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On a correspondence between p-adic Siegel–Eisenstein series and genus theta series

by

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Introduction. In a series of papers [9], [5], and [10], the second author attempted to generalize the notion of Serre's p-adic Eisenstein series and obtained several interesting formulas. For example, in the Siegel modular case [9], a correspondence between p-adic Siegel–Eisenstein series and theta series was reported. More precisely, certain p-adic Siegel–Eisenstein series were shown to coincide with the genus theta series of level p. This indicates a remarkable relationship between p-adic Siegel modular forms and Siegel modular forms on the congruence subgroup $\Gamma_0(p)$ of the so-called Nebentypus (in the sense of Hecke).

In this paper, we show that a similar phenomenon occurs for *Haupttypus*. Namely, we construct a p-adic Siegel–Eisenstein series which coincides with the genus theta series of discriminant p^2 and level p. As an application, we show that the constructed weight 2 form is congruent to a Siegel modular form of weight $p+1$ on the full Siegel modular group.

1. Definitions and notation

1.1. *Siegel modular forms.* Let $\mathbb{H}_n$ be the Siegel upper-half space of degree n; then $\Gamma^{(n)} := \mathrm{Sp}_n(\mathbb{R}) \cap M_{2n}(\mathbb{Z})$ acts discontinuously on $\mathbb{H}_n$. For a congruence subgroup Γ' of $\Gamma^{(n)}$, we denote by $M_k(\Gamma')$ the corresponding space of Siegel modular forms of weight k. Later we mainly deal with the case $\Gamma' = \Gamma^{(n)}$ or $\Gamma_0^{(n)}(N)$ where

$$\Gamma_0^{(n)}(N) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(n)} \,\middle|\, C \equiv O_n \pmod{N} \right\}.$$

In both cases, $F \in M_k(\Gamma')$ has a Fourier expansion of the form

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$$F(Z) = \sum_{0 \le T \in \Lambda_n} a_F(T) \exp[2\pi\sqrt{-1}\,\mathrm{tr}(TZ)],$$

where

$$\Lambda_n = \mathrm{Sym}_n^*(\mathbb{Z}) := \{T = (t_{ij}) \in \mathrm{Sym}_n(\mathbb{Q}) \mid t_{ii}, 2t_{ij} \in \mathbb{Z}\}$$

(the lattice in $\mathrm{Sym}_n(\mathbb{R})$ of half-integral, symmetric matrices).

Taking $q_{ij} := \exp(2\pi\sqrt{-1}\,z_{ij})$ with $Z = (z_{ij}) \in \mathbb{H}_n$, we write

$$q^T := \exp[2\pi\sqrt{-1}\,\mathrm{tr}(TZ)] = \prod_{1\le i<j\le n} q_{ij}^{2t_{ij}} \prod_{i=1}^{n} q_i^{t_i},$$

where $q_i = q_{ii}$, $t_i = t_{ii}$ $(i = 1, \dots, n)$. Using this notation, we obtain the generalized q-expansion:

$$F = \sum_{0 \le T \in \Lambda_n} a_F(T)\, q^T = \sum_{t_i} \Big(\sum_{t_{ij}} a_F(T) \prod_{i<j} q_{ij}^{2t_{ij}} \Big) \prod_{i=1}^{n} q_i^{t_i}$$
$$\in \mathbb{C}[q_{ij}^{-1}, q_{ij}][\![q_1, \dots, q_n]\!].$$

1.2. *Siegel–Eisenstein series.* Define

$$\Gamma_\infty^{(n)} := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(n)} \,\middle|\, C = O_n \right\}.$$

For an even integer $k > n + 1$, define a series by

$$E_k^{(n)}(Z) := \sum_{\binom{* \ *}{C\ D} \in \Gamma_\infty^{(n)} \backslash \Gamma^{(n)}} \det(CZ + D)^{-k}, \qquad Z \in \mathbb{H}_n.$$

This series is an element of $M_k(\Gamma^{(n)})$ called the *Siegel–Eisenstein series* of weight k for $\Gamma^{(n)}$. We write the Fourier expansion as

$$E_k^{(n)} = \sum_{0 \le T \in \Lambda_n} a_k^{(n)}(T) q^T.$$

It is known that all $a_k^{(n)}(T)$ are rational. The explicit formula for $a_k^{(n)}(T)$ has been studied by several people, for example, in [6], [7], [1] for $n = 2$, and in [4] in general. For later purposes, we introduce the explicit formula for $a_k^{(2)}(T)$ according to [1]. For simplicity, we write the Fourier expansion of $E_k^{(2)}$ as

$$E_k^{(2)} = \sum_{0 \le T \in \Lambda_2} a_k(T) q^T.$$

To describe the Fourier coefficient $a_k(T)$ explicitly, we introduce some notation.

For $0 < T \in \Lambda_2$, we write

$$-\det(2T) = D(T) f(T)^2$$

where $f(T) \in \mathbb{N}$ and $D(T)$ is the discriminant of the imaginary quadratic field $\mathbb{Q}(\sqrt{-\det(2T)})$. Moreover, we define the character $\chi_{D(T)}$ by

$$\chi_{D(T)} := \left(\frac{D(T)}{*}\right).$$

Set

$$\varepsilon(T) := \max\{l \in \mathbb{N} \mid l^{-1}T \in \Lambda_2\}.$$

It is easy to see that $\varepsilon(T) \mid f(T)$.

Proposition 1 (Eichler–Zagier [1]). *Let T be a positive-semidefinite element in Λ_2.*

(1) (1) *If* $\operatorname{rank}(T) = 2$, *then*

$$a_k(T) = \frac{-4kB_{k-1,\chi_{D(T)}}}{B_k B_{2k-2}} F_k(T),$$

$$F_k(T) = \sum_{0<d|\varepsilon(T)} d^{k-1} \sum_{0<f|\frac{f(T)}{d}} \mu(f)\chi_{D(T)}(f)f^{k-2}\sigma_{2k-3}\left(\frac{f(T)}{fd}\right),$$

where μ is the Möbius function, $\sigma_m(n) = \sum_{0<d|n} d^m$, and B_m (resp. $B_{m,\chi}$) is the mth Bernoulli (resp. generalized Bernoulli) number.

(2) *If* $\operatorname{rank}(T) = 1$, *then*

$$a_k(T) = \frac{-2k}{B_k}\sigma_{k-1}(\varepsilon(T)).$$

(3) $a_k(O_2) = 1$.

1.3. *Genus theta series.* Fix $0 < S \in \Lambda_m$ and define

$$\theta^{(n)}(S;Z) = \sum_{X \in M_{m,n}(\mathbb{Z})} \exp[2\pi\sqrt{-1}\,\operatorname{tr}(S[X]Z)], \qquad Z \in \mathbb{H}_n,$$

where $S[X] := {}^tXSX$.

Let $\{S_1, \ldots, S_h\}$ be a set of representatives of unimodular equivalence classes of the genus containing S. The *genus theta series* associated with S is defined by

$$\operatorname{genus}\Theta^{(n)}(S)(Z) := \left(\sum_{i=1}^{h} \frac{\theta^{(n)}(S_i;Z)}{E(S_i)}\right) \Big/ \left(\sum_{i=1}^{h} \frac{1}{E(S_i)}\right),$$

where $E(S_i)$ is the order of the unit group of S_i.

We write the Fourier expansion of the genus theta series as

$$\operatorname{genus}\Theta^{(n)}(S) = \sum_{0 \le T \in \Lambda_n} b^{(n)}(T)q^T.$$

The Siegel main formula for quadratic forms asserts that the Fourier coefficient $b^{(n)}(T)$ can be expressed as an infinite product of the local densities:

$$b^{(n)}(T) = \prod_{q \leq \infty} \alpha_q(S, T).$$

Here the local density $\alpha_q(S, T)$ (q: finite prime) is defined as

$$\alpha_q(S,T) = \lim_{a \to \infty} q^{a(n(n+1)/2 - mn)} A_{q^a}(S,T),$$
$$A_{q^a}(S,T) = \sharp\{X \in M_{m,n}(\mathbb{Z}/q^a\mathbb{Z}) \mid S[X] \equiv T \pmod{q^a \Lambda_n}\}.$$

The definition of the infinite part $\alpha_\infty(S, T)$ can be found in [13] and the explicit form is given as follows:

$$\alpha_\infty(S,T) = \det(S)^{-n/2} \det(T)^{(m-n-1)/2} \gamma_{mn},$$
$$\gamma_{mn} = \frac{\pi^{mn/2}}{2^{n(n-1)/2} \Gamma_n(m/2)}, \tag{1.1}$$
$$\Gamma_n(s) = \pi^{n(n-1)/4} \Gamma(s)\Gamma(s - (1/2)) \cdots \Gamma(s - ((n-1)/2)).$$

It should be noted that γ_{mn} above differs from Siegel's original formula ([13, §10, Beispiele]) by a factor of 2, because we use the lattice $\mathrm{Sym}_n^*(\mathbb{Z})$ instead of $\mathrm{Sym}_n(\mathbb{Z})$.

1.4. *p-adic Siegel–Eisenstein series.* Let $\{k_m\}_{m=1}^\infty$ be an increasing sequence of even positive integers which is p-adically convergent. If the corresponding sequence of Siegel–Eisenstein series

$$\{E_{k_m}^{(n)}\} \subset \mathbb{Q}[q_{ij}^{-1}, q_{ij}][\![q_1, \ldots, q_n]\!]$$

converges p-adically to an element of $\mathbb{Q}_p[q_{ij}^{-1}, q_{ij}][\![q_1, \ldots, q_n]\!]$, then we call the limit $\lim_{m \to \infty} E_{k_m}^{(n)}$ a *p-adic Siegel–Eisenstein series.*

2. Main result. Let p be an odd prime number. Then there exists a positive definite, integral, quaternary quadratic form $S^{(p)}$ of discriminant p^2 and level p:

$$0 < S^{(p)} \in \Lambda_4, \quad \det(2S^{(p)}) = p^2, \quad p(2S^{(p)})^{-1} \in 2\,\mathrm{Sym}_4^*(\mathbb{Z}).$$

THEOREM 1. *Let p be an odd prime and $S^{(p)}$ be as above. If we define the sequence $\{k_m\}$ by*

$$k_m = k_m(p) := 2 + (p-1)p^{m-1},$$

then the corresponding sequence of Siegel–Eisenstein series $\{E_{k_m}^{(2)}\}$ defines a p-adic Siegel–Eisenstein series. Moreover,

$$\lim_{m \to \infty} E_{k_m}^{(2)} = \mathrm{genus}\, \Theta^{(2)}(S^{(p)}). \tag{2.1}$$

In particular, the p-adic Siegel–Eisenstein series $\lim_{m\to\infty} E_{k_m}^{(2)}$ *becomes a "true" Siegel modular form of weight* 2 *on* $\Gamma_0^{(2)}(p)$ *of Haupttypus.*

3. Proof of the main theorem. We shall prove identity (2.1) of Theorem 1 by showing that corresponding Fourier coefficients on each side are equal. Let

$$E_{k_m}^{(2)} = \sum_{0\le T\in\Lambda_2} a_{k_m}(T)q^T$$

be the Fourier expansion of the Siegel–Eisenstein series $E_{k_m}^{(2)}$ (cf. §1.2). We shall show that the sequence $\{a_{k_m}(T)\}_{m=1}^{\infty} \subset \mathbb{Q}$ has a limit in $\mathbb{Q}$:

$$\lim_{m\to\infty} a_{k_m}(T) =: \widetilde{a}(T) \in \mathbb{Q}.$$

As before (cf. §1.3), we write the Fourier expansion of the genus theta series $\operatorname{genus}\Theta(S^{(p)})$ as

$$\operatorname{genus}\Theta(S^{(p)}) = \operatorname{genus}\Theta^{(2)}(S^{(p)}) = \sum_{0\le T\in\Lambda_2} b(T)q^T.$$

The proof of our theorem is reduced to showing that the identity

$$\widetilde{a}(T) = b(T) \tag{3.1}$$

holds for every $T \in \Lambda_2$.

3.1. *An explicit formula for* $\widetilde{a}(T)$

Proposition 2. *Let* T *be a positive-semidefinite element in* Λ_2.

(1) *If* $\operatorname{rank}(T) = 2$, *then*

$$\widetilde{a}(T) = \frac{-288}{(1-p)^2}(1-\chi_{D(T)}(p))B_{1,\chi_{D(T)}}\widetilde{F}(T),$$

$$\widetilde{F}(T) = \sum_{\substack{0<d\mid\varepsilon(T)\\(d,p)=1}} d \sum_{\substack{0<f\mid\frac{f(T)}{d}\\(f,p)=1}} \mu(f)\chi_{D(T)}(f)\sigma_1^*\left(\frac{f(T)}{fd}\right),$$

where $\varepsilon(T)$ *and* $f(T)$ *are positive integers defined in* §1.2 *and*

$$\sigma_1^*(m) = \sum_{\substack{0<d\mid m\\(d,p)=1}} d.$$

(2) *If* $\operatorname{rank}(T) = 1$, *then*

$$\widetilde{a}(T) = \frac{24}{p-1}\sigma_1^*(\varepsilon(T)) = \frac{24}{p-1}\sum_{\substack{0<d\mid\varepsilon(T)\\(d,p)=1}} d.$$

(3) $\widetilde{a}(O_2) = 1$.

Proof. We need to show (1) and (2). First we assume that $\mathrm{rank}(T) = 2$. By Proposition 1(1), it follows that

$$a_{k_m}(T) = \frac{-4k_m B_{k_m-1,\chi_{D(T)}}}{B_{k_m}B_{2k_m-2}}\, F_{k_m}(T),$$
$$F_{k_m}(T) = \sum_{0<d|\varepsilon(T)} d^{k_m-1} \sum_{0<f|\frac{f(T)}{d}} \mu(f)\chi_{D(T)}(f) f^{k_m-2}\sigma_{2k_m-3}\left(\frac{f(T)}{fd}\right).$$

By Kummer's congruence for the Bernoulli numbers, we obtain

$$(1-p^{k_m-1})\,\frac{B_{k_m}}{k_m} \equiv (1-p)\,\frac{B_2}{2} \pmod{p^m},$$
$$(1-p^{2k_m-3})\,\frac{B_{2k_m-2}}{2k_m-2} \equiv (1-p)\,\frac{B_2}{2} \pmod{p^m}.$$

We consider the limit of the generalized Bernoulli number. By Corollary 5 of [2], we obtain

$$\frac{(1-\chi_{D(T)}(p)p^{k_m-2})B_{k_m-1,\chi_{D(T)}} - B_{0,\chi_{D(T)}\omega}}{k_m-1} \equiv (1-\chi_{D(T)}(p))B_{1,\chi_{D(T)}} - B_{0,\chi_{D(T)}\omega} \pmod{p^m},$$

where ω is the Teichmüller character. Since

$$B_{0,\chi} = \begin{cases} 0 & \text{if } \chi \neq \chi_0, \\ \varphi(n)/n & \text{if } \chi = \chi_0 \end{cases}$$

for a character χ modulo n in general, we have

$$(1-\chi_{D(T)}(p)p^{k_m-2})\,\frac{B_{k_m-1,\chi_{D(T)}}}{k_m-1} \equiv (1-\chi_{D(T)}(p))B_{1,\chi_{D(T)}} \pmod{p^{m-\delta}},$$

where

$$\delta = \delta(m,p,\chi_{D(T)}) := \begin{cases} 3 & \text{if } p=3,\ \chi_{D(T)} = \chi_{-3}, \text{ and } m=1, \\ 2 & \text{if } p=3,\ \chi_{D(T)} = \chi_{-3}, \text{ and } m\geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

In any case, we have

$$\lim_{m\to\infty} B_{k_m-1,\chi_{D(T)}} = (1-\chi_{D(T)}(p))B_{1,\chi_{D(T)}}.$$

Combining these congruences, we get

$$\lim_{m\to\infty} \frac{-4k_m B_{k_m-1,\chi_{D(T)}}}{B_{k_m}B_{2k_m-2}} = \frac{-8}{\{(1-p)^2B_2\}^2}\,(1-\chi_{D(T)}(p))\,B_{1,\chi_{D(T)}}$$
$$= \frac{-288}{(1-p)^2}\,(1-\chi_{D(T)}(p))B_{1,\chi_{D(T)}}.$$

Using Euler's congruence, we obtain

$$\lim_{m\to\infty} F_{k_m}(T)$$
$$= \lim_{m\to\infty} \sum_{0<d|\varepsilon(T)} d^{k_m-1} \sum_{0<f|\frac{f(T)}{d}} \mu(f)\chi_{D(T)}(f) f^{k_m-2}\sigma_{2k_m-3}\left(\frac{f(T)}{fd}\right)$$
$$= \sum_{\substack{0<d|\varepsilon(T)\\(d,p)=1}} d \sum_{\substack{0<f|\frac{f(T)}{d}\\(f,p)=1}} \mu(f)\chi_{D(T)}(f)\sigma_1^*\left(\frac{f(T)}{fd}\right).$$

This completes the proof of (1).

(2) If $\mathrm{rank}(T) = 1$, then T is unimodular equivalent to the matrix $\left(\begin{smallmatrix}\varepsilon(T) & 0\\ 0 & 0\end{smallmatrix}\right)$. Therefore

$$a_{k_m}(T) = \frac{-2k_m}{B_{k_m}}\,\sigma_{k_m-1}(\varepsilon(T)).$$

It follows that

$$\widetilde{a}(T) = \lim_{m\to\infty} a_{k_m}(T) = \lim_{m\to\infty}\left(\frac{-2k_m}{B_{k_m}}\right)\lim_{m\to\infty}\sigma_{k_m-1}(\varepsilon(T))$$
$$= \frac{-4}{(1-p)B_2}\,\sigma_1^*(\varepsilon(T)) = \frac{24}{p-1}\,\sigma_1^*(\varepsilon(T)).$$

This completes the proof of Proposition 2. ∎

3.2. *An explicit formula for* $b(T)$. As stated in §1.3, the coefficient $b(T)$ can be expressed as the product of local densities. In our case,

$$b(T) = \prod_{q\le\infty} \alpha_q(S^{(p)},T) = \prod_{q\,\text{prime}} \alpha_q(S^{(p)},T)\cdot\alpha_\infty(S^{(p)},T).$$

We calculate $\alpha_q(S^{(p)},T)$ by using formulas of Kaufhold [6] and Yang [14]. In particular, Yang's formula [14, Theorem 7.1] plays an essential role in our calculation.

Proposition 3. *Let T be a positive-semidefinite element in Λ_2.*

(1) *If* $\mathrm{rank}(T) = 2$, *then*

$$\alpha_q(S^{(p)},T) = \begin{cases} \dfrac{(1-q^{-2})^2}{1-\chi_{D(T)}(q)q^{-1}}\displaystyle\sum_{l=0}^{\varepsilon_q}\Big(\sum_{m=0}^{f_q-l} q^{-m} - \chi_{D(T)}(q)q^{-1}\sum_{m=0}^{f_q-l-1} q^{-m}\Big) & \text{if } q\neq p,\\ \dfrac{1-\chi_{D(T)}(p)}{1-\chi_{D(T)}(p)p^{-1}}\,\dfrac{(p+1)^2}{p^{f_p+2}} & \text{if } q = p,\\ 2^3\pi^3p^{-2}|D(T)|^{1/2}f(T) & \text{if } q=\infty, \end{cases}$$

where $\varepsilon_q := \mathrm{ord}_q(\varepsilon(T))$ *and* $f_q := \mathrm{ord}_q(f(T))$.

(2) *If* $\mathrm{rank}(T) = 1$, *then*

$$\alpha_q(S^{(p)}, T) = \begin{cases} (1-q^{-2}) \displaystyle\sum_{l=0}^{\varepsilon_q} q^{-l} & \text{if } q \neq p, \\ \dfrac{1+p}{p^{1+\varepsilon_p}} & \text{if } q = p, \\ 2^2\, p^{-1} \varepsilon(T)\, \pi^2 & \text{if } q = \infty. \end{cases}$$

Proof. (1) Assume that $\mathrm{rank}(T) = 2$. If $q \neq p$, then we can calculate $\alpha_q(S^{(p)}, T)$ by using Kaufhold's formula for the Siegel series $b_q(T) = b_q(s, T)$ (cf. [6, Hilfssatz 10] for the case $|T| \neq 0$):

$$\alpha_q(S^{(p)}, T) = b_q(s, T)|_{s=2}.$$

Hence we have

$$\alpha_q(S^{(p)}, T) = \frac{(1-q^{-2})^2}{1-\chi_{D(T)}(q)q^{-1}} \sum_{l=0}^{\varepsilon_q} \Big(\sum_{m=0}^{f_q-l} q^{-m} - \chi_{D(T)}(q) q^{-1} \sum_{m=0}^{f_q-l-1} q^{-m} \Big).$$

It is essential in the proof to treat the case $q = p$. We can calculate $\alpha_p(S^{(p)}, T)$ by using Yang's formula ([14, Theorem 7.1]). We should remark that there are minor misprints and typographical errors in his original published formula, which he subsequently revised. We make use of the revised version.

Yang's result asserts that $\alpha_p(S^{(p)}, T)$ is essentially expressed by twelve values: $I_{1,i}$ $(1 \leq i \leq 4)$ and $I_{2,i}$ $(1 \leq i \leq 8)$, and

$$\alpha_p(S^{(p)}, T) = 1 + \sum_{i=1}^{4} I_{1,i} + (1-p^{-1}) \sum_{i=1}^{8} I_{2,i} + p^{-1} I_{2,6}$$

([14, Theorem 7.1]). We assume that T is unimodular equivalent to

$$\begin{pmatrix} \alpha_1 p^a & 0 \\ 0 & \alpha_2 p^b \end{pmatrix} \qquad (a \leq b,\ \alpha_1, \alpha_2 \in \mathbb{Z}_p^*).$$

The proof for the case $q = p$ is reduced to the following:

Lemma 3.1.
(a) *If* $a \not\equiv b \pmod 2$, *then*

$$\alpha_p(S^{(p)}, T) = (1+p)^2 p^{-(a+b+3)/2} = \frac{(1+p)^2}{p^{2+f_p}}.$$

(b) *If* $a \equiv b \pmod 2$, *then*

$$\alpha_p(S^{(p)}, T) = (1-\chi_{D(T)}(p))(1+p)p^{-(a+b+2)/2} = (1-\chi_{D(T)}(p))\, \frac{1+p}{p^{1+f_p}}.$$

Proof. According to Yang's result, we can calculate $\alpha_p(S^{(p)}, T)$ by separately considering the following three cases:

(i) $a = b$,
(ii) $a < b$ and $a \not\equiv b \pmod 2$,
(iii) $a < b$ and $a \equiv b \pmod 2$.

In fact, the calculations are similar for each case, so we shall prove the formula only for case (iii). In this case, the values $I_{i,j}$ are as follows:

$$I_{1,1} = -(1-p^{-2})ap, \qquad I_{1,2} = p^{-1},$$

$$I_{1,3} = -(1-p^{-1}) \sum_{k=a+2}^{b} g_a(k) p^{(a-k+2)/2}, \quad I_{1,4} = -\left(\frac{-\alpha_1\alpha_2}{p}\right) p^{(a-b)/2},$$

$$I_{2,1} = (1-p^{-2}) \sum_{k=1}^{a-1} k\, p^{2-a+k}, \qquad I_{2,2} = -\sum_{k=1}^{a} p^{-k},$$

$$I_{2,3} = (1-p^{-1}) \sum_{k_2=1}^{a} \sum_{k_1=a+2}^{b} g_a(k_1) p^{(a-k_1-2k_2+4)/2},$$

$$I_{2,4} = \left(\frac{-\alpha_1\alpha_2}{p}\right) \sum_{k=1}^{a} p^{(a-b-2k+2)/2},$$

$$I_{2,5} = -\sum_{k=a+2}^{b} g_a(k) p^{-(a+k)/2}, \qquad I_{2,6} = -\left(\frac{-\alpha_1\alpha_2}{p}\right) p^{-(a+b+2)/2},$$

$$I_{2,7} = 0, \qquad I_{2,8} = \sum_{k=1}^{a} p^{2-k},$$

where $g_a(k)$ is 1 if $k-a$ is even, and 0 if $k-a$ is odd. We can simplify the above formulas:

$$I_{1,1} = -ap + ap^{-1}, \qquad I_{1,2} = p^{-1},$$

$$I_{1,3} = -1 + p^{(a-b)/2}, \qquad I_{1,4} = -\left(\frac{-\alpha_1\alpha_2}{p}\right) p^{(a-b)/2},$$

$$(1-p^{-1})I_{2,1} = -1 + ap - p + p^{1-a} - ap^{-1} + p^{-a},$$

$$(1-p^{-1})I_{2,2} = -p^{-1} + p^{-1-a},$$

$$(1-p^{-1})I_{2,3} = 1 - p^{(a-b)/2} - p^{-a} + p^{-(a+b)/2},$$

$$(1-p^{-1})I_{2,4} = \left(\frac{-\alpha_1\alpha_2}{p}\right) p^{(a-b)/2} - \left(\frac{-\alpha_1\alpha_2}{p}\right) p^{-(a+b)/2},$$

$$(1-p^{-1})I_{2,5} = -p^{-1-a} + p^{-(a+b+2)/2},$$

$$(1-p^{-1})I_{2,6} + p^{-1}I_{2,6} = I_{2,6} = -\left(\frac{-\alpha_1\alpha_2}{p}\right) p^{-(a+b+2)/2},$$

$$(1-p^{-1})I_{2,7} = 0, \qquad (1-p^{-1})I_{2,8} = p - p^{1-a}.$$

From these formulas, we have

$$\begin{aligned}\alpha_p(S^{(p)},T) &= 1+\sum_{i=1}^{4} I_{1,i} + (1-p^{-1})\sum_{i=1}^{8} I_{2,i} + p^{-1}I_{2,6}\\ &= p^{-(a+b)/2}+p^{-(a+b+2)/2}-\left(\frac{-\alpha_1\alpha_2}{p}\right)p^{-(a+b)/2}\\ &\quad -\left(\frac{-\alpha_1\alpha_2}{p}\right)p^{-(a+b+2)/2}\\ &= \left(1-\left(\frac{-\alpha_1\alpha_2}{p}\right)\right)(p^{-(a+b)/2}+p^{-(a+b+2)/2})\\ &= \left(1-\left(\frac{-\alpha_1\alpha_2}{p}\right)\right)\frac{1+p}{p^{1+(a+b)/2}}\\ &= (1-\chi_{D(T)}(p))\,\frac{1+p}{p^{1+f_p}},\end{aligned}$$

since $\left(\frac{-\alpha_1\alpha_2}{p}\right)=\chi_{D(T)}(p)$ and $f_p=(a+b)/2$. This proves the lemma for case (iii). ■

We continue the proof of Proposition 3(1). We calculate $\alpha_\infty(S^{(p)},T)$. By the general formula for $\alpha_\infty(S,T)$ (cf. (1.1) of §1.3), we have

$$\alpha_\infty(S^{(p)},T)=\det(S^{(p)})^{-1}\det(T)^{1/2}\,\frac{\pi^4}{2\Gamma(2)}=2^3\pi^3p^{-2}|D(T)|^{1/2}f(T).$$

Here we made use of the identities

$$\det(S^{(p)})=2^{-4}p^2,\qquad \det(T)=2^{-2}|D(T)|f(T)^2.$$

This completes the proof of Proposition 3(1).

(2) Next we assume that $\mathrm{rank}(T)=1$. We recall that T is unimodular equivalent to $\left(\begin{smallmatrix}\varepsilon(T) & 0\\ 0 & 0\end{smallmatrix}\right)$. If $q\neq p$, then we have

$$\alpha_q(S^{(p)},T)=\alpha_q(S^{(p)},\varepsilon(T))=(1-q^{-2})\sum_{l=0}^{\varepsilon_q}q^{-l}$$

(e.g. cf. [13, Hilfssatz 16]). It is also essential here to deal with the case $q=p$. We use Yang's formula again. In his notation, $\alpha_p(S^{(p)},T)$ is given by

$$\begin{aligned}\alpha_p(S^{(p)},T)&=\alpha_p(S^{(p)},\varepsilon(T))\\ &= 1+(1-p^{-1})\sum_{k=1}^{\varepsilon_p}v_kp^{d(k)}+f_1(\varepsilon(T))v_{\varepsilon_p+1}p^{d(\varepsilon_p+1)}\end{aligned}$$

(cf. [14, p. 317]). Since

$$v_k=-1,\qquad d(k)=-k+1\quad(\text{for } k\geq 1),\qquad f_1(\varepsilon(T))=-p^{-1}$$

in our case, we obtain

$$\alpha_p(S^{(p)},T) = 1-(1-p^{-1})\sum_{k=1}^{\varepsilon_p} p^{-k+1} + p^{-1}p^{-\varepsilon_p} = p^{-\varepsilon_p} + p^{-1-\varepsilon_p} = \frac{1+p}{p^{1+\varepsilon_p}}.$$

Finally, we calculate $\alpha_\infty(S^{(p)},T)$. Again by (1.1) of §1.3,

$$\alpha_\infty(S^{(p)},T) = \alpha_\infty(S^{(p)},\varepsilon(T)) = (2^{-4}p^2)^{-1/2}\varepsilon(T)\,\frac{\pi^2}{\Gamma(2)} = 2^2p^{-1}\varepsilon(T)\pi^2.$$

This completes the proof of Proposition 3. ∎

Corollary 3.1.

(1) *If* $\operatorname{rank}(T)=2$, *then*

$$b(T) = \frac{-288}{(p-1)^2}\,(1-\chi_{D(T)}(p))\,B_{1,\chi_{D(T)}} f_*(T) \prod_{\substack{q\,prime\\ q\neq p}} G_q(T),$$

$$G_q(T) = \sum_{l=0}^{\varepsilon_q}\Big(\sum_{m=0}^{f_q-l} q^{-m} - \chi_{D(T)}(q)q^{-1}\sum_{m=0}^{f_q-l-1} q^{-m}\Big), \qquad f_*(T) := f(T)/p^{f_p}.$$

(2) *If* $\operatorname{rank}(T)=1$, *then*

$$b(T) = \frac{24}{p-1}\,\sigma_1^*(\varepsilon(T)).$$

(3) $b(O_2)=1$.

Proof. (1) We substitute the formulas for α_q obtained in Proposition 3 into

$$b(T) = \prod_{q\le\infty} \alpha_q(S^{(p)},T).$$

Consequently,

$$\begin{aligned} b(T) &= \prod_{\substack{q\,\text{prime}\\ q\neq p}} \alpha_q(S^{(p)},T)\alpha_p(S^{(p)},T)\alpha_\infty(S^{(p)},T)\\ &= \frac{L(1;\chi_{D(T)})}{\zeta(2)^2}\,\frac{1-\chi_{D(T)}(p)p^{-1}}{(1-p^{-2})^2}\prod_{q\neq p} G_q(T)\\ &\quad\times \frac{1-\chi_{D(T)}(p)}{1-\chi_{D(T)}(p)p^{-1}}\,\frac{(p+1)^2}{p^{f_p+2}}\,2^3\pi^3p^{-2}|D(T)|^{1/2}f(T)\\ &= \frac{-288}{(p-1)^2}\,(1-\chi_{D(T)}(p))\,B_{1,\chi_{D(T)}}(p^{-f_p}f(T))\prod_{q\neq p} G_q(T). \end{aligned}$$

Here we made use of the formulas

$$L(1;\chi_{D(T)}) = -\pi|D(T)|^{1/2}B_{1,\chi_{D(T)}}, \qquad \zeta(2)=\pi^2/6.$$

(2) From Proposition 3(2), we obtain

$$\begin{aligned} b(T) &= \prod_{\substack{q \text{ prime} \\ q\neq p}} \alpha_q(S^{(p)},T)\alpha_p(S^{(p)},T)\alpha_\infty(S^{(p)},T) \\ &= \frac{1}{\zeta(2)}\,\frac{1}{1-p^{-2}}\prod_{q\neq p}\Bigl(\sum_{l=0}^{\varepsilon_q} q^{-l}\Bigr)\frac{1+p}{p^{1+\varepsilon_p}}\,2^2p^{-1}\varepsilon(T)\pi^2 \\ &= \frac{24}{p-1}(p^{-\varepsilon_p}\varepsilon(T))\prod_{q\neq p}\Bigl(\sum_{l=0}^{\varepsilon_q} q^{-l}\Bigr) = \frac{24}{p-1}\,\sigma_1^*(\varepsilon(T)). \end{aligned}$$

(3) The identity $b(O_2) = 1$ is an easy consequence of the definition of the genus theta series. ∎

3.3. *Coincidence between* $\widetilde{a}(T)$ *and* $b(T)$. Comparing Proposition 2 and Corollary 3.1, we see that $\widetilde{a}(T) = b(T)$ for $\mathrm{rank}(T) \leq 1$. It remains to prove this in the case where $\mathrm{rank}(T) = 2$. The proof for this case is reduced to showing the following lemma.

Lemma 3.2. *Recall that*

$$\widetilde{F}(T) = \sum_{\substack{0<d\mid\varepsilon(T)\\(d,p)=1}} d \sum_{\substack{0<f\mid\frac{f(T)}{d}\\(f,p)=1}} \mu(f)\chi_{D(T)}(f)\sigma_1^*\Bigl(\frac{f(T)}{fd}\Bigr),$$

$$G_q(T) = \sum_{l=0}^{\varepsilon_q}\Bigl(\sum_{m=0}^{f_q-l} q^{-m} - \chi_{D(T)}(q)q^{-1}\sum_{m=0}^{f_q-l-1} q^{-m}\Bigr)$$

(*cf. Proposition* 2(1) *and Corollary* 3.1(1)). *With these definitions*,

$$\widetilde{F}(T) = f_*(T)\prod_{q\neq p} G_q(T). \tag{3.2}$$

Proof. We remark that $\widetilde{F}(T)$ has a finite product expression of the form

$$\widetilde{F}(T) = \prod_{q\neq p}\sum_{0<d\mid q^{\varepsilon_q}} d \sum_{0<f\mid q^{f_q-d_q}} \mu(f)\chi_{D(T)}(f)\sigma_1\Bigl(\frac{q^{f_q}}{fd}\Bigr),$$

where $d = \prod q^{d_q}$. Therefore the proof of (3.2) is reduced to showing that

$$\begin{aligned} &\sum_{0<d\mid q^{\varepsilon_q}} d \sum_{0<f\mid q^{f_q-d_q}} \mu(f)\chi_{D(T)}(f)\sigma_1\Bigl(\frac{q^{f_q}}{fd}\Bigr) \\ &\qquad\qquad = q^{f_q}\sum_{l=0}^{\varepsilon_q}\Bigl(\sum_{m=0}^{f_q-l} q^{-m} - \chi_{D(T)}(q)q^{-1}\sum_{m=0}^{f_q-l-1} q^{-m}\Bigr) \end{aligned}$$

for each prime $q \neq p$. To see this, we show the equality of terms corresponding to $d = q^{d_q}$ (on the left hand side) and $l = d_q$ (on the right hand side) for each $0 \leq d_q \leq \varepsilon_q$:

$$(3.3)\qquad q^{d_q} \sum_{0<f|q^{f_q-d_q}} \mu(f)\chi_{D(T)}(f)\sigma_1\left(\frac{q^{f_q-d_q}}{f}\right) = q^{f_q}\Big(\sum_{m=0}^{f_q-d_q} q^{-m} - \chi_{D(T)}(q)q^{-1}\sum_{m=0}^{f_q-d_q-1} q^{-m}\Big).$$

Since $\mu(q^2) = \mu(q^3) = \cdots = 0$, the left hand side of (3.3) is equal to

$$\begin{aligned} &q^{d_q}\mu(1)\chi_{D(T)}(1)\sigma_1(q^{f_q-d_q}) + q^{d_q}\mu(q)\chi_{D(T)}(q)\sigma_1(q^{f_q-d_q-1}) \\ &\quad = q^{d_q}\,\frac{q^{f_q-d_q+1}-1}{q-1} - \chi_{D(T)}(q)q^{d_q}\,\frac{q^{f_q-d_q}-1}{q-1} \\ &\quad = \frac{q^{f_q+1}-q^{d_q}}{q-1} - \chi_{D(T)}(q)\,\frac{q^{f_q}-q^{d_q}}{q-1}. \end{aligned}$$

On the other hand, the right hand side of (3.3) becomes

$$\begin{aligned} &q^{f_q}\,\frac{1-q^{-(f_q-d_q+1)}}{1-q^{-1}} - \chi_{D(T)}(q)q^{f_q-1}\,\frac{1-q^{-(f_q-d_q)}}{1-q^{-1}} \\ &\quad = \frac{q^{f_q+1}-q^{d_q}}{q-1} - \chi_{D(T)}(q)\,\frac{q^{f_q}-q^{d_q}}{q-1}. \end{aligned}$$

This proves (3.3) and thus Lemma 3.2 is now proved. ■

We have now completed the proof of Theorem 1. ■

4. Remarks

4.1. *Modular forms of weight 2.* In general, we denote by $M_k(\Gamma')_R$ the subset of $M_k(\Gamma')$ consisting of modular forms whose Fourier coefficients belong to a subring $R \subset \mathbb{C}$.

In [12], Serre proved the following result:

THEOREM 2 (Serre). *Let $p \geq 3$ be a prime number. For any $f \in M_2(\Gamma_0^{(1)}(p))_{\mathbb{Z}_{(p)}}$, there exists a modular form $g \in M_{p+1}(\Gamma^{(1)})_{\mathbb{Z}_{(p)}}$ satisfying*

$$f \equiv g \pmod p.$$

It is believed that this is true for any degree.

CONJECTURE. Assume that p is a sufficiently large prime relative to the degree. For any $F \in M_2(\Gamma_0^{(n)}(p))_{\mathbb{Z}_{(p)}}$, there exists a modular form $G \in M_{p+1}(\Gamma^{(n)})_{\mathbb{Z}_{(p)}}$ satisfying

$$F \equiv G \pmod p.$$

Our p-adic Siegel–Eisenstein series has the following property.

PROPOSITION 4. *Let $p \geq 3$ be a prime number. The constructed p-adic Siegel–Eisenstein series $F_2(p) := \lim_{m\to\infty} E^{(2)}_{k_m} \in M_2(\Gamma^{(2)}_0(p))$ satisfies the congruence*

$$F_2(p) \equiv E^{(2)}_{p+1} \pmod{p}$$

where $E^{(2)}_{p+1} \in M_{p+1}(\Gamma^{(2)})$ is the ordinary Siegel–Eisenstein series of weight $p+1$ for $\Gamma^{(2)}$.

Proof. For $p = 3$, the congruence may be checked by direct calculation. In fact, we can show that

$$\lim_{m\to\infty} E^{(2)}_{k_m} \equiv 1 \pmod{3} \quad \text{and} \quad E^{(2)}_{p+1} = E^{(2)}_4 \equiv 1 \pmod{3}.$$

Therefore, we may assume that p is prime and strictly greater than 3. By an argument similar to that in the proof of Proposition 2, we can show that

$$\lim_{m\to\infty} E^{(2)}_{k_m} \equiv E^{(2)}_{k_l} \pmod{p^l}.$$

As a special case, we obtain

$$F_2(p) = \lim_{m\to\infty} E^{(2)}_{k_m} \equiv E^{(2)}_{k_1} = E^{(2)}_{p+1} \pmod{p}. \blacksquare$$

4.2. *Comparison with the case of Nebentypus.* In the case of Nebentypus [9], we considered the p-adic Siegel–Eisenstein series

$$F_1(p) := \lim_{m\to\infty} E^{(n)}_{1+\frac{p-1}{2}\cdot p^{m-1}}.$$

One of the main results of [9] is as follows. Let p be a prime with $p \equiv 3 \pmod{4}$, $p > 3$. Then $F_1(p)$ coincides with the genus theta series of level p. In particular, $F_1(p)$ becomes a modular form of $M_1(\Gamma^{(n)}_0(p), \chi_p)$ (the space of modular forms of weight 1 and Nebentypus χ_p). As a consequence, the square $(F_1(p))^2 \in M_2(\Gamma^{(n)}_0(p))$ satisfies

$$(F_1(p))^2 \equiv (E^{(n)}_{(p+1)/2})^2 \pmod{p}.$$

4.3. *Generalization.* In this note, we proved

$$\lim_{m\to\infty} E^{(2)}_{k_m} = \text{genus}\, \Theta^{(2)}(S^{(p)}).$$

We conjecture that this will also be true for any degree, i.e.

$$\lim_{m\to\infty} E^{(n)}_{k_m} = \text{genus}\, \Theta^{(n)}(S^{(p)}).$$

We have numerical examples for which the above identity holds.

Our result concerns the weight 2 Siegel modular forms. For the case of general weights, there is an interesting result due to Y. Mizuno [8]. He considers the p-adic Siegel–Eisenstein series

$$F_k^{(2)}(p) := \lim_{m\to\infty} E_{k+(p-1)p^{m-1}} \qquad (k \text{ even} \geq 2)$$

and shows modularity, i.e., that $F_k^{(2)}(p) \in M_k(\Gamma_0^{(2)}(p))$.

Let $\Lambda = \mathbb{Z}_p[\![\Gamma]\!]$ be the Iwasawa algebra associated to $\Gamma = \mathbb{Z}_p^\times$. Then there is a Λ-adic Eisenstein series E of level 1 with the following properties:

(1) The coefficients of the q-expansion belong to the total fraction ring of Λ with the constant term described by the Kubota–Leopoldt p-adic zeta function.

(2) The specialization at sufficiently large $k \in \mathbb{Z}\setminus p\mathbb{Z}$ is the (p-stabilized) classical Eisenstein series of level 1 and weight k.

(See, for instance, Panchishkin [11], Hida [3].)

Our main theorem shows that the specialization of E at $k = 2$ is classical. The case of $k = 2$ is excluded in (2), thus the theorem provides an example beyond the general theory. The classicality at $k = 1, 2$ of Λ-adic Siegel modular forms of degree 2 would be false in general.

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