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A remark on trigonometric sums

by

Huixue Lao (Jinan)

1. Introduction and main results. Let $\mathcal{P}$ be the set of all primes, and $\mathcal{P}_2$ the set of integers of the form p_1p_2 where $p_1, p_2 \in \mathcal{P}$, $p_1 \neq p_2$. We shall write $e(\alpha)$ instead of $e^{2\pi i\alpha}$. Let $\pi(x)$ be the number of primes up to x. In [3] Kátai considered the following trigonometric sums:

$$S(x,\alpha|X_p) = \sum_{\substack{p_1p_2<x \\ p_1<p_2}} X_{p_1}X_{p_2}e(\alpha p_1p_2), \tag{1.1}$$

where X_p are complex numbers satisfying $|X_p| \le 1$ and p_1, p_2 run over the prime numbers. Let

$$\pi_2(x) = \sum_{\substack{p_1p_2<x \\ p_1<p_2}} 1. \tag{1.2}$$

Kátai showed that

$$\max_{\substack{|X_p|\le 1 \\ p\in\mathcal{P}}} \frac{S(x,\alpha|X_p)}{\pi_2(x)} \to 0 \quad (x\to\infty) \tag{1.3}$$

provided that α is an irrational number satisfying the following condition:

Condition δ. *There exists $x_0 > 0$ such that for all $x \ge x_0$ there exists a rational number a/q with $(a,q) = 1$ satisfying $x^{2/3+\delta} < q < x^{1-\delta}$ and $|\alpha - a/q| \le 1/q^2$. Here δ is an arbitrary small positive number.*

The aim of this note is to give a stronger version of Kátai's result. To this end, we recall the definition of the irrationality measure for a real number α.

Definition 1.1. Let α be a real number, and let $R(\alpha)$ be the set of positive real numbers μ for which

$$0 < |\alpha - p/q| < 1/q^{\mu} \tag{1.4}$$

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has (at most) finitely many solutions p/q for q and p integers. Then the *irrationality measure* of α is defined as

$$\mu(\alpha) = \inf_{\mu \in R(\alpha)} \mu.$$

If the set $R(\alpha)$ is empty, then we set $\mu(\alpha) = \infty$.

By Dirichlet's well-known rational approximation lemma, for every irrational number α, we have $\mu(\alpha) \geq 2$. For every irrational algebraic number α, Roth [4] proved in 1955 that $\mu(\alpha) = 2$. And it is well-known that for almost all real numbers the irrationality measure is 2.

In this note we shall prove the following theorem.

THEOREM 1.1. *For any irrational number α with $\mu(\alpha) < \infty$, we have*

$$\max_{\substack{|X_p| \leq 1 \\ p \in \mathcal{P}}} \frac{S(x, \alpha | X_p)}{\pi_2(x)} \to 0 \quad (x \to \infty). \tag{1.5}$$

2. Preliminaries. We need the following two lemmas.

LEMMA 2.1. *Let $\Lambda(n)$ be the von Mangoldt function*

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

If

$$|\alpha - a/q| < 1/q^2, \qquad (a, q) = 1,$$

then

$$S(\alpha) = \sum_{n \leq x} \Lambda(n) e(\alpha n) \ll (xq^{-1/2} + x^{4/5} + x^{1/2} q^{1/2})(\log x)^4.$$

Proof. See Davenport [1, Chapter 25].

LEMMA 2.2. *Let α be any irrational number with $\mu(\alpha) < \infty$. Fix $\eta \in R(\alpha)$ and $0 < \varepsilon < 1/12(\eta - 1)$. Then for sufficiently large x we have*

$$\max_{1 \leq H \leq x^{2\varepsilon}} \Big| \sum_{p \leq x} e(H\alpha p) \Big| \ll x^{1-4\varepsilon}.$$

Proof. First we show that for any $1 \leq H \leq x^{2\varepsilon}$ there exist integers a and q such that

$$\left| H\alpha - \frac{a}{q} \right| < \frac{1}{q x^{1-9\varepsilon}} \quad \text{with } (a, q) = 1,\ x^{9\varepsilon} \leq q \leq x^{1-9\varepsilon}. \tag{2.1}$$

Any irrational $H\alpha$ has just one infinite simple continued fraction. Let a/q and a'/q' be the two consecutive convergents to that continued fraction such that

$$q \leq x^{1-9\varepsilon} < q'. \tag{2.2}$$

Recall the well-known property of continued fractions:

$$\left| H\alpha - \frac{a}{q} \right| < \frac{1}{qq'} \quad \text{with } (a,q)=1. \tag{2.3}$$

If $1 \le q \le x^{9\varepsilon}$, then $qH \le x^{11\varepsilon} \le (x^{1-9\varepsilon})^{12\varepsilon}$ for sufficiently large x. Thus by (2.2) and (2.3),

$$\|q\alpha H\| \le (q')^{-1} < (x^{1-9\varepsilon})^{-1} \le (qH)^{-1/12\varepsilon},$$

where $\|\cdot\|$ denotes the distance to the nearest integer. But for $\eta \in R(\alpha)$, the inequality

$$\|qH\alpha\| < (qH)^{1-\eta}$$

has (at most) finitely many integer solutions qH. When x is sufficiently large, this contradicts the choice of η and ε. Thus we have $x^{9\varepsilon} \le q \le x^{1-9\varepsilon}$.

On noting (2.1), by Lemma 2.1 we have

$$\max_{1\le H\le x^{2\varepsilon}} \Big| \sum_{p\le x} e(H\alpha p) \Big| \ll \{x(x^{9\varepsilon})^{-1/2} + x^{4/5} + x^{1/2}(x^{1-9\varepsilon})^{1/2}\}(\log x)^3 \ll x^{1-4\varepsilon}.$$

3. Proof of Theorem 1.1. Following the arguments of Kátai, to prove Theorem 1.1 it suffices to show that

$$S_1(x,\alpha) := \sum_{\substack{p_1p_2<x \\ p_1<Y,\, p_2>\sqrt{x}}} X_{p_1}X_{p_2}e(p_1p_2\alpha) = o(\pi_2(x)), \tag{3.1}$$

where $Y = e^{(\log x)^{1-\delta_x}}$, δ_x is a function of x for which $\delta_x \to 0$, and

$$\pi_2(x) = \sum_{\substack{p_1p_2<x \\ p_1<p_2}} 1 \sim \frac{x}{\log x}\log\log x.$$

We have

$$S_1(x,\alpha) = \sum_{p_2} X_{p_2} \sum_{p_1\le \min(x/p_2,Y)} X_{p_1}e(p_1p_2\alpha).$$

Then by the Cauchy–Schwarz inequality,

$$|S_1(x,\alpha)|^2 \le \sum_{p_2}|X_{p_2}|^2 \sum_{p_2}\Big| \sum_{p_1\le\min(x/p_2,Y)} X_{p_1}e(p_1p_2\alpha)\Big|^2 \le \pi(x)\sum\nolimits_1, \tag{3.2}$$

where

$$\sum\nolimits_1 = \sum_{\substack{p_1,p_1'\in\mathcal{P} \\ p_1,p_1'\le Y}} X_{p_1}\overline{X}_{p_1'} \sum_{\sqrt{x}\le p_2\le x/\max(p_1,p_1')} e((p_1-p_1')p_2\alpha). \tag{3.3}$$

From (3.3), we have

$$\sum\nolimits_1 \le \sum\nolimits_2 + \sum\nolimits_3, \tag{3.4}$$

where

$$\sum\nolimits_2 = \sum_{p_1=p_1'\le Y}\ \sum_{\sqrt{x}\le p_2\le x/\max(p_1,p_1')} 1 \ll \frac{x}{\log x}\sum_{p\le Y}\frac{1}{p} \ll \frac{x}{\log x}\log\log x, \tag{3.5}$$

and

$$\sum\nolimits_3 = 2\sum_{p_1'<p_1\le Y}\Big|\sum_{\sqrt{x}\le p_2\le x/\max(p_1,p_1')} e((p_1-p_1')p_2\alpha)\Big|. \tag{3.6}$$

For the inner sum in (3.6), by Lemma 2.2 we have

$$\sum_{\sqrt{x}\le p_2\le x/\max(p_1,p_1')} e((p_1-p_1')p_2\alpha) \ll x^{1-4\varepsilon} \tag{3.7}$$

for every $p_1\ne p_1'$ and $p_1, p_1' < Y$. Thus

$$\sum\nolimits_3 \ll Y^2x^{1-4\varepsilon} = o(\pi_2(x)). \tag{3.8}$$

From (3.2), (3.5) and (3.8), we complete the proof of Theorem 1.1.

4. Another similar result. Recently Indlekofer and Kátai [2] proved another result about trigonometric sums

$$S(x|\alpha; Y_m, X_p) = \sum_{m_jp\le x} Y_{m_j}X_pe(\alpha m_jp), \tag{4.1}$$

where the m_j are integers depending on x with $m_1 < \cdots < m_t \le x^{\delta_x}$, $\delta_x \to 0$ as $x\to\infty$, p runs over the primes $p\ge\sqrt{x}$, and $|Y_{m_j}|\le 1$, $|X_p|\le 1$.

Assume further that as $x\to\infty$,

$$\sum_{j=1}^{t}\frac{1}{m_j}\to\infty.$$

Then they proved that provided that the irrational number α satisfies the Condition δ (see Section 1), we have

$$\max_{Y_m,X_p}|S(x|\alpha; Y_m, X_p)| = o_x(1)\sum_{j=1}^{t}\pi(x/m_j). \tag{4.2}$$

We remark that our previous arguments also give the following stronger result:

Proposition 4.1. *For any irrational number α with $\mu(\alpha)<\infty$, if*

$$\sum_{j=1}^{t}\frac{1}{m_j}\to\infty \qquad \text{as } x\to\infty,$$

then

$$\max_{Y_m, X_p} |S(x|\alpha; Y_m, X_p)| = o_x(1) \sum_{j=1}^{t} \pi\left(\frac{x}{m_j}\right).$$

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School of Mathematics and System Sciences
Shandong University
Jinan, Shandong, 250100, P.R. China

School of Mathematical Sciences
Shandong Normal University
Jinan, Shandong, 250014, P.R. China
E-mail: laohuixue@sina.com

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