1(B) = \Omega(B^{21/4}), \qquad F_2(B) = \Omega(B^{5/4}), \qquad B\to\infty.$$

Furthermore, by Theorems 1.2 and 1.3 we easily get the following asymptotic formulae for the mean values of certain character sums.

COROLLARY 1.2. *Let B be an integer with* $1 \le B \le \sqrt{p}$. *Then*

$$\frac{1}{p-1}\sum_{\chi\ne\chi_0}\Bigl|\sum_{n\le B} n\chi(n)\Bigr|^4 = \frac{4}{9\pi^2}B^6\log B + \biggl(C_1 - \frac{B^2}{16p}\biggr)B^6 + O(B^{\frac{2211}{416}+\varepsilon}),$$

$$\frac{1}{p-1}\sum_{\chi\ne\chi_0}\Bigl|\sum_{n\le B}(-1)^n\chi(n)\Bigr|^4 = \frac{4}{\pi^2}B^2\log B + C_2B^2 + O(B^{\frac{547}{416}+\varepsilon}).$$

Notation. For any real number t, $[t]$ denotes the greatest integer not more than t, and $\{t\} = t - [t]$. We use $d(n)$ to denote the Dirichlet divisor function, and define $\sigma_\alpha(n) = \sum_{d\mid n} d^\alpha$ so that $d(n) = \sigma_0(n)$. The Möbius and Euler functions are denoted by $\mu(n)$ and $\phi(n)$ respectively. We denote by γ the Euler constant. Let ε be a small positive constant, and let (a,b) denote the greatest common divisor of a and b.

2. A new proof of (1.3). In this section we give a new and short proof of S. Shi's result (1.3).

Suppose first that $B \geq 2$ is an integer. We begin with the formula (4) of S. Shi [7], which reads

$$N(B) = 2\sum_{n=1}^{B} \phi(n)\left[\frac{B}{n}\right]^2 - B^2 + O(B), \tag{2.1}$$

where $\phi(n)$ is the Euler function.

The main ingredient of our proof is the following elementary formula:

$$[t]^2 = 2\frac{[t]^2+[t]}{2} - [t] = 2\sum_{m\leq t} m - \sum_{m\leq t} 1,\ \ t \geq 1. \tag{2.2}$$

Inserting (2.2) into (2.1) we get

$$\begin{aligned} N(B) &= 2\sum_{n=1}^{B} \phi(n)\Big(2\sum_{m\leq B/n} m - \sum_{m\leq B/n} 1\Big) - B^2 + O(B) \\ &= 4\Sigma_1 - 2\Sigma_2 - B^2 + O(B), \end{aligned} \tag{2.3}$$

where

$$\Sigma_1 = \sum_{n=1}^{B} \phi(n) \sum_{m\leq B/n} m = \sum_{nm\leq B} \phi(n)m,$$

$$\Sigma_2 = \sum_{n=1}^{B} \phi(n) \sum_{m\leq B/n} 1 = \sum_{nm\leq B} \phi(n).$$

It is easily seen that (2.3) also holds when B is not an integer by noting that $N(B) = N([B])$. So from now on, we suppose $B \geq 2$ is any real number.

The identity $n = \sum_{n=ml} \phi(l)$ implies that

$$\Sigma_2 = \sum_{n\leq B} n = \frac{1}{2}B^2 + O(B). \tag{2.4}$$

Pillai [6] proved that

$$g(n) = n\sum_{d|n} \frac{\phi(d)}{d} = \sum_{n=lm} \phi(l)m,$$

which implies that

$$\Sigma_1 = G_0(B) = \sum_{n\leq B} g(n). \tag{2.5}$$

So from (2.3)–(2.5) we immediately get (1.8). Furthermore by (1.4), (1.5) and (1.9) we easily have

$$E(B) = 4E_0(B) + O(B). \tag{2.6}$$

LEMMA 2.1. *Let* $E_\alpha(x) = G_\alpha(x) - N_\alpha(x) - C(\alpha)$. *Then*

$$E_\alpha(x) = x^{1-\alpha} \sum_{n\le x} \frac{\mu(n)}{n} \Delta\left(\frac{x}{n}\right) + O(x^{1-\alpha}\log x). \tag{2.7}$$

Proof. This is equation (26) of [8]. ∎

From (2.6) and (2.7) we get

$$N(B) = \frac{12}{\pi^2} B^2 \log B + CB^2 + E(B), \tag{2.8}$$

where

$$E(B) = 4B \sum_{n\le B} \frac{\mu(n)}{n} \Delta\left(\frac{B}{n}\right) + O(B\log B). \tag{2.9}$$

With the help of (1.2) we deduce from (2.8) that

$$E(B) \ll B^{\frac{547}{416}} \log^{\frac{26749}{8320}} B.$$

For completeness, we give a direct proof of (2.8) in this section. The argument here is slightly better since the error term $O(B\log B)$ in (2.9) can be replaced by $O(B)$.

From the relation

$$\sum_{n=ml} \phi(l)m = \sum_{n=lm} \mu(l)md(m)$$

we get

$$\Sigma_1 = \sum_{n\le B} \mu(n) \sum_{m\le B/n} md(m).$$

For any $t \ge 1$ by partial summation we have

$$\begin{aligned} \sum_{m\le t} md(m) &= \int_1^t u\, d\Big(\sum_{m\le u} d(m)\Big) = \int_1^t u(\log u + 2\gamma)\, du + \int_1^t u\, d\Delta(u) \\ &= \frac{t^2}{2}(\log t + 2\gamma) - \frac{t^2}{4} + t\Delta(t) - \int_1^t \Delta(u)\, du + c_0 \\ &= \frac{t^2\log t}{2} + \left(\gamma - \frac{1}{4}\right)t^2 + t\Delta(t) - \frac{t}{4} + O(t^{3/4}), \end{aligned} \tag{2.10}$$

where c_0 is a constant and we used the estimate

$$\int_1^t \Delta(u)\, du = \frac{t}{4} + O(t^{3/4}),$$

which is a corollary of results of Voronoĭ [10, part II, Section IV, Chapter 39, pp. 500–501].

Inserting (2.10) into Σ_1 we get

(2.11)

$$\begin{aligned}\Sigma_1 &= \sum_{n\le B}\mu(n)\biggl(\frac{1}{2}\frac{B^2}{n^2}\log\frac{B}{n}+\Bigl(\gamma-\frac{1}{4}\Bigr)\frac{B^2}{n^2}+\frac{B}{n}\Delta\Bigl(\frac{B}{n}\Bigr)-\frac{B}{4n}+O\Bigl(\frac{B^{3/4}}{n^{3/4}}\Bigr)\biggr)\\
&= \frac{B^2\log B}{2}\sum_{n\le B}\frac{\mu(n)}{n^2}-\frac{B^2}{2}\sum_{n\le B}\frac{\mu(n)\log n}{n^2}+\Bigl(\gamma-\frac{1}{4}\Bigr)B^2\sum_{n\le B}\frac{\mu(n)}{n^2}\\
&\quad+B\sum_{n\le B}\frac{\mu(n)}{n}\Delta\Bigl(\frac{B}{n}\Bigr)-\frac{B}{4}\sum_{n\le B}\frac{\mu(n)}{n}+O(B)\\
&= \frac{3}{\pi^2}B^2\log B+\Bigl(\frac{6\gamma}{\pi^2}-\frac{6}{4\pi^2}-\frac{\zeta'(2)}{2\zeta^2(2)}\Bigr)B^2+B\sum_{n\le B}\frac{\mu(n)}{n}\Delta\Bigl(\frac{B}{n}\Bigr)+O(B),\end{aligned}$$

where we used the well-known estimate

$$\sum_{n\le u}\mu(n)\ll ue^{-c\sqrt{\log u}}\qquad(c>0)$$

and the fact that

$$\frac{6}{\pi^2}=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2}.$$

Now from (2.3), (2.4) and (2.11) we get

$$E(B)=4B\sum_{n\le B}\frac{\mu(n)}{n}\Delta\Bigl(\frac{B}{n}\Bigr)+O(B).$$

3. Proof of Theorem 1.2. First we study the links between $M_1(B)$ and Pillai's function.

LEMMA 3.1. *Define $h(n) := \sum_{j=1}^{n} n/(j,n)$. Then*

$$M_1(B)=\frac{4}{9}\sum_{n\le B}n^4g(n)+\frac{4}{9}\sum_{n\le B}n^3h(n)+O(B^5).$$

Proof. It is not hard to show that

$$\begin{aligned}M_1(B) &= \sum_{n_1\le B}\sum_{n_2\le B}\sum_{\substack{n_3\le B\\ n_1n_2=n_3n_4}}\sum_{n_4\le B}n_1n_2n_3n_4=\sum_{a\le B}\sum_{\substack{c\le B\\ (a,c)=1}}\Bigl(\sum_{n_1\le B}\sum_{\substack{n_3\le B\\ n_1/n_3=a/c}}n_1n_3\Bigr)^2\\
&= \Bigl(\sum_{n\le B}n^2\Bigr)^2+2\sum_{1<a\le B}\sum_{\substack{1\le c\le a\\ (c,a)=1}}\Bigl(\sum_{n_1\le B}\sum_{\substack{n_3\le B\\ n_1/n_3=a/c}}n_1n_3\Bigr)^2\end{aligned}$$

$$= \Big(\sum_{n\le B} n^2\Big)^2 + 2\sum_{1<a\le B}\sum_{\substack{1\le c\le a\\(c,a)=1}}\Big(\sum_{m\le B}\sum_{\substack{n_1\le B\\n_1=ma}}\sum_{\substack{n_3\le B\\n_3=mc}} n_1n_3\Big)^2$$

$$= \Big(\sum_{n\le B} n^2\Big)^2 + 2\sum_{1<a\le B} a^2 \sum_{\substack{1\le c\le a\\(c,a)=1}} c^2\Big(\sum_{m\le B/a} m^2\Big)^2.$$

Define

$$\phi_k(n) = \sum_{\substack{1\le c\le n\\(c,n)=1}} c^k;$$

we know that

$$\phi_2(n) = \frac{1}{3}n^2\phi(n) + \frac{n}{6}\prod_{p|n}(1-p) \quad \text{for } n>1.$$

Let $N=[B]$. We have

$$M_1(B) = \frac{N^2(N+1)^2(2N+1)^2}{36} + 2\sum_{1<a\le N} a^2\bigg(\frac{1}{3}a^2\phi(a) + \frac{a}{6}\prod_{p|a}(1-p)\bigg)$$
$$\times \frac{\big[\frac{N}{a}\big]^2\big(\big[\frac{N}{a}\big]+1\big)^2\big(2\big[\frac{N}{a}\big]+1\big)^2}{36}$$
$$= 2\sum_{1\le a\le N} a^2\bigg(\frac{1}{3}a^2\phi(a) + \frac{a}{6}\prod_{p|a}(1-p)\bigg)\frac{\big[\frac{N}{a}\big]^2\big(\big[\frac{N}{a}\big]+1\big)^2\big(2\big[\frac{N}{a}\big]+1\big)^2}{36}.$$

Noting that

$$\sum_{a=1}^{n} a^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2, \qquad \sum_{a=1}^{n} a^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2,$$

we get

$$\frac{\big[\frac{N}{a}\big]^2\big(\big[\frac{N}{a}\big]+1\big)^2\big(2\big[\frac{N}{a}\big]+1\big)^2}{36}$$
$$= \frac{1}{9}\Big[\frac{N}{a}\Big]^6 + \frac{1}{3}\Big[\frac{N}{a}\Big]^5 + \frac{13}{36}\Big[\frac{N}{a}\Big]^4 + \frac{1}{6}\Big[\frac{N}{a}\Big]^3 + \frac{1}{36}\Big[\frac{N}{a}\Big]^2$$
$$= \frac{2}{3}\bigg(\frac{1}{6}\Big[\frac{N}{a}\Big]^6 + \frac{1}{2}\Big[\frac{N}{a}\Big]^5 + \frac{5}{12}\Big[\frac{N}{a}\Big]^4 - \frac{1}{12}\Big[\frac{N}{a}\Big]^2\bigg)$$
$$+ \frac{1}{3}\bigg(\frac{1}{4}\Big[\frac{N}{a}\Big]^4 + \frac{1}{2}\Big[\frac{N}{a}\Big]^3 + \frac{1}{4}\Big[\frac{N}{a}\Big]^2\bigg)$$
$$= \frac{2}{3}\sum_{b\le N/a} b^5 + \frac{1}{3}\sum_{b\le N/a} b^3.$$

Therefore

$$M_1(B) = \sum_{1\le a\le N} a^2\left(\frac{2}{3}a^2\phi(a) + \frac{1}{3}a\prod_{p|a}(1-p)\right)\left(\frac{2}{3}\sum_{b\le N/a} b^5 + \frac{1}{3}\sum_{b\le N/a} b^3\right)$$
$$= \frac{4}{9}\sum_{ab\le N} a^4\phi(a)b^5 + \frac{2}{9}\sum_{ab\le N} a^4\phi(a)b^3$$
$$+ \frac{2}{9}\sum_{ab\le N} a^3\prod_{p|a}(1-p)\cdot b^5 + \frac{1}{9}\sum_{ab\le N} a^3\prod_{p|a}(1-p)\cdot b^3.$$

On the other hand, we have

$$\begin{aligned} g(n) &= \sum_{j=1}^{n}(j,n) = \sum_{d|n} d\phi(n/d) = \sum_{km=n}\mu(k)md(m), \\ h(n) &= \sum_{j=1}^{n}\frac{n}{(j,n)} = \sum_{d|n} d\phi(d) = \sum_{km=n}\mu(k)k\sigma_2(m). \end{aligned} \tag{3.1}$$

Therefore

$$\sum_{ab\le N} a^4\phi(a)b^5 = \sum_{n\le N} n^4\sum_{km=n} u(k)md(m) = \sum_{n\le N} n^4 g(n),$$
$$\sum_{ab\le N} a^4\phi(a)b^3 = \sum_{n\le N} n^3\sum_{km=n}\mu(k)k\sigma_2(m) = \sum_{n\le N} n^3 h(n),$$
$$\sum_{ab\le N} a^3\prod_{p|a}(1-p)\cdot b^5 = \sum_{n\le N} n^3\sum_{km=n}\mu(k)k\sigma_2(m) = \sum_{n\le N} n^3 h(n).$$

It is easy to prove that

$$\sum_{ab\le N} a^3\prod_{p|a}(1-p)\cdot b^3 \ll \sum_{a\le N} a^4\sum_{b\le N/a} b^3 \ll \sum_{a\le N} a^4\cdot\frac{N^4}{a^4} \ll N^5.$$

Thus we have

$$M_1(B) = \frac{4}{9}\sum_{ab\le N} a^4\phi(a)b^5 + \frac{2}{9}\sum_{ab\le N} a^4\phi(a)b^3$$
$$+ \frac{2}{9}\sum_{ab\le N} a^3\prod_{p|a}(1-p)\cdot b^5 + O(N^5)$$
$$= \frac{4}{9}\sum_{n\le B} n^4 g(n) + \frac{4}{9}\sum_{n\le B} n^3 h(n) + O(B^5). \ \blacksquare$$

Now we prove Theorem 1.2. On noting that $\sum_{n\le x}\sigma_2(n) = \frac{\zeta(3)}{3}x^3 + O(x^2)$, we get

$$\sum_{n\le B}h(n)=\sum_{n\le B}\sum_{km=n}\mu(k)k\sigma_2(m)=\sum_{k\le B}\mu(k)k\sum_{m\le B/k}\sigma_2(m)$$
$$=\sum_{k\le B}\mu(k)k\biggl(\frac{\zeta(3)}{3}\cdot\frac{B^3}{k^3}+O\biggl(\frac{B^2}{k^2}\biggr)\biggr)=\frac{\zeta(3)}{3\zeta(2)}B^3+O(B^2\log B).$$

Then by Abel's identity we have

$$\sum_{n\le B}n^3h(n)=\Bigl(\sum_{n\le B}h(n)\Bigr)B^3-3\int_0^B\Bigl(\sum_{n\le t}h(n)\Bigr)t^2\,dt \tag{3.2}$$
$$=\frac{\zeta(3)}{3\zeta(2)}B^6+O(B^5\log B)-3\int_1^B\biggl(\frac{\zeta(3)}{3\zeta(2)}t^3+O(t^2\log(2t))\biggr)t^2\,dt$$
$$=\frac{\zeta(3)}{6\zeta(2)}B^6+O(B^5\log B).$$

Now from Lemma 3.1, (3.2), (1.4) and (1.5) we get

$$M_1(B)=\frac{4}{9}\sum_{n\le B}n^4g(n)+\frac{2\zeta(3)}{27\zeta(2)}B^6+O(B^5\log B)$$
$$=\frac{4}{9}N_{-4}(B)+\frac{4}{9}E_{-4}(B)+\frac{2\zeta(3)}{27\zeta(2)}B^6+O(B^5\log B)$$
$$=\frac{4}{9\pi^2}B^6\log B+\frac{4}{9\pi^2}\biggl(2\gamma-\frac{1}{6}-\frac{\zeta'(2)}{\zeta(2)}+\zeta(3)\biggr)B^6+O(B^{\frac{2211}{416}+\varepsilon})$$
$$=\frac{4}{9\pi^2}B^6\log B+C_1B^6+O(B^{\frac{2211}{416}+\varepsilon}).$$

On the other hand, letting $F_1(B)=M_1(B)-\frac{4}{9\pi^2}B^6\log B-C_1B^6$, we have

$$F_1(B)=\frac{4}{9}E_{-4}(B)+O(B^5\log B).$$

Therefore by (1.6) and the Cauchy–Schwarz inequality we obtain

$$\int_2^T F_1^2(B)\,dB=\frac{8D_0}{1863\pi^2}T^{23/2}+O(T^{45/4+\varepsilon}).$$

4. Some lemmas. To prove Theorem 1.3, we need the following lemmas.

LEMMA 4.1. *Let $N>1$ be an integer. Then*

$$\sum_{\substack{ab\le N\\ 2\nmid a}}\phi(a)b=\frac{1}{3\zeta(2)}N^2\log N+\frac{1}{3\zeta(2)}\biggl(\frac{2\log 2}{3}+2\gamma-\frac{1}{2}-\frac{\zeta'(2)}{\zeta(2)}\biggr)N^2$$
$$+N\sum_{k\le N}\frac{\mu^*(k)}{k}\Delta\biggl(\frac{N}{k}\biggr)+O(N\log^2 N),$$

where

$$\mu^*(k) = \begin{cases} \mu(k) & \text{if } 2 \nmid k, \\ 2\mu(k) & \text{if } 2 \mid k. \end{cases}$$

Proof. Define

$$\chi_2(n) = \begin{cases} 1 & \text{if } 2 \nmid n, \\ 0 & \text{if } 2 \mid n, \end{cases} \quad \text{and} \quad \alpha(n) = \begin{cases} 1 & n = 1, \\ -4 & n = 2, \\ 4 & n = 4, \\ 0 & \text{otherwise.} \end{cases}$$

We get

$$\sum_{n=1}^{\infty} \frac{\chi_2(n)nd(n)}{n^s} = \left(1 - \frac{1}{2^{s-1}}\right)^2 \zeta^2(s-1), \qquad \sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s} = \left(1 - \frac{1}{2^{s-1}}\right)^2,$$

$$\sum_{n=1}^{\infty} \frac{nd(n)}{n^s} = \zeta^2(s-1).$$

Then

$$\sum_{\substack{m\le y \\ 2\nmid m}} md(m) = \sum_{m\le y} \chi_2(m)md(m) = \sum_{km\le y} \alpha(k)md(m) = \sum_{k\le y} \alpha(k) \sum_{m\le y/k} md(m)$$

$$= \sum_{m\le y} md(m) - 4 \sum_{m\le y/2} md(m) + 4 \sum_{m\le y/4} md(m).$$

So from (2.10) we have

$$\begin{aligned}\sum_{\substack{m\le y \\ 2\nmid m}} md(m) &= \frac{1}{2}y^2 \log y + y^2\left(\gamma - \frac{1}{4}\right) + y\varDelta(y) \\ &\quad - 4\left(\frac{1}{8}y^2 \log \frac{y}{2} + \frac{y^2}{4}\left(\gamma - \frac{1}{4}\right) + \frac{y}{2}\varDelta\left(\frac{y}{2}\right)\right) \\ &\quad + 4\left(\frac{1}{32}y^2 \log \frac{y}{4} + \frac{y^2}{16}\left(\gamma - \frac{1}{4}\right) + \frac{y}{4}\varDelta\left(\frac{y}{4}\right)\right) + O(y) \\ &= \frac{1}{8}y^2 \log y + \left(\frac{1}{4}\log 2 + \frac{\gamma}{4} - \frac{1}{16}\right)y^2 \\ &\quad + y\varDelta(y) - 2y\varDelta\left(\frac{y}{2}\right) + y\varDelta\left(\frac{y}{4}\right) + O(y).\end{aligned}$$

Therefore by (3.1) we get

$$\sum_{\substack{ab\le N \\ 2\nmid a}} \phi(a)b = \sum_{n\le N} \sum_{\substack{d\mid n \\ 2\nmid d}} \phi(d)\frac{n}{d} = \sum_{\substack{2^s n\le N \\ s\ge 0 \\ 2\nmid n}} 2^s \sum_{d\mid n} \phi(d)\frac{n}{d}$$

$$= \sum_{\substack{2^s n\le N \\ s\ge 0 \\ 2\nmid n}} 2^s \sum_{km=n} \mu(k) \cdot md(m)$$

$$= \sum_{\substack{2^s\le N\\ s\ge 0}} 2^s \sum_{\substack{n\le N/2^s\\ 2\nmid n}} \sum_{km=n} \mu(k)\cdot md(m) = \sum_{\substack{2^s\le N\\ s\ge 0}} 2^s \sum_{\substack{k\le N/2^s\\ 2\nmid k}} \mu(k) \sum_{\substack{m\le N/2^s k\\ 2\nmid m}} md(m)$$

$$= \sum_{\substack{2^s\le N\\ s\ge 0}} 2^s \sum_{\substack{k\le N/2^s\\ 2\nmid k}} \mu(k)\biggl(\frac{1}{8}\Bigl(\frac{N}{2^s k}\Bigr)^2 \log\Bigl(\frac{N}{2^s k}\Bigr) + \Bigl(\frac{1}{4}\log 2 + \frac{\gamma}{4} - \frac{1}{16}\Bigr)\Bigl(\frac{N}{2^s k}\Bigr)^2$$

$$+ \frac{N}{2^s k}\Delta\Bigl(\frac{N}{2^s k}\Bigr) - \frac{N}{2^{s-1}k}\Delta\Bigl(\frac{N}{2^{s+1}k}\Bigr) + \frac{N}{2^s k}\Delta\Bigl(\frac{N}{2^{s+2}k}\Bigr)\biggr)$$

$$+ O\biggl(\sum_{\substack{2^s\le N\\ s\ge 0}} 2^s \sum_{\substack{k\le N/2^s\\ 2\nmid k}} \frac{N}{2^s k}\biggr)$$

$$= \frac{1}{8}N^2\log N \sum_{\substack{2^s\le N\\ s\ge 0}} \frac{1}{2^s} \sum_{\substack{k\le N/2^s\\ 2\nmid k}} \frac{\mu(k)}{k^2} - \frac{1}{8}N^2 \sum_{\substack{2^s\le N\\ s\ge 0}} \frac{\log 2^s}{2^s} \sum_{\substack{k\le N/2^s\\ 2\nmid k}} \frac{\mu(k)}{k^2}$$

$$- \frac{1}{8}N^2 \sum_{\substack{2^s\le N\\ s\ge 0}} \frac{1}{2^s} \sum_{\substack{k\le N/2^s\\ 2\nmid k}} \frac{\mu(k)\log k}{k^2}$$

$$+ \Bigl(\frac{1}{4}\log 2 + \frac{\gamma}{4} - \frac{1}{16}\Bigr)N^2 \sum_{\substack{2^s\le N\\ s\ge 0}} \frac{1}{2^s} \sum_{\substack{k\le N/2^s\\ 2\nmid k}} \frac{\mu(k)}{k^2}$$

$$+ N \sum_{\substack{2^s\le N\\ s\ge 0}} \sum_{\substack{k\le N/2^s\\ 2\nmid k}} \frac{\mu(k)}{k}\Delta\Bigl(\frac{N}{2^s k}\Bigr) - 2N \sum_{\substack{2^s\le N\\ s\ge 0}} \sum_{\substack{k\le N/2^s\\ 2\nmid k}} \frac{\mu(k)}{k}\Delta\Bigl(\frac{N}{2^{s+1}k}\Bigr)$$

$$+ N \sum_{\substack{2^s\le N\\ s\ge 0}} \sum_{\substack{k\le N/2^s\\ 2\nmid k}} \frac{\mu(k)}{k}\Delta\Bigl(\frac{N}{2^{s+2}k}\Bigr) + O(N\log^2 N)$$

$$= \frac{1}{3\zeta(2)}N^2\log N + \frac{1}{3\zeta(2)}\Bigl(\frac{2\log 2}{3} + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)}\Bigr)N^2$$

$$+ N \sum_{\substack{2^s\le N\\ s\ge 0}} \sum_{\substack{k\le N/2^s\\ 2\nmid k}} \frac{\mu(k)}{k}\Delta\Bigl(\frac{N}{2^s k}\Bigr) - 2N \sum_{\substack{2^s\le N\\ s\ge 0}} \sum_{\substack{k\le N/2^s\\ 2\nmid k}} \frac{\mu(k)}{k}\Delta\Bigl(\frac{N}{2^{s+1}k}\Bigr)$$

$$+ N \sum_{\substack{2^s\le N\\ s\ge 0}} \sum_{\substack{k\le N/2^s\\ 2\nmid k}} \frac{\mu(k)}{k}\Delta\Bigl(\frac{N}{2^{s+2}k}\Bigr) + O(N\log^2 N)$$

$$= \frac{1}{3\zeta(2)}N^2\log N + \frac{1}{3\zeta(2)}\Bigl(\frac{2\log 2}{3} + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)}\Bigr)N^2$$

$$+ N \sum_{\substack{2^s \le N \\ s \ge 0}} \sum_{\substack{k \le N/2^s \\ 2 \nmid k}} \frac{\mu(k)}{k} \varDelta\left(\frac{N}{2^s k}\right) - 2N \sum_{\substack{2^s \le N/2 \\ s \ge 0}} \sum_{\substack{k \le N/2^{s+1} \\ 2 \nmid k}} \frac{\mu(k)}{k} \varDelta\left(\frac{N}{2^{s+1} k}\right)$$
$$+ N \sum_{\substack{2^s \le N/4 \\ s \ge 0}} \sum_{\substack{k \le N/2^{s+2} \\ 2 \nmid k}} \frac{\mu(k)}{k} \varDelta\left(\frac{N}{2^{s+2} k}\right) + O(N \log^2 N).$$

Noting that

$$N \sum_{\substack{2^s \le N \\ s \ge 0}} \sum_{\substack{k \le N/2^s \\ 2 \nmid k}} \frac{\mu(k)}{k} \varDelta\left(\frac{N}{2^s k}\right) - 2N \sum_{\substack{2^s \le N/2 \\ s \ge 0}} \sum_{\substack{k \le N/2^{s+1} \\ 2 \nmid k}} \frac{\mu(k)}{k} \varDelta\left(\frac{N}{2^{s+1} k}\right)$$
$$+ N \sum_{\substack{2^s \le N/4 \\ s \ge 0}} \sum_{\substack{k \le N/2^{s+2} \\ 2 \nmid k}} \frac{\mu(k)}{k} \varDelta\left(\frac{N}{2^{s+2} k}\right)$$
$$= N \sum_{\substack{k \le N \\ 2 \nmid k}} \frac{\mu(k)}{k} \varDelta\left(\frac{N}{k}\right) + N \sum_{\substack{k \le N/2 \\ 2 \nmid k}} \frac{\mu(k)}{k} \varDelta\left(\frac{N}{2k}\right) - 2N \sum_{\substack{k \le N/2 \\ 2 \nmid k}} \frac{\mu(k)}{k} \varDelta\left(\frac{N}{2k}\right)$$
$$= N \sum_{\substack{k \le N \\ 2 \nmid k}} \frac{\mu(k)}{k} \varDelta\left(\frac{N}{k}\right) - N \sum_{\substack{k \le N/2 \\ 2 \nmid k}} \frac{\mu(k)}{k} \varDelta\left(\frac{N}{2k}\right)$$
$$= N \sum_{k \le N} \frac{\mu^*(k)}{k} \varDelta\left(\frac{N}{k}\right),$$

we have

$$\sum_{\substack{ab \le N \\ 2 \nmid a}} \phi(a) b = \frac{1}{3\zeta(2)} N^2 \log N + \frac{1}{3\zeta(2)} \left(\frac{2 \log 2}{3} + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)}\right) N^2$$
$$+ N \sum_{k \le N} \frac{\mu^*(k)}{k} \varDelta\left(\frac{N}{k}\right) + O(N \log^2 N). \ \blacksquare$$

Lemma 4.2. *Let $N > 1$ be an integer. Then*

$$\sum_{\substack{ab \le N \\ 2 \nmid a}} \phi(a) = \frac{1}{3} N^2 + O(N), \qquad \sum_{\substack{ab \le N \\ 2 \nmid a,\, 2 \nmid b}} \phi(a) = \frac{1}{4} N^2 + O(N),$$
$$\sum_{\substack{ab \le N \\ 2 \nmid a,\, 2 \mid b}} \phi(a) = \frac{1}{12} N^2 + O(N),$$
$$\sum_{\substack{ab \le N \\ 2 \mid a,\, 2 \nmid b}} \phi(a) = \frac{1}{8} N^2 + O(N), \qquad \sum_{\substack{ab \le N \\ 2 \mid a,\, 2 \mid b}} \phi(a) = \frac{1}{24} N^2 + O(N).$$

Proof. Define

$$\beta(n)=\begin{cases}1 & n=1,\\ -1 & n=2,2^2,2^3,\dots,\\ 0 & \text{otherwise.}\end{cases}$$

On noting that

$$\sum_{n=1}^{\infty}\Big(\sum_{\substack{d|n\\ 2\nmid d}}\phi(d)\Big)n^{-s}=\zeta(s-1)\left(1-\frac{1}{2^s-1}\right),\qquad \sum_{n=1}^{\infty}\frac{\beta(n)}{n^s}=\left(1-\frac{1}{2^s-1}\right),$$

we have

$$\sum_{\substack{ab\le N\\ 2\nmid a}}\phi(a)=\sum_{n\le N}\sum_{\substack{d|n\\ 2\nmid d}}\phi(d)=\sum_{km\le N}\beta(k)m=\sum_{k\le N}\beta(k)\sum_{m\le N/k}m$$

$$=\sum_{m\le N}m-\sum_{\substack{2^s\le N\\ s\ge 1}}\sum_{m\le N/2^s}m=\frac{1}{3}N^2+O(N).$$

As $\sum_{ab\le N}\phi(a)=\sum_{n\le N}\sum_{d|N}\phi(d)=\sum_{n\le N}n=\frac{1}{2}N^2+O(N)$, we get

$$\sum_{\substack{ab\le N\\ 2|a}}\phi(a)=\sum_{ab\le N}\phi(a)-\sum_{\substack{ab\le N\\ 2\nmid a}}\phi(a)=\frac{1}{6}N^2+O(N).$$

Therefore

$$\sum_{\substack{ab\le N\\ 2\nmid a,\,2|b}}\phi(a)=\sum_{\substack{ab\le N/2\\ 2\nmid a}}\phi(a)=\frac{1}{12}N^2+O(N),$$

$$\sum_{\substack{ab\le N\\ 2|a,\,2|b}}\phi(a)=\sum_{\substack{ab\le N/2\\ 2|a}}\phi(a)=\frac{1}{24}N^2+O(N),$$

$$\sum_{\substack{ab\le N\\ 2\nmid a,\,2\nmid b}}\phi(a)=\sum_{\substack{ab\le N\\ 2\nmid a}}\phi(a)-\sum_{\substack{ab\le N\\ 2\nmid a,\,2|b}}\phi(a)=\frac{1}{4}N^2+O(N),$$

$$\sum_{\substack{ab\le N\\ 2|a,\,2\nmid b}}\phi(a)=\sum_{\substack{ab\le N\\ 2|a}}\phi(a)-\sum_{\substack{ab\le N\\ 2|a,\,2|b}}\phi(a)=\frac{1}{8}N^2+O(N).\ \blacksquare$$

LEMMA 4.3. *For any integer $N>1$, we have*

$$\sum_{\substack{1\le a\le N\\ 2\nmid a}}\phi(a)\left[\frac{N}{a}\right]^2=\frac{2}{3\zeta(2)}N^2\log N+\frac{2}{3\zeta(2)}\left(\frac{2\log 2}{3}+2\gamma-\frac{1}{2}-\frac{\zeta'(2)}{\zeta(2)}-\frac{\zeta(2)}{2}\right)N^2$$

$$+2N\sum_{k\le N}\frac{\mu^*(k)}{k}\Delta\left(\frac{N}{k}\right)+O(N\log^2 N).$$

Proof. By Lemmas 4.1 and 4.2 we easily get

$$\sum_{\substack{1\le a\le N\\ 2\nmid a}} \phi(a)\left[\frac{N}{a}\right]^2 = 2\sum_{\substack{a\le N\\ 2\nmid a}} \phi(a)\frac{\left[\frac{N}{a}\right]\left(\left[\frac{N}{a}\right]+1\right)}{2} - \sum_{\substack{a\le N\\ 2\nmid a}} \phi(a)\left[\frac{N}{a}\right]$$
$$= 2\sum_{\substack{ab\le N\\ 2\nmid a}} \phi(a)b - \sum_{\substack{ab\le N\\ 2\nmid a}} \phi(a) = \frac{2}{3\zeta(2)}N^2\log N$$
$$+\frac{2}{3\zeta(2)}\left(\frac{2\log 2}{3} + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)} - \frac{\zeta(2)}{2}\right)N^2$$
$$+2N\sum_{k\le N}\frac{\mu^*(k)}{k}\Delta\left(\frac{N}{k}\right) + O(N\log^2 N). \blacksquare$$

LEMMA 4.4. *For any integer $N > 1$, we have*

$$\sum_{\substack{1\le a\le N\\ 2\nmid a\\ 2\nmid[N/a]}} \phi(a) = \frac{1}{6}N^2 + O(N), \qquad \sum_{\substack{1\le a\le N\\ 2\mid a\\ 2\nmid[N/a]}} \phi(a) = \frac{1}{12}N^2 + O(N).$$

Proof. Noting that

$$\left[\frac{Y}{2}+\frac{1}{2}\right] - \left[\frac{Y}{2}\right] = \begin{cases} 1 & \text{if } 2\nmid [Y],\\ 0 & \text{if } 2\mid [Y],\end{cases}$$

from Lemma 4.2 we have

$$\sum_{\substack{1\le a\le N\\ 2\nmid a\\ 2\nmid[N/a]}} \phi(a) = \sum_{\substack{1\le a\le N\\ 2\nmid a}} \phi(a)\left(\left[\frac{N}{2a}+\frac{1}{2}\right] - \left[\frac{N}{2a}\right]\right)$$
$$= \sum_{\substack{1\le a\le N\\ 2\nmid a}} \phi(a)\left[\frac{N}{2a}+\frac{1}{2}\right] - \sum_{\substack{1\le a\le N\\ 2\nmid a}} \phi(a)\left[\frac{N}{2a}\right]$$
$$= \sum_{\substack{1\le ab\le N\\ 2\nmid a,\, 2\nmid b}} \phi(a) - \sum_{\substack{1\le ab\le N\\ 2\nmid a,\, 2\mid b}} \phi(a) = \frac{1}{6}N^2 + O(N),$$

and

$$\sum_{\substack{1\le a\le N\\ 2\mid a\\ 2\nmid[N/a]}} \phi(a) = \sum_{\substack{1\le a\le N\\ 2\mid a}} \phi(a)\left(\left[\frac{N}{2a}+\frac{1}{2}\right] - \left[\frac{N}{2a}\right]\right)$$
$$= \sum_{\substack{1\le a\le N\\ 2\mid a}} \phi(a)\left[\frac{N}{2a}+\frac{1}{2}\right] - \sum_{\substack{1\le a\le N\\ 2\mid a}} \phi(a)\left[\frac{N}{2a}\right]$$
$$= \sum_{\substack{1\le ab\le N\\ 2\mid a,\, 2\nmid b}} \phi(a) - \sum_{\substack{1\le ab\le N\\ 2\mid a,\, 2\mid b}} \phi(a) = \frac{1}{12}N^2 + O(N). \blacksquare$$

LEMMA 4.5. *Let $y = T^{1-\varepsilon}$, and define*

$$\Upsilon(B) = \frac{\sqrt{2}B^{5/4}}{\pi} \sum_{k\le y} \frac{\mu^*(k)}{k^{5/4}} \sum_{n\le y} \frac{d(n)}{n^{3/4}} \cos\left(4\pi\sqrt{\frac{nB}{k}} - \frac{\pi}{4}\right).$$

Then

$$\int_T^{2T} \Upsilon^2(B)\,dB = \frac{D_*}{\pi^2}\int_T^{2T} B^{5/2}\,dB + O(T^{3+\varepsilon}),$$

where

$$D_* := \sum_{n=1}^{\infty} h_*^2(n) n^{-3/2}, \qquad h_*(n) := \sum_{n=kl} \frac{\mu^*(k)d(l)}{k^{1/2}}.$$

Proof. By the elementary formula

$$\cos u \cos v = \frac{1}{2}(\cos(u-v) + \cos(u+v))$$

we may write

$$\begin{aligned}(4.1)\qquad \Upsilon^2(B) &= \frac{2B^{5/2}}{\pi^2} \sum_{k_1\le y}\sum_{k_2\le y} \frac{\mu^*(k_1)\mu^*(k_2)}{(k_1k_2)^{5/4}} \sum_{n_1\le y}\sum_{n_2\le y} \frac{d(n_1)d(n_2)}{(n_1n_2)^{3/4}} \\ &\quad\times \cos\left(4\pi\sqrt{\frac{n_1B}{k_1}} - \frac{\pi}{4}\right)\cos\left(4\pi\sqrt{\frac{n_2B}{k_2}} - \frac{\pi}{4}\right) \\ &= S_1(B) + S_2(B) + S_3(B),\end{aligned}$$

where

$$S_1(B) = \frac{B^{5/2}}{\pi^2} \sum_{k_1\le y}\sum_{\substack{k_2\le y \\ k_2n_1=k_1n_2}}\sum_{n_1\le y}\sum_{n_2\le y} \frac{\mu^*(k_1)\mu^*(k_2)}{(k_1k_2)^{5/4}} \cdot \frac{d(n_1)d(n_2)}{(n_1n_2)^{3/4}},$$

$$\begin{aligned}S_2(B) &= \frac{B^{5/2}}{\pi^2} \sum_{k_1\le y}\sum_{\substack{k_2\le y \\ k_2n_1\ne k_1n_2}}\sum_{n_1\le y}\sum_{n_2\le y} \frac{\mu^*(k_1)\mu^*(k_2)}{(k_1k_2)^{5/4}} \cdot \frac{d(n_1)d(n_2)}{(n_1n_2)^{3/4}} \\ &\quad\times \cos\left(4\pi\sqrt{B}\left(\sqrt{\frac{n_1}{k_1}} - \sqrt{\frac{n_2}{k_2}}\right)\right),\end{aligned}$$

$$\begin{aligned}S_3(B) &= \frac{B^{5/2}}{\pi^2} \sum_{k_1\le y}\sum_{k_2\le y}\sum_{n_1\le y}\sum_{n_2\le y} \frac{\mu^*(k_1)\mu^*(k_2)}{(k_1k_2)^{5/4}} \cdot \frac{d(n_1)d(n_2)}{(n_1n_2)^{3/4}} \\ &\quad\times \sin\left(4\pi\sqrt{B}\left(\sqrt{\frac{n_1}{k_1}} + \sqrt{\frac{n_2}{k_2}}\right)\right).\end{aligned}$$

We have

$$(4.2)\qquad \int_T^{2T} S_1(B)\,dB = \frac{A(T)}{\pi^2}\int_T^{2T} B^{5/2}\,dB,$$

where

$$A(T) = \sum_{k_1\le y}\sum_{\substack{k_2\le y\\ k_2n_1=k_1n_2}}\sum_{n_1\le y}\sum_{n_2\le y}\frac{\mu^*(k_1)\mu^*(k_2)}{(k_1k_2)^{5/4}}\cdot\frac{d(n_1)d(n_2)}{(n_1n_2)^{3/4}}.$$

It can be written as

$$A(T) = \sum_{n\le y^2}\frac{\alpha^2(n;y)}{n^{3/2}},$$

where

$$\alpha(n;y) = \sum_{\substack{n=kl\\ k\le y,\, l\le y}}\frac{\mu^*(k)d(l)}{k^{1/2}}.$$

Define

$$h_*(n) = \sum_{n=kl}\frac{\mu^*(k)d(l)}{k^{1/2}},\qquad \beta(n) = 2\sum_{n=kl}\frac{d(l)}{k^{1/2}}.$$

It is obvious that $h_*(n) = \alpha(n;y)$ for $n \le y$, and $\sum_{n\le x}\beta^2(n) \ll x^{1+\varepsilon}$. Thus

$$\begin{aligned}(4.3)\qquad A(T) &= \sum_{n\le y} h_*^2(n)n^{-3/2} + O\Big(\sum_{y<n\le y^2}\beta^2(n)n^{-3/2}\Big)\\ &= \sum_{n=1}^{\infty} h_*^2(n)n^{-3/2} + O\Big(\sum_{n>y}\beta^2(n)n^{-3/2}\Big)\\ &= \sum_{n=1}^{\infty} h_*^2(n)n^{-3/2} + O(y^{-1/2+\varepsilon}).\end{aligned}$$

So from (4.2) and (4.3) we get

$$(4.4)\qquad \int_T^{2T} S_1(B)\,dB = \frac{1}{\pi^2}\Big(\sum_{n=1}^{\infty} h_*^2(n)n^{-3/2}\Big)\int_T^{2T} B^{5/2}\,dB + O(T^{3+\varepsilon}).$$

Now we consider $\int_T^{2T} S_2(B)\,dB$. By the first derivative test (see Lemma 4.3 of [9]) we get

(4.5)

$$\begin{aligned}\int_T^{2T} S_2(B)\,dB &\ll T^3\sum_{k_1\le y}\sum_{\substack{k_2\le y\\ k_2n_1\ne k_1n_2}}\sum_{n_1\le y}\sum_{n_2\le y}\frac{d(n_1)d(n_2)}{(k_1k_2)^{5/4}(n_1n_2)^{3/4}}\cdot\frac{1}{|\sqrt{n_1/k_1}-\sqrt{n_2/k_2}|}\\ &\ll T^3\sum_{k_1\le y}\sum_{\substack{k_2\le y\\ k_2n_1\ne k_1n_2}}\sum_{n_1\le y}\sum_{n_2\le y}\frac{d(n_1)d(n_2)}{(k_1k_2n_1n_2)^{3/4}}\cdot\frac{1}{|\sqrt{k_2n_1}-\sqrt{k_1n_2}|}\end{aligned}$$

$$\ll T^3 \sum_{\substack{k_1\le y\\ |\sqrt{k_2n_1}-\sqrt{k_1n_2}|\ge\frac12(k_1k_2n_1n_2)^{1/4}}}\sum_{k_2\le y}\sum_{n_1\le y}\sum_{n_2\le y} \frac{d(n_1)d(n_2)}{(k_1k_2n_1n_2)^{3/4}}\cdot\frac{1}{|\sqrt{k_2n_1}-\sqrt{k_1n_2}|}$$

$$+ T^3 \sum_{\substack{k_1\le y\\ 0<|\sqrt{k_2n_1}-\sqrt{k_1n_2}|<\frac12(k_1k_2n_1n_2)^{1/4}}}\sum_{k_2\le y}\sum_{n_1\le y}\sum_{n_2\le y} \frac{d(n_1)d(n_2)}{(k_1k_2n_1n_2)^{3/4}}\cdot\frac{1}{|\sqrt{k_2n_1}-\sqrt{k_1n_2}|}$$

$$\ll T^3 \sum_{k_1\le y}\sum_{k_2\le y}\sum_{n_1\le y}\sum_{n_2\le y}\frac{d(n_1)d(n_2)}{k_1k_2n_1n_2}$$

$$+ T^3 \sum_{k_1\le y}\sum_{\substack{k_2\le y\\ k_2n_1\ne k_1n_2}}\sum_{n_1\le y}\sum_{n_2\le y}\frac{d(n_1)d(n_2)}{(k_1k_2n_1n_2)^{1/2}}\cdot\frac{1}{|k_2n_1-k_1n_2|}$$

$$\ll T^3 \sum_{k_1\le y}\sum_{k_2\le y}\sum_{n_1\le y}\sum_{n_2\le y}\frac{d(n_1)d(n_2)}{k_1k_2n_1n_2} + T^3\sum_{n\le y^2}\frac{(\sum_{n=kl} d(l))^2}{n} \ll T^{3+\varepsilon}.$$

For $\int_T^{2T} S_3(B)\,dB$, by the first derivative test again we get

$$\int_T^{2T} S_3(B)\,dB \tag{4.6}$$
$$\ll T^3 \sum_{k_1\le y}\sum_{k_2\le y}\sum_{n_1\le y}\sum_{n_2\le y}\frac{d(n_1)d(n_2)}{(k_1k_2)^{5/4}(n_1n_2)^{3/4}}\cdot\frac{1}{|\sqrt{n_1/k_1}+\sqrt{n_2/k_2}|}$$
$$\ll T^3 \sum_{k_1\le y}\sum_{k_2\le y}\sum_{n_1\le y}\sum_{n_2\le y}\sum_{n_2\le y}\frac{d(n_1)d(n_2)}{(k_1k_2)^{5/4}(n_1n_2)^{3/4}(n_1/k_1)^{1/4}(n_2/k_2)^{1/4}}$$
$$\ll T^3 \sum_{k_1\le y}\sum_{k_2\le y}\sum_{n_1\le y}\sum_{n_2\le y}\frac{d(n_1)d(n_2)}{k_1k_2n_1n_2} \ll T^{3+\varepsilon}.$$

Now from (4.1), (4.4)–(4.6) we immediately have

$$\int_T^{2T} \varUpsilon^2(B)\,dB = \frac{1}{\pi^2}\Big(\sum_{n=1}^{\infty} h_*^2(n)n^{-3/2}\Big)\int_T^{2T} B^{5/2}\,dB + O(T^{3+\varepsilon})$$
$$= \frac{D_*}{\pi^2}\int_T^{2T} B^{5/2}\,dB + O(T^{3+\varepsilon}). \ \blacksquare$$

5. Proof of Theorem 1.3. Let $N=[B]$. We have

$$M_2(B) = \sum_{n_1\le B}\sum_{\substack{n_2\le B\\ n_1n_2=n_3n_4}}\sum_{n_3\le B}\sum_{n_4\le B}(-1)^{n_1+n_2+n_3+n_4} \tag{5.1}$$
$$= \sum_{a\le N}\sum_{\substack{c\le N\\ (a,c)=1}}\Big(\sum_{\substack{n_1\le N\\ n_1/n_3=a/c}}\sum_{n_3\le N}(-1)^{n_1+n_3}\Big)^2$$

$$= N^2 + 2 \sum_{1<a\le N} \sum_{\substack{1\le c\le a\\ (c,a)=1}} \Big(\sum_{m\le N/a} (-1)^{(a+c)m} \Big)^2$$

$$= N^2 + 2 \sum_{\substack{1<a\le N\\ 2\nmid a}} \sum_{\substack{1\le c\le a\\ 2\nmid c\\ (c,a)=1}} \left[\frac{N}{a}\right]^2 + 2 \sum_{\substack{1<a\le N\\ 2\nmid a\\ 2\nmid [N/a]}} \sum_{\substack{1\le c\le a\\ 2\mid c\\ (c,a)=1}} 1 + 2 \sum_{\substack{1<a\le N\\ 2\mid a\\ 2\nmid [N/a]}} \sum_{\substack{1\le c\le a\\ 2\nmid c\\ (c,a)=1}} 1.$$

Since $(C,a) = (c,a)$ if $c + C = a$, it follows that if a is an odd integer with $a > 1$, then

$$\sum_{\substack{1\le c\le a\\ 2\mid c\\ (c,a)=1}} 1 = \sum_{\substack{1\le C\le a\\ 2\nmid C\\ (C,a)=1}} 1 = \phi(a)/2.$$

It therefore follows from (5.1), Lemmas 4.3 and 4.4, and Huxley's bound (1.2) that

$$\begin{aligned}
(5.2)\qquad M_2(B) &= N^2 + \sum_{\substack{1<a\le N\\ 2\nmid a}} \phi(a) \left[\frac{N}{a}\right]^2 + \sum_{\substack{1<a\le N\\ 2\nmid a\\ 2\nmid [N/a]}} \phi(a) + 2 \sum_{\substack{1<a\le N\\ 2\mid a\\ 2\nmid [N/a]}} \phi(a) \\
&= \sum_{\substack{1\le a\le N\\ 2\nmid a}} \phi(a) \left[\frac{N}{a}\right]^2 + \sum_{\substack{1\le a\le N\\ 2\nmid a\\ 2\nmid [N/a]}} \phi(a) + 2 \sum_{\substack{1\le a\le N\\ 2\mid a\\ 2\nmid [N/a]}} \phi(a) + O(1) \\
&= \frac{2}{3\zeta(2)} N^2 \log N + \frac{2}{3\zeta(2)} \left(\frac{2\log 2}{3} + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)} \right) N^2 \\
&\quad + 2N \sum_{k\le N} \frac{\mu^*(k)}{k} \Delta\left(\frac{N}{k}\right) + O(N \log^2 N) \\
&= \frac{4}{\pi^2} N^2 \log N + \frac{2}{3\zeta(2)} \left(\frac{2\log 2}{3} + 2\gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)} \right) N^2 \\
&\quad + O(N^{547/416+\varepsilon}) \\
&= \frac{4}{\pi^2} B^2 \log B + C_2 B^2 + O(B^{547/416+\varepsilon}).
\end{aligned}$$

As in the statement of Theorem 1.3, let $F_2(B) = M_2(B) - \frac{4}{\pi^2} B^2 \log B - C_2 B^2$. Then from (5.2) we have

$$F_2(B) = 2B \sum_{k\le B} \frac{\mu^*(k)}{k} \Delta\left(\frac{B}{k}\right) + O(B \log^2 B).$$

To prove Theorem 1.3, it suffices to evaluate the integral $\int_T^{2T} F_2^2(B)\,dB$. Letting $y = T^{1-\varepsilon}$, we get

$$F_2(B) = 2B \sum_{k\le y} \frac{\mu^*(k)}{k} \Delta\left(\frac{B}{k}\right) + O(T^{1+\epsilon}).$$

Then from the well-known Voronoĭ formula (see (12.4.4) of [9])

$$\Delta(x) = \frac{x^{1/4}}{\pi\sqrt{2}} \sum_{n\le N} \frac{d(n)}{n^{3/4}} \cos\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right) + O(x^{\varepsilon} + x^{1/2+\varepsilon}N^{-1/2})$$

we have

$$\begin{aligned} F_2(B) &= \frac{\sqrt{2}B^{5/4}}{\pi} \sum_{k\le y} \frac{\mu^*(k)}{k^{5/4}} \sum_{n\le y} \frac{d(n)}{n^{3/4}} \cos\left(4\pi\sqrt{\frac{nB}{k}} - \frac{\pi}{4}\right) + O(T^{1+\varepsilon}) \\ &= \Upsilon(B) + O(T^{1+\varepsilon}). \end{aligned}$$

Now from Lemma 4.5 we get

$$\int_T^{2T} F_2^2(B)\,dB = \int_T^{2T} \Upsilon^2(B)\,dB + O(T^{13/4+\varepsilon}) = \frac{D_*}{\pi^2} \int_T^{2T} B^{5/2}\,dB + O(T^{13/4+\varepsilon}),$$

which implies Theorem 1.3 by a splitting argument.

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