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The joint distribution of q-additive functions

by

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1. Introduction. Let $q > 1$ be a given integer. A real-valued function f, defined on the non-negative integers, is said to be *q-additive* if $f(0) = 0$ and

$$f(n) = \sum_{j\ge 0} f(a_{q,j}(n)q^j) \quad \text{for} \quad n = \sum_{j\ge 0} a_{q,j}(n)q^j,$$

where $a_{q,j}(n) \in E_q := \{0, 1, \ldots, q-1\}$. A special q-additive function is the *sum-of-digits* function

$$s_q(n) = \sum_{j\ge 0} a_{q,j}(n).$$

The statistical behaviour of the sum-of-digits function and, more generally, of q-additive functions has been very well studied by several authors.

The most general result concerning the *mean value* of q-additive functions is due to Manstavičius [20] (extending earlier work of Coquet [3]). Let

$$m_{k,q} := \frac{1}{q}\sum_{c\in E_q} f(cq^k), \qquad m_{2;k,q}^2 := \frac{1}{q}\sum_{c\in E_q} f^2(cq^k)$$

and

$$M_q(x) := \sum_{k=0}^{[\log_q x]} m_{k,q}, \qquad B_q^2(x) = \sum_{k=0}^{[\log_q x]} m_{2;k,q}^2.$$

Then

$$\frac{1}{x}\sum_{n<x} (f(n) - M_q(x))^2 \le cB_q^2(x), \tag{1.1}$$

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which implies

$$\frac{1}{x}\sum_{n<x} f(n) = M_q(x) + O(B_q(x)).$$

For the sum-of-digits function $s_q(n)$ much more precise results are known, e.g. Delange [5] proved (for integral x) that

$$\frac{1}{x}\sum_{n<x} s_q(n) = \frac{q-1}{2}\log_q x + \gamma(\log_q x),$$

where γ is a continuous, nowhere differentiable and periodic function with period 1. (Higher moments of $a_q(n)$ were considered by Kirschenhofer [19] and by Kennedy and Cooper [17] (for the variance) and by Grabner, Kirschenhofer, Prodinger and Tichy [12].)

There also exist distributional results for q-additive functions. In 1972 Delange [4] proved an analogue to the Erdős–Wintner theorem. There exists a distribution function $F(y)$ such that, as $x \to \infty$,

$$\frac{1}{x}\#\{n < x \mid f(n) < y\} \to F(y) \tag{1.2}$$

if and only if the two series $\sum_{k\ge 0} m_{k,q}$, $\sum_{k\ge 0} m_{2;k,q}^2$ converge. This theorem was generalized by Kátai [16] who proved that there exists a distribution function $F(y)$ such that, as $x \to \infty$,

$$\frac{1}{x}\#\{n < x \mid f(n) - M_q(x) < y\} \to F(y)$$

if and only if the series $\sum_{k\ge 0} m_{2;k,q}^2$ converges.

The most general theorem known concerning a central limit theorem is again due to Manstavičius [20]. Suppose that, as $x \to \infty$,

$$\max_{cq^j<x} |f(cq^j)| = o(B_q(x))$$

and that $D_q(x) \to \infty$, where

$$D_q^2(x) = \sum_{k=0}^{\log_q x} \sigma_{k,q}^2 \quad \text{and} \quad \sigma_{k,q}^2 := \frac{1}{q}\sum_{c\in E_q} f^2(cq^k) - m_{k,q}^2.$$

Then, as $x \to \infty$,

$$\frac{1}{x}\#\left\{n < x \,\middle|\, \frac{f(n) - M_q(x)}{D_q(x)} < y\right\} \to \Phi(y),$$

where Φ is the normal distribution function.

Similar distribution results for the sum-of-digits function of number systems related to substitution automata were considered by Dumont and Thomas [8]. For number systems whose bases satisfy linear recurrences we refer to [6].

Furthermore, Bassily and Kátai [1] studied the distribution of q-additive functions on polynomial sequences.

Theorem 1. *Let f be a q-additive function such that $f(cq^j) = O(1)$ as $j \to \infty$ and $c \in E_q$. Assume that $D_q(x)/(\log x)^\eta \to \infty$ as $x \to \infty$ for some $\eta > 0$ and let $P(x)$ be a polynomial with integer coefficients, degree r, and positive leading term. Then, as $x \to \infty$,*

$$\frac{1}{x}\#\left\{n < x \,\middle|\, \frac{f(P(n)) - M_q(x^r)}{D_q(x^r)} < y\right\} \to \Phi(y),$$

$$\frac{1}{\pi(x)}\#\left\{p < x \,\middle|\, \frac{f(P(p)) - M_q(x^r)}{D_q(x^r)} < y\right\} \to \Phi(y).$$

This result relies on the fact that suitably modified centralized moments converge (cf. Lemma 4). Note also that this theorem was only stated (and proved) for $\eta = 1/3$. However, a short inspection of the proof shows that $\eta > 0$ is sufficient.

2. Joint distributions. It is a natural question to ask whether there are analogue results for the joint distribution of q_l-additive functions $f_l(n)$ (if $q_1, \dots, q_d > 1$ are pairwise coprime integers). For example, Hildebrand [14] announced that one always has

$$\frac{1}{x}\#\{n < x \mid f_l(n) < y_l,\ 1 \le l \le d\} \to F_1(y_1)\dots F_d(y_d)$$

if f_l satisfies (1.2) for all $l = 1, \dots, d$ and that there is a joint central limit theorem of the form

$$\frac{1}{x}\#\left\{n < x \,\middle|\, \frac{f_l(n) - M_{q_l}(x)}{D_{q_l}(x)} < y_l,\ 1 \le l \le d\right\} \to \Phi(y_1)\dots\Phi(y_d)$$

if $B_{q_l}(x) \to \infty$ and $B_{q_l}(x^\eta) \sim B_{q_l}(x)$ for every $\eta > 0$ as $x \to \infty$. (Note that the sum-of-digits function $s_q(n)$ is not covered by this result.)

In this paper we will first extend the above result of Bassily and Kátai to the joint distribution of q_l-additive functions f_l $(1 \le l \le d)$ on specific polynomial sequences if $q_1, \dots, q_d$ are pairwise coprime.

Theorem 2. *Let $q_1, \dots, q_d > 1$ be pairwise coprime integers and let f_l, $1 \le l \le d$, be q_l-additive functions such that $f_l(cq_l^j) = O(1)$ as $j \to \infty$ and $c \in E_{q_l}$. Assume that $D_{q_l}(x)/(\log x)^\eta \to \infty$ as $x \to \infty$, $1 \le l \le d$, for some $\eta > 0$ and let $P_l(x)$ be polynomials with integer coefficients of different degrees r_l and positive leading terms, $1 \le l \le d$. Then, as $x \to \infty$,*

$$\frac{1}{x}\#\left\{n < x \,\middle|\, \frac{f_l(P_l(n)) - M_{q_l}(x^{r_l})}{D_{q_l}(x^{r_l})} < y_l,\ 1 \le l \le d\right\} \to \Phi(y_1)\dots\Phi(y_d),$$

$$\frac{1}{\pi(x)}\#\left\{p < x \,\middle|\, \frac{f_l(P_l(p)) - M_{q_l}(x^{r_l})}{D_{q_l}(x^{r_l})} < y_l,\ 1 \le l \le d\right\} \to \Phi(y_1)\dots\Phi(y_d).$$

Corollary 1. *Let* $q_1, \ldots, q_d > 1$ *be pairwise coprime integers and let* $P_l(x)$ *be polynomials with integer coefficients of different degrees* r_l *and positive leading terms*, $1 \le l \le d$. *Then*, *as* $x \to \infty$,

$$\frac{1}{x}\#\left\{n < x \,\middle|\, \frac{s_{q_l}(P_l(n)) - \frac{q_l-1}{2}\log_{q_l} x^{r_l}}{\sqrt{\frac{q_l^2-1}{12}\log_{q_l} x^{r_l}}} < y_l,\ 1 \le l \le d\right\} \to \Phi(y_1)\ldots\Phi(y_d),$$

$$\frac{1}{\pi(x)}\#\left\{p<x \,\middle|\, \frac{s_{q_l}(P_l(p)) - \frac{q_l-1}{2}\log_{q_l} x^{r_l}}{\sqrt{\frac{q_l^2-1}{12}\log_{q_l} x^{r_l}}} < y_l,\ 1 \le l \le d\right\} \to \Phi(y_1)\ldots\Phi(y_d).$$

This theorem contains an *unnatural condition*, namely that one has to consider polynomials $P_l(x)$ with different degrees r_l. It would seem that this condition is not necessary. However, this is the crux of the matter. By using a variation of Bassily and Kátai's proof (combined with Baker's theorem on linear forms of logarithms) we could handle the case $d = 2$ with linear polynomals $P_l(x) = A_l x + B_l$.

Theorem 3. *Let* $q_1, q_2 > 1$ *be coprime integers and let* f_l *be* q_l-*additive functions such that* $f_l(cq_l^j) = O(1)$ *as* $j \to \infty$ *and* $c \in E_{q_l}$, $l = 1, 2$. *Assume that* $D_{q_l}(x)/(\log x)^\eta \to \infty$ *as* $x \to \infty$, $l = 1, 2$, *for some* $\eta > 0$. *Let* $P_l(x) = A_l x + B_l$, $l = 1, 2$, *be arbitrary linear polynomials with integer coefficients and positive leading terms* A_l *coprime to* q_l. *Then*, *as* $x \to \infty$,

$$\frac{1}{x}\#\left\{n < x \,\middle|\, \frac{f_l(P_l(n)) - M_{q_l}(x)}{D_{q_l}(x)} < y_l,\ l = 1, 2\right\} \to \Phi(y_1)\Phi(y_2).$$

Corollary 2. *Let* $q_1, q_2 > 1$ *be coprime integers. Then*, *as* $x \to \infty$,

$$\frac{1}{x}\#\left\{n < x \,\middle|\, \frac{s_{q_l}(n) - \frac{q_l-1}{2}\log_{q_l} x}{\sqrt{\frac{q_l^2-1}{12}\log_{q_l} x}} < y_l,\ l = 1, 2\right\} \to \Phi(y_1)\Phi(y_2).$$

Interestingly, there is even a local version of Corollary 2.

Theorem 4. *Let* $q_1, q_2 > 1$ *be coprime integers and set* $d = \gcd(q_1 - 1, q_2 - 1)$. *Then*, *as* $x \to \infty$,

$$\frac{1}{x}\#\{n < x \mid s_{q_1}(n) = k_1,\ s_{q_2}(n) = k_2\}$$
$$= d\prod_{l=1}^{2}\left(\frac{1}{\sqrt{2\pi\frac{q_l^2-1}{12}\log_{q_l} x}}\exp\left(-\frac{\left(k_l - \frac{q_l-1}{2}\log_{q_l} x\right)^2}{2\frac{q_l^2-1}{12}\log_{q_l} x}\right)\right) + o((\log x)^{-1})$$

uniformly for all integers $k_1, k_2 \ge 0$ *with* $k_1 \equiv k_2 \bmod d$.

Note that $s_{q_l}(n) \equiv n \bmod (q_l - 1)$. Thus we always have $s_{q_1}(n) \equiv s_{q_2}(n) \bmod d$ and consequently

$$\#\{n < x \mid s_{q_1}(n) = k_1, s_{q_2}(n) = k_2\} = 0$$

if $k_1 \not\equiv k_2 \bmod d$.

There are some other results indicating that the q_l-ary digital expansions are *asymptotically independent* for different bases q_l; e.g. Kim [18] ([1]) showed that for all integers $c_1, \dots, c_d$,

$$\frac{1}{x}|\{n < x \mid s_{q_j}(n) \equiv c_j \bmod m_j \ (1 \le j \le d)\}| = \frac{1}{m_1 \dots m_d} + O(x^{-\delta})$$

with

$$\delta = \frac{1}{120 d^2 q^2 m^2},$$

where $q_1, \dots, q_d > 1$ are pairwise coprime integers and $m_1, \dots, m_d$ are positive integers such that

$$\gcd(q_j - 1, m_j) = 1 \qquad (1 \le j \le d);$$

$q = \max\{q_1, \dots, q_d\}$, $m = \max\{m_1, \dots, m_d\}$ and the O-constant depends only on d and q. (This result sharpens a result by Bésineau [2] and solves a conjecture of Gelfond [11].)

Drmota and Larcher [7] used a variation of Kim's method to prove that a d-dimensional sequence $(\alpha_1 s_{q_1}(n), \dots, \alpha_d s_{q_d}(n))_{n \ge 0}$ is uniformly distributed modulo 1 if and only if $\alpha_1, \dots, \alpha_d$ are irrational. (Grabner, Liardet and Tichy [13] could prove a similar theorem by ergodic means.)

Another problem has been considered by Senge and Straus [26]. They proved that if q_1 and q_2 are coprime and c is any given positive constant then there are only finitely many $n \ge 0$ such that

$$s_{q_1}(n) \le c \quad \text{and} \quad s_{q_2}(n) \le c.$$

This result was later generalized and sharpened by Stewart [28], Schlickewei [22, 23] and by Pethő and Tichy [21]. The proofs use Baker's method for linear forms of logarithms and the p-adic version of Schmidt's subspace theorem by Schlickewei applied to S-unit equations.

One would get a much deeper insight into all these results if one could prove a local version of Theorem 2, e.g. asymptotic expansions or general estimates for the numbers

$$\frac{1}{x}\#\{n < x \mid s_q(n^2) = k\}$$

or for

$$\frac{1}{\pi(x)}\#\{p < x \mid s_q(p) = k\}$$

([1]) For brevity we restrict to the sum-of-digits function $s_q(n)$.

(and of course multivariate versions). It seems that problems of this kind are extremely difficult, e.g. it is an open question whether there are infinitely many primes p with even sum-of-digits function $s_2(p)$. The best known results concerning these questions are due to Fouvry and Mauduit [9, 10] who proved that

$$\frac{1}{x}\#\{n < x \mid n \in \mathbb{P} \vee (n = n_1 \cdot n_2 \wedge n_1, n_2 \in \mathbb{P}), s_q(n) \equiv 0 \bmod 2\} \geq c > 0$$

for some constant $c > 0$. ($\mathbb{P}$ denotes the set of primes.)

These questions are also related to two other conjectures of Gelfond [11], namely that $s_q(P(n))$ and $s_q(p)$ are uniformly distributed modulo m.

Remark. Schmidt [25] and Schmid [24] discussed the joint distribution of $s_2(k_l n)$ for different odd integers k_l, $1 \leq l \leq d$. (The distribution modulo m was investigated by Solinas [27].) It is surely possible to extend their result to the joint distribution of $f_l(P_l(n))$, $1 \leq l \leq d$, where f_l are q_l-additive functions, P_l are (certain) integer polynomials, and $q_l > 1$ arbitrary integers (e.g. all equal). However, we will not discuss this question here.

3. Proof of Theorem 2. As already mentioned, Theorem 2 is a direct generalization of Bassily and Kátai's result of [1]. Therefore we can proceed as in [1].

The first two lemmata on exponential sums are stated in [1]; a proof can also be found in [15].

Lemma 1. *Let $f(y)$ be a polynomial of degree k of the form*

$$f(y) = \frac{a}{b}y^k + \alpha_1 y^{k-1} + \ldots + \alpha_k$$

with $\gcd(a, b) = 1$. *Let τ be a positive number satisfying*

$$\tau \geq 2^{3(k-2)} \quad \text{and} \quad (\log x)^\tau < b < x^k (\log x)^{-\tau}.$$

Then, as $x \to \infty$,

$$\frac{1}{x}\sum_{n<x} e(f(n)) = O((\log x)^{-\tau}).$$

Lemma 2. *Let $f(y)$ be as in Lemma 1 and τ_0, τ arbitrary positive numbers satisfying*

$$\tau \geq 2^{6k}\tau_0 \quad \text{and} \quad (\log x)^\tau < b < x^k (\log x)^{-\tau}.$$

Then, as $x \to \infty$,

$$\frac{1}{\pi(x)}\sum_{p<x} e(f(p)) = O((\log x)^{-\tau_0}).$$

The third lemma is proved in [1] with the help of Lemmata 1 and 2 and the inequality of Erdős–Turán.

LEMMA 3. *Let* $0 < \Delta < 1$ *and*

$$U_{b,q,\Delta} := [0, \Delta] \cup \bigcup_{b=1}^{q-1} [b/q - \Delta, b/q + \Delta] \cup [1 - \Delta, 1].$$

Suppose that $P(x)$ *is an integer polynomial of degree* r *with positive leading term. Then for every* $\varepsilon > 0$ *and arbitrary* $\lambda > 0$ *we have uniformly for* $(\log_q x)^\varepsilon < j < r \log_q x - (\log_q x)^\varepsilon$ *and* $0 < \Delta < 1/(2q)$, *as* $x \to \infty$,

$$\frac{1}{x} \#\left\{ n < x \,\middle|\, \left\{ \frac{P(n)}{q^{j+1}} \right\} \in U_{b,q,\Delta} \right\} \ll \Delta + (\log x)^{-\lambda},$$
$$\frac{1}{\pi(x)} \#\left\{ p < x \,\middle|\, \left\{ \frac{P(p)}{q^{j+1}} \right\} \in U_{b,q,\Delta} \right\} \ll \Delta + (\log x)^{-\lambda}.$$

We will also make use of the following limiting relations for *centralized moments* of q-additive functions (see [1]).

LEMMA 4. *Let* f *be a* q*-additive function such that* $f(cq^j) = O(1)$ *as* $j \to \infty$ *and* $c \in E_q$ *and let* $P(x)$ *be a polynomial with integer coefficients, degree* r, *and positive leading term. Furthermore, suppose that for some* $\eta > 0$ *we have* $D_q(x^r)/(\log x)^\eta \to \infty$ *as* $x \to \infty$. *Define* f_1 *for* $n < x^r$ *by*

$$f_1(n) = \sum_{(\log_q x)^\eta \le j \le r \log_q x - (\log_q x)^\eta} f(a_{q,j}(n) q^j)$$

and set

$$M_{q,1}(x^r) := \sum_{(\log_q x)^\eta \le k \le r \log_q x - (\log_q x)^\eta} m_{k,q},$$
$$D_{q,1}^2(x^r) := \sum_{(\log_q x)^\eta \le k \le r \log_q x - (\log_q x)^\eta} \sigma_{k,q}^2.$$

Then, *as* $x \to \infty$,

$$\frac{1}{x} \# \sum_{n<x} \left(\frac{f_1(P(n)) - M_{q,1}(x^r)}{D_{q,1}(x^r)} \right)^k \to \int_{-\infty}^{\infty} z^k \, d\Phi(z),$$
$$\frac{1}{\pi(x)} \# \sum_{p<x} \left(\frac{f_1(P(p)) - M_{q,1}(x^r)}{D_{q,1}(x^r)} \right)^k \to \int_{-\infty}^{\infty} z^k \, d\Phi(z).$$

In [1] this property is only proved for $\eta = 1/3$. However, as already mentioned, it is also true for any $\eta > 0$.

PROPOSITION 1. *Let* $N_l = [\log_{q_l} x]$, $1 \le l \le d$, *let* $\lambda > 0$ *be an arbitrary constant and* h_l, $1 \le l \le d$, *be positive integers. Furthermore*, *let* $P_l(x)$, $1 \le l \le d$, *be integer polynomials with non-negative leading terms and different degrees* $r_l \ge 1$. *Then for integers*

$$(3.1)\qquad N_l^\eta \le k_1^{(l)} < k_2^{(l)} < \ldots < k_{h_l}^{(l)} \le r_l N_l - N_l^\eta \qquad (1 \le l \le d)$$

(*with some* $\eta > 0$) *we have, as* $x \to \infty$,

$$(3.2)\qquad \frac{1}{x}\#\{n < x \mid a_{q_l,k_j^{(l)}}(P_l(n)) = b_j^{(l)},\ 0 \le j \le h_l,\ 1 \le l \le d\} = \frac{1}{q_1^{h_1}\cdots q_d^{h_d}} + O((\log x)^{-\lambda})$$

and

$$(3.3)\qquad \frac{1}{\pi(x)}\#\{p < x \mid a_{q_l,k_j^{(l)}}(P_l(p)) = b_j^{(l)},\ 0 \le j \le h_l,\ 1 \le l \le d\} = \frac{1}{q_1^{h_1}\cdots q_d^{h_d}} + O((\log x)^{-\lambda})$$

uniformly for $b_j^{(l)} \in E_{q_l}$ *and* $k_j^{(l)}$ *in the given range, where the implicit constant of the error term may depend on* q_l, *on the polynomials* P_l, *on* h_l *and on* λ.

Proof. We follow [1]. Let $f_{b,q,\Delta}(x)$ be defined by

$$f_{b,q,\Delta}(x) := \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} \mathbf{1}_{[b/q,(b+1)/q]}(\{x+z\})\,dz,$$

where $\mathbf{1}_A$ is the characteristic function of the set A and $\{x\} = x - [x]$ the fractional part of x. The Fourier coefficients of the Fourier series $f_{b,q,\Delta}(x) = \sum_{m\in\mathbb{Z}} d_{m,b,q,\Delta} e(mx)$ are given by

$$d_{0,b,q,\Delta} = 1/q$$

and for $m \ne 0$ by

$$d_{m,b,q,\Delta} = \frac{e(-mb/q) - e(-m(b+1)/q)}{2\pi i m} \cdot \frac{e(m\Delta/2) - e(-m\Delta/2)}{2\pi i m\Delta}.$$

Note that $d_{m,b,q,\Delta} = 0$ if $m \ne 0$ and $m \equiv 0 \bmod q$ and that

$$|d_{m,b,q,\Delta}| \le \min\left(\frac{1}{\pi|m|}, \frac{1}{\Delta\pi m^2}\right).$$

By definition we have

$$0 \le f_{b,q,\Delta}(x) \le 1$$

and

$$f_{b,q,\Delta}(x) = \begin{cases} 1 & \text{if } x \in [b/q+\Delta, (b+1)/q - \Delta], \\ 0 & \text{if } x \in [0,1] \setminus [b/q - \Delta, (b+1)/q + \Delta]. \end{cases}$$

So if we set

$$t(y_1,\ldots,y_d) := \prod_{l=1}^{d}\prod_{j=1}^{h_l} f_{b_j^{(l)},q_l,\Delta}\left(\frac{y_l}{q_l^{k_j^{(l)}+1}}\right)$$

then for $\varDelta < 1/(2q)$ we get

$$\Big|\#\{n < x \mid a_{q_l,k_j^{(l)}}(P_l(n)) = b_j^{(l)},\ 0 \le j \le h_l,\ 1 \le l \le d\} - \sum_{n<x} t(P_1(n),\dots,P_d(n))\Big|$$

$$\le \sum_{l=1}^{d}\sum_{j=1}^{h_l} \#\left\{n < x \;\middle|\; \left\{\frac{P_l(n)}{q_l^{k_j^{(l)}+1}}\right\} \in U_{b_j^{(l)},q_l,\varDelta}\right\} \ll \varDelta x + x(\log x)^{-\lambda}$$

and

$$\Big|\#\{p < x \mid a_{q_l,k_j^{(l)}}(P_l(p)) = b_j^{(l)},\ 0 \le j \le h_l,\ 1 \le l \le d\} - \sum_{p<x} t(P_1(p),\dots,P_d(p))\Big|$$

$$\le \sum_{l=1}^{d}\sum_{j=1}^{h_l} \#\left\{n < x \;\middle|\; \left\{\frac{P_l(p)}{q_l^{k_j^{(l)}+1}}\right\} \in U_{b_j^{(l)},q_l,\varDelta}\right\} \ll \varDelta\pi(x) + \pi(x)(\log x)^{-\lambda},$$

where $U_{b_j^{(l)},q_l,\varDelta}$ is given in Lemma 3.

For convenience, let $\mathbf{m}_l = (m_1^{(l)},\dots,m_{h_l}^{(l)})$ denote h_l-dimensional integer vectors and $\mathbf{v}_l = (q_l^{-k_1^{(l)}-1},\dots,q_l^{-k_{h_l}^{(l)}-1})$, $1 \le l \le d$. Furthermore set

$$T_{\mathbf{m}_1,\dots,\mathbf{m}_d} := \prod_{l=1}^{d}\prod_{j=1}^{h_l} d_{m_j^{(l)},b_j^{(l)},q_l,\varDelta}.$$

Then $t(P_1(n),\dots,P_d(n))$ has Fourier series expansion

$$t(y_1,\dots,y_d) = \sum_{\mathbf{m}_1,\dots,\mathbf{m}_d} T_{\mathbf{m}_1,\dots,\mathbf{m}_d} e(\mathbf{m}_1\cdot\mathbf{v}_1 y_1 + \dots + \mathbf{m}_d\cdot\mathbf{v}_d y_d).$$

Thus, we are led to consider the exponential sums

$$S_1 = \sum_{\mathbf{m}_1,\dots,\mathbf{m}_d} T_{\mathbf{m}_1,\dots,\mathbf{m}_d} \sum_{n<x} e(\mathbf{m}_1\cdot\mathbf{v}_1 P_1(n) + \dots + \mathbf{m}_d\cdot\mathbf{v}_d P_d(n)), \tag{3.4}$$

$$S_2 = \sum_{\mathbf{m}_1,\dots,\mathbf{m}_d} T_{\mathbf{m}_1,\dots,\mathbf{m}_d} \sum_{p<x} e(\mathbf{m}_1\cdot\mathbf{v}_1 P_1(p) + \dots + \mathbf{m}_d\cdot\mathbf{v}_d P_d(p)). \tag{3.5}$$

Let us consider for a moment just the first sum S_1. If $\mathbf{m}_1,\dots,\mathbf{m}_d$ are all zero then

$$T_{\mathbf{m}_1,\dots,\mathbf{m}_d} \sum_{n<x} e(\mathbf{m}_1\cdot\mathbf{v}_1 P_1(n) + \dots + \mathbf{m}_d\cdot\mathbf{v}_d P_d(n)) = \frac{x + O(1)}{q_1^{h_1}\cdots q_d^{h_d}},$$

which provides the leading term. Furthermore, if there exist l and j with $m_j^{(l)} \ne 0$ and $m_j^{(l)} \equiv 0 \bmod q_l$ then $T_{\mathbf{m}_1,\dots,\mathbf{m}_d} = 0$. So it remains to consider the case where there exist l and j with $m_j^{(l)} \not\equiv 0 \bmod q_l$. Here the exponent is of the form

$$\mathbf{m}_1 \cdot \mathbf{v}_1 P_1(n) + \ldots + \mathbf{m}_d \cdot \mathbf{v}_d P_d(n) = \frac{a_1}{b_1} P_1(n) + \ldots + \frac{a_d}{b_d} P_d(n)$$

in which we assume that $\gcd(a_l, b_l) = 1$, $1 \le l \le d$. The first observation is that for any l for which there exists j with $m_j^{(l)} \not\equiv 0 \bmod q_l$ there exists $\eta_l > 0$ (only depending on q_l) such that $b_l \ge q_l^{\eta_l k_s^{(l)}}$ if $m_s^{(l)} \ne 0$, $m_s^{(l)} \not\equiv 0 \bmod q_l$ and $m_{s+1}^{(l)} = m_{s+2}^{(l)} = \ldots = m_{h_l}^{(l)} = 0$ (cf. [1]). For the reader's convenience we repeat the argument. Suppose that the prime factorization of q_l is given by $q_l = p_1^{e_1} \ldots p_k^{e_k}$. If $m_s^{(l)} \not\equiv 0 \bmod q_l$ then there exists t such that $m_s^{(l)} \not\equiv 0 \bmod p_t^{e_t}$. Now we have

$$b_l (m_s^{(l)} + q_l^{k_s^{(l)} - k_{s-1}^{(l)}} m_{s-1}^{(l)} + \ldots + q_l^{k_s^{(l)} - k_1^{(l)}} m_1^{(l)}) = a_l q_l^{k_s^{(l)} + 1}.$$

Hence $b_l \equiv 0 \bmod p_t^{k_s^{(l)} e_t}$ and consequently $b_l \ge p_t^{k_s^{(l)} e_t} \ge q_l^{\eta_l k_s^{(l)}}$. Note that we also have $b_l \le q_l^{\eta_l k_{h_l}^{(l)}}$.

Now let D denote the set of $l \in \{1, \ldots, d\}$ such that there exists j with $m_j^{(l)} \not\equiv 0 \bmod q_l$. Since all degrees r_l are different there exists a unique l_0 with $r_{l_0} = \max\{r_l \mid l \in D\}$. We now want to apply Lemma 1 with $k = r_{l_0}$ and $b = b_{l_0}$. If $k_j^{(l)}$ are in the range (3.1) then for every $\tau > 0$ there exists $x_0(\tau)$ such that for $x \ge x_0(\tau)$,

$$(\log x)^\tau < b_{l_0} < x^{r_{l_0}} (\log x)^{-\tau}.$$

Consequently, we can apply Lemma 1 to obtain

$$\begin{aligned} &\frac{1}{x} \#\{n < x \mid a_{q_l, k_j^{(l)}}(P(n)) = b_j^{(l)},\ 0 \le j \le h_l,\ 1 \le l \le d\} \\ &\qquad = \frac{1}{q_1^{h_1} \ldots q_d^{h_d}} + O\Big((\log x)^{-\lambda} \sum_{\mathbf{m} \ne \mathbf{0}} |T_{\mathbf{m}_1, \ldots, \mathbf{m}_d}|\Big) + O(\varDelta + (\log x)^{-\lambda}), \end{aligned}$$

where $\mathbf{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_d)$. Since

$$\sum_{\mathbf{m} \ne \mathbf{0}} |T_{\mathbf{m}_1, \ldots, \mathbf{m}_d}| \le (2 + 2\log(1/\varDelta))^{h_1 + \ldots + h_d}$$

it is possible to choose $\varDelta = (\log x)^{-\lambda_1}$ for a sufficiently large constant λ_1 such that (3.2) holds.

The proof of (3.3) runs along the same lines. ∎

Corollary 3. *Let $N_l = [\log_{q_l} x]$, $1 \le l \le d$, and $\lambda, \eta > 0$. Then for integers $k_j^{(l)}$ satisfying*

$$N_l^\eta \le k_j^{(l)} < r_l N_l - N_l^\eta \quad (1 \le j \le h_l,\ 1 \le l \le d)$$

and $b_j^{(l)} \in E_{q_l}$, *we uniformly have, as* $x \to \infty$,

$$\frac{1}{x}\#\{n < x \mid a_{q_l,k_j^{(l)}}(P_l(n)) = b_j^{(l)},\ 0 \le j \le h_l,\ 1 \le l \le d\}$$
$$= \prod_{l=1}^{d}\left(\frac{1}{x}\#\{n < x \mid a_{q_l,k_j^{(l)}}(P_l(n)) = b_j^{(l)},\ 0 \le j \le h_l\}\right) + O((\log x)^{-\lambda})$$

and

$$\frac{1}{\pi(x)}\#\{p < x \mid a_{q_l,k_j^{(l)}}(P_l(p)) = b_j^{(l)},\ 0 \le j \le h_l,\ 1 \le l \le d\}$$
$$= \prod_{l=1}^{d}\left(\frac{1}{\pi(x)}\#\{p < x \mid a_{q_l,k_j^{(l)}}(P_l(p)) = b_j^{(l)},\ 0 \le j \le h_l\}\right) + O((\log x)^{-\lambda}).$$

Proof. If there exist l and j_1, j_2 with $k_{j_1}^{(l)} = k_{j_2}^{(l)}$ but $b_{j_1}^{(l)} \ne b_{j_2}^{(l)}$ then both sides are zero.

So it remains to consider the case where for every l the integers $k_j^{(l)}$, $1 \le j \le h_l$, are different, and without loss of generality we can assume that they are increasing. Hence we can directly apply Proposition 1. ∎

Corollary 4. *For any choice of integers* k_l, $1 \le l \le d$, *we have, as* $x \to \infty$,

$$\frac{1}{x}\sum_{n<x}\prod_{l=1}^{d}\left(\frac{f_{l,1}(P_l(n)) - M_{q_l,1}(x^{r_l})}{D_{q_l,1}(x^{r_l})}\right)^{k_l}$$
$$-\prod_{l=1}^{d}\left(\frac{1}{x}\sum_{n<x}\left(\frac{f_{l,1}(P_l(n)) - M_{q_l,1}(x^{r_l})}{D_{q_l,1}(x^{r_l})}\right)^{k_l}\right) \to 0$$

and

$$\frac{1}{\pi(x)}\sum_{p<x}\prod_{l=1}^{d}\left(\frac{f_{l,1}(P_l(p)) - M_{q_l,1}(x^{r_l})}{D_{q_l,1}(x^{r_l})}\right)^{k_l}$$
$$-\prod_{l=1}^{d}\left(\frac{1}{\pi(x)}\sum_{p<x}\left(\frac{f_{l,1}(P_l(p)) - M_{q_l,1}(x^{r_l})}{D_{q_l,1}(x^{r_l})}\right)^{k_l}\right) \to 0.$$

Proof. In order to demonstrate how this property can be derived, we consider the case $d = 2$ and $k_1 = k_2 = 2$. Set $A_l = [(\log_{q_l} x)^\eta]$ and $B_l = [\log_{q_l} x - (\log_{q_l} x)^\eta]$ and observe that

$$f_{l,1}(P_l(n)) - M_{q_l,1}(x^{r_l}) = \sum_{j=A_l}^{B_l}\sum_{b\in E_{q_l}}\left(f_l(bq_l^j)\delta(a_{q_l,j}(P_l(n)), b) - \frac{m_{j,q_l}}{q_l}\right),$$

where $\delta(x,y)$ denotes the Kronecker delta. Hence we have

$$\frac{1}{x}\sum_{n<x}\left(\frac{f_{1,1}(P_1(n))-M_{q_1,1}(x^{r_1})}{D_{q_1,1}(x^{r_1})}\right)^2\left(\frac{f_{2,1}(P_2(n))-M_{q_2,1}(x^{r_2})}{D_{q_2,1}(x^{r_2})}\right)^2$$

$$= \sum_{j_1=A_1}^{B_1}\sum_{j_2=A_1}^{B_1}\sum_{j_3=A_2}^{B_2}\sum_{j_4=A_2}^{B_2}\sum_{b_1\in E_{q_1}}\sum_{b_2\in E_{q_1}}\sum_{b_3\in E_{q_2}}\sum_{b_4\in E_{q_2}}\frac{1}{D_{q_1,1}^2(x^{r_1})D_{q_2,1}^2(x^{r_2})}$$

$$\times\frac{1}{x}\sum_{n<x}\left(f_1(b_1q_1^{j_1})\delta(a_{q_1,j_1}(P_1(n)),b_1)-\frac{m_{j_1,q_1}}{q_1}\right)$$

$$\times\left(f_1(b_2q_1^{j_2})\delta(a_{q_1,j_2}(P_1(n)),b_2)-\frac{m_{j_2,q_1}}{q_1}\right)$$

$$\times\left(f_2(b_3q_2^{j_3})\delta(a_{q_2,j_3}(P_2(n)),b_3)-\frac{m_{j_3,q_2}}{q_2}\right)$$

$$\times\left(f_2(b_4q_2^{j_4})\delta(a_{q_2,j_4}(P_2(n)),b_4)-\frac{m_{j_4,q_2}}{q_2}\right).$$

By Corollary 3 it follows that

$$\frac{1}{x}\sum_{n<x}\left(f_1(b_1q_1^{j_1})\delta(a_{q_1,j_1}(P_1(n)),b_1)-\frac{m_{j_1,q_1}}{q_1}\right)$$

$$\times\left(f_1(b_2q_1^{j_2})\delta(a_{q_1,j_2}(P_1(n)),b_2)-\frac{m_{j_2,q_1}}{q_1}\right)$$

$$\times\left(f_2(b_3q_2^{j_3})\delta(a_{q_2,j_3}(P_2(n)),b_3)-\frac{m_{j_3,q_2}}{q_2}\right)$$

$$\times\left(f_2(b_4q_2^{j_4})\delta(a_{q_2,j_4}(P_2(n)),b_4)-\frac{m_{j_4,q_2}}{q_2}\right)$$

$$= f_1(b_1q_1^{j_1})f_1(b_2q_1^{j_2})f_2(b_3q_2^{j_3})f_2(b_4q_2^{j_4})$$

$$\times\frac{1}{x}\#\{n<x \mid a_{q_1,j_1}(P_1(n))=b_1,\ a_{q_1,j_2}(P_1(n))=b_2,$$

$$a_{q_2,j_3}(P_2(n))=b_3,\ a_{q_2,j_4}(P_2(n))=b_4\}$$

$$-f_1(b_1q_1^{j_1})f_1(b_2q_1^{j_2})f_2(b_3q_2^{j_3})$$

$$\times\frac{1}{x}\#\{n<x \mid a_{q_1,j_1}(P_1(n))=b_1, a_{q_1,j_2}(P_1(n))=b_2,\ a_{q_2,j_3}(P_2(n))=b_3\}$$

$$\times\frac{m_{j_4,q_2}}{q_2}\mp\dots+\frac{m_{j_1,q_1}}{q_1}\cdot\frac{m_{j_2,q_1}}{q_1}\cdot\frac{m_{j_3,q_2}}{q_2}\cdot\frac{m_{j_4,q_2}}{q_2}$$

$$
\begin{aligned}
&= \left(f_1(b_1 q_1^{j_1}) f_1(b_2 q_1^{j_2}) \frac{1}{x} \#\{n < x \mid a_{q_1,j_1}(P_1(n)) = b_1,\ a_{q_1,j_2}(P_1(n)) = b_2\} \right) \\
&\quad \times \Bigg(f_2(b_3 q_2^{j_3}) f_2(b_4 q_2^{j_4}) \\
&\qquad\qquad \times \frac{1}{x} \#\{n < x \mid a_{q_2,j_3}(P_2(n)) = b_3,\ a_{q_2,j_4}(P_2(n)) = b_4\} \Bigg) \\
&\quad - \Bigg(f_1(b_1 q_1^{j_1}) f_1(b_2 q_1^{j_2}) \\
&\qquad\qquad \times \frac{1}{x} \#\{n < x \mid a_{q_1,j_1}(P_1(n)) = b_1,\ a_{q_1,j_2}(P_1(n)) = b_2\} \Bigg) \\
&\quad \times \left(f_2(b_3 q_2^{j_3}) \frac{1}{x} \#\{n < x \mid a_{q_2,j_3}(P_2(n)) = b_3\} \right) \frac{m_{j_4,q_2}}{q_2} \\
&\quad \mp \ldots + \left(\frac{m_{j_1,q_1}}{q_1} \cdot \frac{m_{j_2,q_1}}{q_1} \right) \left(\frac{m_{j_3,q_2}}{q_2} \cdot \frac{m_{j_4,q_2}}{q_2} \right) + O((\log x)^{-\lambda}) \\
&= \Bigg(\frac{1}{x} \sum_{n<x} \left(f_1(b_1 q_1^{j_1}) \delta(a_{q_1,j_1}(P_1(n)), b_1) - \frac{m_{j_1,q_1}}{q_1} \right) \\
&\quad \times \left(f_1(b_2 q_1^{j_2}) \delta(a_{q_1,j_2}(P_1(n)), b_2) - \frac{m_{j_2,q_1}}{q_1} \right) \Bigg) \\
&\quad \times \Bigg(\frac{1}{x} \sum_{n<x} \left(f_2(b_3 q_2^{j_3}) \delta(a_{q_2,j_3}(P_2(n)), b_3) - \frac{m_{j_3,q_2}}{q_2} \right) \\
&\quad \times \left(f_2(b_4 q_2^{j_4}) \delta(a_{q_2,j_4}(P_2(n)), b_4) - \frac{m_{j_4,q_2}}{q_2} \right) \Bigg) \\
&\quad + O((\log x)^{-\lambda}).
\end{aligned}
$$

So we directly obtain the claimed result with an error term of the form $O((\log x)^{-\lambda+4-4\eta})$. ∎

By combining Lemma 4, Corollary 4, and the Fréchet–Shohat theorem it follows that, as $x \to \infty$,

$$
\frac{1}{x} \# \left\{ n < x \,\middle|\, \frac{f_{l,1}(P_l(n)) - M_{q_l,1}(x^{r_l})}{D_{q_l,1}(x^{r_l})} < y_l,\ 1 \le l \le d \right\} \to \Phi(y_1) \ldots \Phi(y_d),
$$

$$
\frac{1}{\pi(x)} \# \left\{ p < x \,\middle|\, \frac{f_{l,1}(P_l(p)) - M_{q_l,1}(x^{r_l})}{D_{q_l,1}(x^{r_l})} < y_l,\ 1 \le l \le d \right\} \to \Phi(y_1) \ldots \Phi(y_d).
$$

Since

$$
\begin{aligned}
M_{q_l}(x^{r_l}) - M_{q_l,1}(x^{r_l}) &= O((\log x)^{\eta}), \\
D_{q_l}(x^{r_l}) - D_{q_l,1}(x^{r_l}) &= O((\log x)^{\eta}),
\end{aligned}
$$

it also follows that

$$\max_{n<x}\left|\frac{f_l(P_l(n))-M_{q_l}(x^{r_l})}{D_{q_l}(x^{r_l})}-\frac{f_{l,1}(P_l(n))-M_{q_l,1}(x^{r_l})}{D_{q_l,1}(x^{r_l})}\right|\to 0$$

as $x\to\infty$. Consequently we finally obtain the limiting relations stated in Theorem 2.

4. Proof of Theorem 3. The proof of Theorem 3 is similar to that of Theorem 2, i.e., we will prove an analogue to Proposition 1. However, the proof requires an additional ingredient, namely a proper version of Baker's theorem on linear forms. More precisely, we will use the following version due to Waldschmidt [29].

LEMMA 5. *Let $\alpha_1,\ldots,\alpha_n$ be non-zero algebraic numbers and $b_1,\ldots,b_n$ integers such that*

$$\alpha_1^{b_1}\ldots\alpha_n^{b_n}\neq 1$$

and let $A_1,\ldots,A_n\geq e$ be real numbers with $\log A_j\geq h(\alpha_j)$, where $h(\cdot)$ denotes the absolute logarithmic height. Set $d=[\mathbb{Q}(\alpha_1,\ldots,\alpha_n):\mathbb{Q}]$. Then

$$|\alpha_1^{b_1}\ldots\alpha_n^{b_n}-1|\geq\exp(-U),$$

where

$$U=2^{6n+32}n^{3n+6}d^{n+2}(1+\log d)(\log B+\log d)\log A_1\ldots\log A_n,$$
$$B=\max\{2,|b_1|,\ldots,|b_n|\}.$$

COROLLARY 5. *Let $q_1,q_2>1$ be coprime integers and m_1, m_2 integers such that $m_1\not\equiv 0\bmod q_1$ and $m_2\not\equiv 0\bmod q_2$. Then there exists a constant $C>0$ such that for all integers $k_1,k_2>1$,*

$$\left|\frac{m_1}{q_1^{k_1}}+\frac{m_2}{q_2^{k_2}}\right|$$
$$\geq\max\left(\frac{|m_1|}{q_1^{k_1}},\frac{|m_2|}{q_2^{k_2}}\right)\cdot e^{-C\log q_1\log q_2\log(\max(k_1,k_2))\cdot\log(\max(|m_1|,|m_2|))}.$$

Proof. Since $q_1,q_2>1$ are coprime integers and $m_1\not\equiv 0\bmod q_1$, $m_2\not\equiv 0\bmod q_2$ we surely have $m_1q_1^{-k_1}+m_2q_2^{-k_2}\neq 0$. So we can apply Lemma 5 for $n=3$, $\alpha_1=q_1$, $\alpha_2=q_2$, $\alpha_3=-m_2/m_1$, $b_1=k_1$, $b_2=-k_2$, $b_3=1$ and directly obtain

$$\left|\frac{m_1}{q_1^{k_1}}+\frac{m_2}{q_2^{k_2}}\right|=|m_1|\cdot q_1^{k_1}\cdot\left|-q_1^{k_1}q_2^{-k_2}\frac{m_2}{m_1}-1\right|$$
$$\geq|m_1|q_1^{k_1}e^{-C\log q_1\log q_2\log(\max(k_1,k_2))\cdot\log\max(|m_1|,|m_2|)}.$$

Since the problem is symmetric it is no loss of generality to assume that $|m_1|q_1^{-k_1}\geq|m_2|q_2^{-k_2}$. ∎

Finally we will use the following (trivial) lemma on exponential sums.

LEMMA 6. *Let α be a real number with $0 < |\alpha| \le 1/2$. Then, as $x \to \infty$,*

$$\sum_{n<x} e(\alpha n) \ll \frac{1}{|\alpha|}.$$

PROPOSITION 2. *Let $P_l(x) = A_l x + B_l$, $l = 1,2$, be linear polynomials with integer coefficients and non-negative leading terms A_l which are coprime to q_l. Set $N_l = [\log_{q_l} x]$, $l = 1,2$, let $\lambda, \eta > 0$ be arbitrary constants and let h_1, h_2 be positive integers. Then for integers*

$$N_l^\eta \le k_1^{(l)} < k_2^{(l)} < \ldots < k_{h_l}^{(l)} \le N_l - N_l^\eta \quad (l = 1,2) \tag{4.1}$$

we have, as $x \to \infty$,

$$\frac{1}{x}\#\{n < x \mid a_{q_l,k_j^{(l)}}(A_l n + B_l) = b_j^{(l)},\ 0 \le j \le h_l,\ l = 1,2\} = \frac{1}{q_1^{h_1} q_2^{h_2}} + O((\log x)^{-\lambda}) \tag{4.2}$$

uniformly for $b_j^{(l)} \in E_{q_l}$ and $k_j^{(l)}$ in the given range, where the implicit constant of the error term may depend on q_l, h_l and λ.

Proof. The proof runs along the same lines as the proof of Proposition 1. The only problem is to estimate the sum

$$\sum_{(\mathbf{m}_1,\mathbf{m}_2)\ne\mathbf{0}} |T_{\mathbf{m}_1,\mathbf{m}_2}| \cdot \left|\frac{1}{x}\sum_{n<x} e((A_1\mathbf{m}_1\cdot\mathbf{v}_1 + A_2\mathbf{m}_2\cdot\mathbf{v}_2)n)\right|,$$

where $\mathbf{m}_l = (m_1^{(l)}, \ldots, m_{h_l}^{(l)})$ and $\mathbf{v}_l = (q_l^{-k_1^{(l)}-1}, \ldots, q_l^{-k_{h_l}^{(l)}-1})$, $l = 1,2$, such that the integers $k_j^{(l)}$ are in the given range (4.1).

First we fix $\Delta = (\log x)^{-\lambda_0}$ with an arbitrary (but fixed) constant $\lambda_0 > 0$. Furthermore, since

$$\sum_{\exists l\,\exists j:|m_j^{(l)}|>(\log x)^{2\lambda_0}} |T_{\mathbf{m}_1,\mathbf{m}_2}| \ll (\log x)^{-\lambda_0}$$

we can restrict to those $\mathbf{m} \ne \mathbf{0}$ for which $|m_j^{(l)}| \le (\log x)^{2\lambda_0}$ for all l, j and $m_j^{(l)} \not\equiv 0 \bmod q_l$ if $m_j^{(l)} \ne 0$.

We also note that it is also sufficient to consider just the case where $m_j^{(l)} \ne 0$ for all j and $l = 1,2$. (Otherwise we just reduce h_1 resp. h_2 to a smaller value and use the same arguments.)

Set $\delta = \eta/(h_1 + h_2 - 1)$. Then there exists an integer k with $0 \le k \le h_1 + h_2 - 2$ such that for all j and $l = 1,2$

$$k_{j+1}^{(l)} - k_j^{(l)} \notin [(\log x)^{k\delta}, (\log x)^{(k+1)\delta}).$$

So fix k with this property. Before discussing the general case, let us consider two extremal ones.

First suppose that

$$k_{j+1}^{(l)} - k_j^{(l)} < (\log x)^{k\delta}$$

for all j and $l = 1, 2$. Set

$$\overline{m}_l = A_l \sum_{j=1}^{h_l} m_j^{(l)} q_l^{k_{h_l}^{(l)} - k_j^{(l)}} \qquad (l = 1, 2).$$

Then we have $\overline{m}_l \not\equiv 0 \bmod q_l$ and $\log|\overline{m}_l| \ll (\log x)^{k\delta}$. Hence, we can apply Corollary 5 to

$$A_1 \mathbf{m}_1 \cdot \mathbf{v}_1 + A_2 \mathbf{m}_2 \cdot \mathbf{v}_2 = \frac{\overline{m}_1}{q_1^{k_{h_1}^{(1)}+1}} + \frac{\overline{m}_2}{q_2^{k_{h_2}^{(1)}+1}}$$

and obtain

$$|A_1 \mathbf{m}_1 \cdot \mathbf{v}_1 + A_2 \mathbf{m}_2 \cdot \mathbf{v}_2| \ge \max(q_1^{-k_{h_1}^{(1)}-1}, q_2^{-k_{h_2}^{(1)}-1}) e^{-C \log\log x\, (\log x)^{k\delta}}$$

for some constant $C > 0$. Since $|A_1 \mathbf{m}_1 \cdot \mathbf{v}_1 + A_2 \mathbf{m}_2 \cdot \mathbf{v}_2| \le 1/2$, from Lemma 6 we get

$$\begin{aligned}
&\left| \frac{1}{x} \sum_{n<x} e((A_1 \mathbf{m}_1 \cdot \mathbf{v}_1 + A_2 \mathbf{m}_2 \cdot \mathbf{v}_2)n) \right| \\
&\qquad \ll \frac{1}{x} q^{\log_q x - (\log x)^{(h_1+h_2-1)\delta}} e^{C \log\log x\, (\log x)^{k\delta}} \\
&\qquad = e^{-(\log x)^{(h_1+h_2-1)\delta}/\log q + C \log\log x\, (\log x)^{k\delta}} \ll (\log x)^{-\lambda}
\end{aligned}$$

for any given $\lambda > 0$.

Next suppose that

$$k_{j+1}^{(l)} - k_j^{(l)} \ge (\log x)^{(k+1)\delta}$$

for all j and $l = 1, 2$. Here we set $\overline{m}_l = A_l m_1^{(l)}$ $(l = 1, 2)$ and obtain

$$\begin{aligned}
&|A_1 \mathbf{m}_1 \cdot \mathbf{v}_1 + A_2 \mathbf{m}_2 \cdot \mathbf{v}_2| \\
&\qquad \ge \left| \frac{\overline{m}_1}{q_1^{k_1^{(1)}+1}} + \frac{\overline{m}_2}{q_2^{k_1^{(2)}+1}} \right| - \left| \sum_{j_1=2}^{h_1} \frac{m_{j_1}^{(1)}}{q_1^{k_{j_1}^{(1)}+1}} \right| - \left| \sum_{j_2=2}^{h_2} \frac{m_{j_2}^{(2)}}{q_2^{k_{j_2}^{(2)}+1}} \right| \\
&\qquad \ge \max(q_1^{-k_{h_1}^{(1)}-1}, q_2^{-k_{h_2}^{(1)}-1}) e^{-C(\log\log x)^2} \\
&\qquad\quad - O((\log x)^{2\lambda_0} \max(q_1^{-k_{h_1}^{(1)}-1}, q_2^{-k_{h_2}^{(1)}-1}) e^{-(\log x)^{(k+1)\delta}}) \\
&\qquad \gg \max(q_1^{-k_{h_1}^{(1)}-1}, q_2^{-k_{h_2}^{(1)}-1}) e^{-C(\log\log x)^2}.
\end{aligned}$$

Thus, we again have

$$\left|\frac{1}{x}\sum_{n<x} e((A_1\mathbf{m}_1\cdot\mathbf{v}_1 + A_2\mathbf{m}_2\cdot\mathbf{v}_2)n)\right| \ll (\log x)^{-\lambda} \tag{4.3}$$

for any given $\lambda > 0$.

In general, we assume that for some s_l $(l = 1, 2)$,

$$k_{j+1}^{(l)} - k_j^{(l)} < (\log x)^{k\delta} \qquad (j < s_l)$$

and

$$k_{s_l+1}^{(l)} - k_{s_l}^{(l)} \ge (\log x)^{(k+1)\delta}.$$

Here we set

$$\overline{m}_l = A_l \sum_{j=1}^{s_l} m_j^{(l)} q_l^{k_{s_l}^{(l)} - k_j^{(l)}} \qquad (l = 1, 2).$$

Then we have (as in the first case) $\overline{m}_l \not\equiv 0 \bmod q_l$ and $\log|\overline{m}_l| \ll (\log x)^{k\delta}$. Furthermore, we can estimate the sums

$$\sum_{j=s_l+1}^{h_l} \frac{m_j^{(l)}}{q_l^{k_j^{(l)}+1}} = O((\log x)^{2\lambda_0} q_l^{-(\log x)^{(k+1)\delta}}).$$

Thus we get

$$\begin{aligned}
&|A_1\mathbf{m}_1\cdot\mathbf{v}_1 + A_2\mathbf{m}_2\cdot\mathbf{v}_2| \\
&\qquad \ge \left|\frac{\overline{m}_1}{q_1^{k_{s_1}^{(1)}+1}} + \frac{\overline{m}_2}{q_2^{k_{s_2}^{(2)}+1}}\right| - \left|\sum_{j_1=s_1+1}^{h_1} \frac{m_{j_1}^{(1)}}{q_1^{k_{j_1}^{(1)}+1}}\right| - \left|\sum_{j_2=s_2+1}^{h_2} \frac{m_{j_2}^{(2)}}{q_2^{k_{j_2}^{(2)}+1}}\right| \\
&\qquad \ge \max(q_1^{-k_{s_1}^{(1)}-1}, q_2^{-k_{s_2}^{(1)}-1}) e^{-C\log\log x\,(\log x)^{k\delta}} \\
&\qquad\quad - O((\log x)^{2\lambda_0} \max(q_1^{-k_{s_1}^{(1)}-1}, q_2^{-k_{s_2}^{(1)}-1}) e^{-(\log x)^{(k+1)\delta}}) \\
&\qquad \gg \max(q_1^{-k_{s_1}^{(1)}-1}, q_2^{-k_{s_2}^{(1)}-1}) e^{-C\log\log x(\log x)^{k\delta}},
\end{aligned}$$

which again implies (4.3).

Hence, we finally get

$$\begin{aligned}
\sum_{(\mathbf{m}_1,\mathbf{m}_2)\ne\mathbf{0}} |T_{\mathbf{m}_1,\mathbf{m}_2}| \cdot \left|\frac{1}{x}\sum_{n<x} e((A_1\mathbf{m}_1\cdot\mathbf{v}_1 + A_2\mathbf{m}_2\cdot\mathbf{v}_2)n)\right| \\
= O((\log x)^{-\lambda_0}) + O((\log x)^{4\lambda_0-\lambda}),
\end{aligned}$$

which completes the proof of Proposition 2. ■

5. Proof of Theorem 4. The proof of Theorem 4 relies on a direct application of proper saddle point approximations.

Set

$$a_{k_1k_2} = \#\{n < x \mid s_{q_1}(n) = k_1,\ s_{q_2}(n) = k_2\}.$$

Then the *empirical characteristic function* is given by

$$\varphi_x(t_1,t_2) = \frac{1}{x}\sum_{n<x} e^{it_1 s_{q_1}(n)+it_2 s_{q_2}(n)} = \frac{1}{x}\sum_{k_1,k_2\ge 0} a_{k_1k_2} e^{it_1k_1+it_2k_2},$$

which implies that the numbers $a_{k_1k_2}$ can be determined by

$$a_{k_1k_2} = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \varphi_x(t_1,t_2)e^{-it_1k_1-it_2k_2}\,dt_1\,dt_2.$$

We first use Theorem 2 to extract the asymptotic leading term of $a_{k_1k_2}$. In fact, we need a little bit more general property.

LEMMA 7. *Set*

$$M_l(x) := \frac{q_l-1}{2}\log_{q_l} x \quad \text{and} \quad D_l(x) := \frac{q_l^2-1}{12}\log_{q_l} x$$

and let $P(x)$ denote the linear polynomial $P(x) = \mathrm{lcm}(q_1-1,q_2-1)x+B$ for some integer B with $0\le B<\mathrm{lcm}(q_1-1,q_2-1)$. Then, for every $\varepsilon>0$ there exists $x_0 = x_0(\varepsilon)$ such that

$$\left|\frac{1}{x}\sum_{n<x} e^{it_1 s_{q_1}(P(n))+it_2 s_{q_2}(P(n))} - e^{i(t_1M_{q_1}(x)+t_2M_{q_2}(x))-\frac{1}{2}(t_1^2D_{q_1}^2(x)+t_2^2D_{q_2}^2(x))}\right| < \varepsilon$$

for all $x\ge x_0$ and for all t_1,t_2 real.

Proof. First we notice that Theorem 2 cannot be directly applied. It may occur that the leading term $A = \mathrm{lcm}(q_1-1,q_2-1)$ of $P(x)$ is not coprime to q_1 resp. to q_2. However, if $A = q_l^{K_l}\overline{A}_l$ (for some $K_l>0$ and $\overline{A}_l$ coprime to q_l) and if B_l has q_l-ary expansion $B_l = B_0+B_1q_l+\ldots+B_{L_l}q_l^{L_l}$ then

$$\begin{aligned}
s_{q_l}(An+B) &= s_{q_l}(q_l^{K_l}\overline{A}_l n + B_0 + B_1q_l + \ldots + B_{L_l}q_l^{L_l})\\
&= s_{q_l}(q_l^{K_l-1}\overline{A}_l n + B_1 + B_2q_l + \ldots + B_{L_l}q_l^{L_l-1}) + B_0\\
&= s_{q_l}(q_l^{K_l-2}\overline{A}_l n + B_2 + B_3q_l + \ldots + B_{L_l}q_l^{L_l-2}) + B_0 + B_1\\
&\ \ \vdots\\
&= s_{q_l}(\overline{A}_l n + \overline{B}_l) + \overline{C}_l
\end{aligned}$$

for some integers $\overline{B}_l,\ \overline{C}_l$. Therefore, the joint (normalized) limiting distribution of $(s_{q_1}(An+B), s_{q_2}(An+B))$ is the same as that of $(s_{q_1}(\overline{A}_1n+\overline{B}_1), s_{q_2}(\overline{A}_2n+\overline{B}_2))$, and $\overline{A}_l$ is coprime to q_l, $l=1,2$. Hence, we can always apply Theorem 2 for properly chosen linear polynomials $P_l(x)$, $l=1,2$.

By Levi's theorem it now follows from Theorem 2 (and the above remark) that for every fixed t_1, t_2 we have, as $x \to \infty$,

$$(5.1) \quad \frac{1}{x}\sum_{n<x} e^{i(t_1 s_{q_1}(P(n))+t_2 s_{q_2}(P(n)))/\sqrt{\log x}} - e^{i(t_1 M_1(x)+t_2 M_{q_2}(x))/\sqrt{\log x}-\frac{1}{2}(t_1^2 D_1^2(x)+t_2^2 D_2^2(x))/(\log x)} \to 0.$$

Moreover, we can show that this convergence is uniform for all t_1, t_2. Since $\Phi(y_1)\Phi(y_2)$ is continuous we know that the *normalized empirical distribution function*

$$\widetilde{F}_x(y_1, y_2) := \frac{1}{x}\#\{n < x \mid s_{q_l}(n) \le M_l(n) + y_l D_l(x),\ l = 1, 2\}$$

converges uniformly to $\Phi(y_1)\Phi(y_2)$. Furthermore, the *variances*

$$\frac{1}{x}\sum_{n<x} \frac{(s_{q_l}(n) - M_l(n))^2}{D_l^2(x)}$$

are bounded (compare with (1.1)). Hence we get

$$\int_{\max\{|y_1|,|y_2|\}\ge A} d\widetilde{F}_x(y_1, y_2) \ll \frac{1}{A}.$$

Thus it follows by elementary means (and by using the definition of the characteristic function) that the convergence in (5.1) is uniform. ∎

The proof of Theorem 2 will also make use of the following estimate on exponential sums.

PROPOSITION 3. *Let $q_1, \ldots, q_d > 1$ be pairwise coprime integers. Then there exists a constant $c > 0$ such that for all real numbers $t_1, \ldots, t_d$,*

$$\left|\frac{1}{x}\sum_{n<x} e(t_1 s_{q_1}(n) + t_2 s_{q_2}(n) + \ldots + t_d s_{q_d}(n))\right| \ll e^{-c\log x \sum_{l=1}^{d} \|(q_l-1)t_l\|^2},$$

where $\|t\| = \min_{k\in\mathbb{Z}} |t - k|$ denotes the distance to the integers.

A proof of Proposition 3 can be found in [7]. It is, more or less, a slight generalization of a corresponding estimate of exponential sums presented by Kim [18].

Now we can start with the proof of Theorem 4.

Proof. For any $K > 0$ and integers s_1, s_2 set

$$C_K(s_1, s_2) := \left\{(t_1, t_2) \in [-\pi, \pi]^2 : \left|t_l - \frac{2\pi s_l}{q_l - 1} \bmod 2\pi\right| \le \frac{K}{\sqrt{\log x}},\ l = 1, 2\right\}.$$

Furthermore set

$$A_K := [-\pi,\pi]^2 \setminus \bigcup_{s_1=0}^{q_1-2} \bigcup_{s_2=0}^{q_2-2} C_K(s_1,s_2).$$

By Proposition 3 for every $\varepsilon > 0$ there exists $K = K(\varepsilon)$ such that

$$\frac{1}{(2\pi)^2} \int_{A_K} |\varphi_x(t_1,t_2)|\, dt_1\, dt_2 \le \frac{\varepsilon}{\log x}.$$

Furthermore, we can choose $K \le c'(-\log \varepsilon)^{1/2}$ (for some constant $c' > 0$). So it remains to consider the integrals

$$\begin{aligned} I_K(s_1,s_2) &:= \frac{1}{(2\pi)^2} \int_{C_K(s_1,s_2)} \left(\frac{1}{x} \sum_{n<x} e^{it_1(s_{q_1}(n)-k_1)+it_2(s_{q_2}(n)-k_2)} \right) dt_1\, dt_2 \\ &= e^{-2\pi i (k_1 \frac{s_1}{q_1-1} + k_2 \frac{s_2}{q_2-1})} \frac{1}{(2\pi)^2} \\ &\quad \times \int_{C_K(0,0)} \left(\frac{1}{x} \sum_{n<x} e^{it_1'(s_{q_1}(n)-k_1)+it_2'(s_{q_2}(n)-k_2)} \right) e^{2\pi i (\frac{s_1}{q_1-1} + \frac{s_2}{q_2-1})n}\, dt_1'\, dt_2'. \end{aligned}$$

By Lemma 7 it is easy to evaluate $I_K(0,0)$ asymptotically. For sufficiently large $x \ge x_0(\varepsilon)$ we have

$$|\varphi_x(t_1,t_2) - e^{i(t_1 M_1(x) + t_2 M_2(x)) - \frac12 (t_1^2 D_1^2(x) + t_2^2 D_2^2(x))}| < \varepsilon$$

for all real t_1, t_2, and consequently

$$\begin{aligned} (5.2)\qquad & I_K(0,0) \\ &= \frac{1}{(2\pi)^2} \int_{C_K(0,0)} e^{it_1(M_1(x)-k_1)+it_2(M_2(x)-k_2) - \frac12 (t_1^2 D_1^2(x) + t_2^2 D_2^2(x))}\, dt_1\, dt_2 \\ &\quad + O\left(\frac{\varepsilon K^2}{\log x} \right) \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{it_1(M_1(x)-k_1)+it_2(M_2(x)-k_2) - \frac12 (t_1^2 D_1^2(x) + t_2^2 D_2^2(x))}\, dt_1\, dt_2 \\ &\quad + O\left(\frac{\varepsilon(-\log \varepsilon)}{\log x} \right) \\ &= \prod_{l=1}^{2} \left(\frac{1}{\sqrt{2\pi} D_{q_l}(x)} \exp\left(-\frac{(k_l - M_{q_l}(x))^2}{2D_{q_l}^2(x)} \right) \right) + O\left(\frac{\varepsilon(-\log \varepsilon)}{\log x} \right). \end{aligned}$$

In order to treat the remaining integrals $I_K(s_1,s_2)$ we recall that d and A denote $d = \gcd(q_1-1, q_2-1)$ and $A = \operatorname{lcm}(q_1-1, q_2-1)$. We represent

s_1, s_2 by

$$s_l = m_l \frac{q_l - 1}{d} + r_l \qquad (0 \le m_l < d,\ 0 \le r_l < (q_l - 1)/d,\ l = 1, 2)$$

and observe that

$$\begin{aligned} \frac{s_1}{q_1 - 1} + \frac{s_2}{q_2 - 1} &= \frac{m_1 + m_2}{d} + \frac{r_1}{q_1 - 1} + \frac{r_2}{q_2 - 1} \\ &= \frac{m_1 + m_2}{d} + \frac{r_1 \frac{q_2 - 1}{d} + r_2 \frac{q_1 - 1}{d}}{A}. \end{aligned}$$

Thus, $\zeta := e^{2\pi i (\frac{s_1}{q_1 - 1} + \frac{s_2}{q_2 - 1})}$ is always an Ath root of unity and $\zeta = 1$ if and only if

$$m_1 + m_2 = d, \qquad r_1 = 0 \text{ and } r_2 = 0. \tag{5.3}$$

Thus, if (5.3) is satisfied, i.e., $s_1 = m_1 \frac{q_1 - 1}{d}$ and $s_2 = (d - m_1)\frac{q_2 - 1}{d}$, we have (recall that $k_1 \equiv k_2 \bmod d$)

$$I_K(s_1, s_2) = e^{-2\pi i \frac{m_1}{d}(k_1 - k_2)} I_K(0, 0) = I_K(0, 0).$$

Hence

$$\sum_{m_1 = 0}^{d-1} I_K\left(m_1 \frac{q_1 - 1}{d}, (d - m_1)\frac{q_2 - 1}{d}\right) = d I_K(0, 0)$$

which fits (by (5.2)) the asymptotic leading term of $a_{k_1 k_2}$.

Finally we have to consider the case where

$$\zeta = e^{2\pi i (\frac{s_1}{q_1 - 1} + \frac{s_2}{q_2 - 1})} \ne 1.$$

Here we have

$$\begin{aligned} I_K(s_1, s_2) &= e^{-2\pi i (k_1 \frac{s_1}{q_1 - 1} + k_2 \frac{s_2}{q_2 - 1})} \\ &\times \sum_{B=0}^{A-1} \zeta^B \int_{C_K(0,0)} \left(\frac{1}{x} \sum_{n' < (x - B)/A} e^{it_1'(s_{q_1}(An' + B) - k_1) + it_2'(s_{q_2}(An' + B) - k_2)} \right) dt_1'\, dt_2'. \end{aligned}$$

As above, it follows by Lemma 7 that for sufficiently large $x \ge x_1(\varepsilon)$ (and of course uniformly for all $B = 0, 1, \ldots, A - 1$)

$$\begin{aligned} &\int_{C_K(0,0)} \left(\frac{1}{x} \sum_{n' < (x - B)/A} e^{it_1'(s_{q_1}(An' + B) - k_1) + it_2'(s_{q_2}(An' + B) - k_2)} \right) dt_1'\, dt_2' \\ &\quad = \frac{1}{A} \prod_{l=1}^{2} \left(\frac{1}{\sqrt{2\pi} D_{q_l}(x)} \exp\left(-\frac{(k_l - M_{q_l}(x))^2}{2 D_{q_l}^2(x)} \right) \right) + O\left(\frac{\varepsilon \log(-\varepsilon)}{\log x} \right). \end{aligned}$$

Thus

$$I_K(s_1, s_2) = O\left(\frac{\varepsilon(-\log \varepsilon)}{\log x} \right).$$

This completes the proof of Theorem 4. ■

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References

[1] N. L. Bassily and I. Kátai, *Distribution of the values of q-additive functions on polynomial sequences*, Acta Math. Hungar. 68 (1995), 353–361.

[2] J. Bésineau, *Indépendence statistique d'ensembles liés à la fonction "sommes des chiffres"*, Acta Arith. 20 (1972), 401–416.

[3] J. Coquet, *Corrélation de suites arithmétiques*, Sém. Delange–Pisot–Poitou, 20e Année 1978/79, Exp. 15, 12 p. (1980).

[4] H. Delange, *Sur les fonctions q-additives ou q-multiplicatives*, Acta Arith. 21 (1972), 285–298.

[5] —, *Sur la fonction sommatoire de la fonction "somme des chiffres"*, Enseign. Math. 21 (1975), 31–47.

[6] M. Drmota and J. Gajdosik, *The distribution of the sum-of-digits function*, J. Théor. Nombres Bordeaux 10 (1998), 17–32.

[7] M. Drmota and G. Larcher, *The sum-of-digits function and uniform distribution modulo* 1, J. Number Theory, to appear.

[8] J. M. Dumont and A. Thomas, *Gaussian asymptotic properties of the sum-of-digits functions*, ibid. 62 (1997), 19–38.

[9] E. Fouvry et C. Mauduit, *Sommes des chiffres et nombres presque premiers*, Math. Ann. 305 (1996), 571–599.

[10] —, —, *Méthodes de crible et fonctions sommes des chiffres*, Acta Arith. 77 (1996), 339–351.

[11] A. O. Gelfond, *Sur les nombres qui ont des propriétés additives et multiplicatives données*, ibid. 13 (1968), 259–265.

[12] P. J. Grabner, P. Kirschenhofer, H. Prodinger, and R. F. Tichy, *On the moments of the sum-of-digits function*, in: Applications of Fibonacci Numbers 5, Kluwer, 1993, 263–271.

[13] P. J. Grabner, P. Liardet and R. F. Tichy, work in preparation.

[14] A. J. Hildebrand, personal communication.

[15] L. K. Hua, *Additive Theory of Prime Numbers*, Transl. Math. Monographs 13, Amer. Math. Soc., Providence, 1965.

[16] I. Kátai, *Distribution of q-additive function*, in: Probability Theory and Applications, Essays to the Memory of J. Mogyorodi, Math. Appl. 80, Kluwer, Dordrecht, 1992, 309–318.

[17] R. E. Kennedy and C. N. Cooper, *An extension of a theorem by Cheo and Yien concerning digital sums*, Fibonacci Quart. 29 (1991), 145–149.

[18] D.-H. Kim, *On the joint distribution of q-additive functions in residue classes*, J. Number Theory 74 (1999), 307–336.

[19] P. Kirschenhofer, *On the variance of the sum of digits function*, in: Lecture Notes in Math. 1452, Springer, 1990, 112–116.

[20] E. Manstavičius, *Probabilistic theory of additive functions related to systems of numerations*, in: Analytic and Probabilistic Methods in Number Theory, VSP, Utrecht, 1997, 413–430.

[21] A. Pethő and R. F. Tichy, *S-unit equations, linear recurrences and digit expansions*, Publ. Math. Debrecen 42 (1993), 145–154.

[22] H. P. Schlickewei, *S-unit equations over number fields*, Invent. Math. 102 (1990), 95–107.

[23] —, *Linear equations in integers with bounded sum of digits*, J. Number Theory 35 (1990), 335–344.

[24] J. Schmid, *The joint distribution of the binary digits of integer multiples*, Acta Arith. 43 (1984), 391–415.

[25] W. M. Schmidt, *The joint distribution of the digits of certain integer s-tuples*, in: Studies in Pure Mathematics in Memory of P. Turán, Birkhäuser, 1983, 605–622.

[26] H. G. Senge and E. G. Straus, *PV-numbers and sets of multiplicity*, Period. Math. Hungar. 3 (1973), 93–100.

[27] J. A. Solinas, *On the joint distribution of digital sums*, J. Number Theory 33 (1989), 132–151.

[28] C. L. Stewart, *On the representation of an integer in two different bases*, J. Reine Angew. Math. 319 (1980), 63–72.

[29] M. Waldschmidt, *Minorations de combinaisons linéaires de logarithmes de nombres algébriques*, Canad. J. Math. 45 (1993), 176–224.

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