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Self-conjugate vector partitions and the parity of the spt-function

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1. Introduction. Let $\mathrm{spt}(n)$ denote the total number of appearances of the smallest parts in the partitions of n. The spt-function satisfies three simple congruences:

$$\mathrm{spt}(5n+4) \equiv 0 \pmod{5}, \tag{1.1}$$

$$\mathrm{spt}(7n+5) \equiv 0 \pmod{7}, \tag{1.2}$$

$$\mathrm{spt}(13n+6) \equiv 0 \pmod{13}. \tag{1.3}$$

These congruences were discovered and proved by the first author [5]. In a recent paper [8], we found new combinatorial interpretations of the congruences mod 5 and 7 in terms of what we called the spt-crank. In this paper we study how the spt-crank is related to the parity of the spt-function.

Let $\mathscr{P}$ denote the set of partitions and $\mathscr{D}$ denote the set of partitions into distinct parts. Following [12], define

$$V = \mathscr{D} \times \mathscr{P} \times \mathscr{P}.$$

We call the elements of V *vector partitions*. In [12], new combinatorial interpretations of Ramanujan's partition congruences mod 5, 7 and 11 were given in terms of these vector partitions. The combinatorial interpretation of the congruences (1.1)–(1.2) is similar. It is in terms of a subset of V,

$$S := \{\vec{\pi} = (\pi_1, \pi_2, \pi_3) \in V : 1 \le s(\pi_1) < \infty \text{ and } s(\pi_1) \le \min(s(\pi_2), s(\pi_3))\}.$$

Here $s(\pi)$ as the smallest part in the partition with the convention that $s(-) = \infty$ for the empty partition. We call the vector partitions in S simply *S-partitions*. For $\vec{\pi} = (\pi_1, \pi_2, \pi_3) \in S$, we define the weight $\omega_1(\vec{\pi}) = (-1)^{\#(\pi_1)-1}$, the $\mathrm{crank}(\vec{\pi}) = \#(\pi_2) - \#(\pi_3)$, and $|\vec{\pi}| = |\pi_1| + |\pi_2| + |\pi_3|$,

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where $|\pi_j|$ is the sum of the parts of π_j, and $\#(\pi_j)$ denotes the number of parts of π_j. The number of S-partitions of n in S with crank m counted according to the weight ω_1 is denoted by $N_S(m,n)$, so that

$$N_S(m,n) = \sum_{\substack{\vec{\pi}\in S,\, |\vec{\pi}|=n \\ \mathrm{crank}(\vec{\pi})=m}} \omega_1(\vec{\pi}). \tag{1.4}$$

We see that

$$S(z,q) := \sum_{n=1}^{\infty}\sum_{m} N_S(m,n) z^m q^n = \sum_{n=1}^{\infty} \frac{q^n (q^{n+1};q)_\infty}{(zq^n;q)_\infty (z^{-1}q^n;q)_\infty}. \tag{1.5}$$

Letting $z=1$ gives

$$\sum_{\vec{\pi}\in S,\, |\vec{\pi}|=n} \omega_1(\vec{\pi}) = \sum_m N_S(m,n) = \mathrm{spt}(n). \tag{1.6}$$

The number of S-partitions of n with crank congruent to m modulo t counted according to the weight ω_1 is denoted by $N_S(m,t,n)$, so that

$$N_S(m,t,n) = \sum_{k=-\infty}^{\infty} N_S(kt+m,n) = \sum_{\substack{\vec{\pi}\in S,\, |\vec{\pi}|=n \\ \mathrm{crank}(\vec{\pi})\equiv m \pmod{t}}} \omega_1(\vec{\pi}). \tag{1.7}$$

The following theorem was our main result in [8], and contains the combinatorial interpretations of (1.1)–(1.2).

Theorem 1.1.

$$N_S(k,5,5n+4) = \frac{\mathrm{spt}(5n+4)}{5} \quad \text{for } 0 \le k \le 4, \tag{1.8}$$

$$N_S(k,7,7n+5) = \frac{\mathrm{spt}(7n+5)}{7} \quad \text{for } 0 \le k \le 6. \tag{1.9}$$

The map $\iota : S \to S$ given by

$$\iota(\vec{\pi}) = \iota(\pi_1,\pi_2,\pi_3) = \iota(\pi_1,\pi_3,\pi_2)$$

is a natural involution. An S-partition $\vec{\pi} = (\pi_1,\pi_2,\pi_3)$ is a fixed-point of ι if and only if $\pi_2 = \pi_3$. We call these fixed-points *self-conjugate S-partitions*. The number of self-conjugate S-partitions counted according to the weight ω_1 is denoted by $N_{\mathrm{SC}}(n)$, so that

$$N_{\mathrm{SC}}(n) = \sum_{\substack{\vec{\pi}\in S,\, |\vec{\pi}|=n \\ \iota(\vec{\pi})=\vec{\pi}}} \omega_1(\vec{\pi}). \tag{1.10}$$

Since ι is an involution that preserves the weight ω_1, we have

$$N_{\mathrm{SC}}(n) \equiv \mathrm{spt}(n) \pmod{2} \tag{1.11}$$

for all n, by (1.6). A standard argument and some calculation give

$$\mathrm{SC}(q) := \sum_{n=1}^{\infty} N_{\mathrm{SC}}(n) q^n = \sum_{n=1}^{\infty} q^n \frac{(q^{n+1};q)_\infty}{(q^{2n};q^2)_\infty} = \frac{1}{(-q;q)_\infty} \sum_{n=1}^{\infty} \frac{q^n(-q;q)_{n-1}}{(1-q^n)}. \tag{1.12}$$

In Section 2 we prove the following theorem.

Theorem 1.2.

$$\frac{1}{(-q;q)_\infty} \sum_{n=1}^{\infty} \frac{q^n(-q;q)_{n-1}}{(1-q^n)} = \sum_{n=0}^{\infty} \frac{1}{(q^2;q^2)_n}((q)_{2n} - (q)_\infty) \tag{1.13}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{n^2}}{(q;q^2)_n}. \tag{1.14}$$

The function in (1.14) is a mock theta function studied by the first author, Dyson and Hickerson [6]. In [6], the arithmetic of the coefficients of the two mock theta functions

$$\sigma(q) = \sum_{n=0}^{\infty} S(n) q^n = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(-q;q)_n}, \tag{1.15}$$

$$\sigma^*(q) = \sum_{n=1}^{\infty} S^*(n) q^n = \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}}{(q;q^2)_n}, \tag{1.16}$$

was studied. The coefficients $S(n)$ and $S^*(n)$ are determined by the prime factorisation of $24n+1$ and $1-24n$ respectively, and are connected with the arithmetic of the field $\mathbb{Q}(\sqrt{6})$. By (1.11)–(1.14) and (1.16) we have

$$N_{\mathrm{SC}}(n) = -S^*(n), \tag{1.17}$$

$$\mathrm{spt}(n) \equiv S^*(n) \pmod 2. \tag{1.18}$$

By combining this with results of [6] we obtain our main result on self-conjugate S-partitions and the parity of the spt-function.

Theorem 1.3.

(i) $N_{\mathrm{SC}}(n) = 0$ *if and only if*

$$p^e \,\|\, 24n-1$$

for some prime $p \not\equiv \pm 1 \pmod{24}$ *and some odd integer* e.

(ii) $\mathrm{spt}(n)$ *is odd if and only if* $24n-1 = p^{4a+1} m^2$ *for some prime* $p \equiv 23 \pmod{24}$ *and some integers* a, m, *where* $(p,m) = 1$.

Remark 1.4. In (ii) above, we have corrected a statement given by Folsom and Ono [11, Theorem 1.2] on the parity of $\mathrm{spt}(n)$.

The details of the proof and discussion of Folsom and Ono's results will be given in Section 2. We note that the proofs of Theorems 3 and 5 in [6] involve only elementary results of arithmetic on $\mathbb{Q}(\sqrt{6})$ together with the method of Bailey chains. This together with the proof of Theorem 1.2 constitutes an elementary q-series proof of the spt-parity result of Theorem 1.3(ii). Folsom and Ono's spt-parity result depends on the theory of weak Maas forms and some heavy calculation with modular forms. In Section 2 we will also connect the value of $\operatorname{spt}(n)$ mod 4 with another mock theta function.

Theorem 1.5. *Let*

$$\Psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n} = \sum_{n=1}^{\infty} \psi(n) q^n.$$

Then

$$\operatorname{spt}(n) \equiv (-1)^{n-1}\psi(n) \pmod 4. \tag{1.19}$$

In Section 3 we obtain some results that we discovered in the process of trying to prove Theorem 1.2. These results include a number of sums of tails identities and generating function identities for the spt-crank and self-conjugate S-partitions.

2. Self-conjugate S-partitions, the parity of the spt-function and mock theta functions

2.1. Proof of Theorem 1.2. We compute

$$\begin{aligned}
\frac{1}{(q^2;q^2)_\infty} \sum_{n=1}^{\infty} \frac{q^n(-q;q)_{n-1}}{(1-q^n)} &= \frac{1}{(q^2;q^2)_\infty} \sum_{n=1}^{\infty} \frac{q^n(q^2;q^2)_{n-1}}{(q)_n} \\
&= \sum_{n=1}^{\infty} \frac{q^n}{(q)_n(q^{2n};q^2)_\infty} = \sum_{n=1}^{\infty} \frac{q^n}{(q)_n} \sum_{k=0}^{\infty} \frac{q^{2nk}}{(q^2;q^2)_k} \qquad \text{(by [4, p. 19, (2.2.5)])} \\
&= \sum_{k=0}^{\infty} \frac{1}{(q^2;q^2)_k} \sum_{n=1}^{\infty} \frac{q^{n(2k+1)}}{(q)_n} = \sum_{k=0}^{\infty} \frac{1}{(q^2;q^2)_k}\left(\frac{1}{(q^{2k+1};q)_\infty} - 1\right),
\end{aligned}$$

again by [4, p. 19, (2.2.5)]. By multiplying by $(q)_\infty$ we have

$$\frac{(q)_\infty}{(q^2;q^2)_\infty} \sum_{n=1}^{\infty} \frac{q^n(-q;q)_{n-1}}{(1-q^n)} = \sum_{k=0}^{\infty} \frac{1}{(q^2;q^2)_k}((q)_{2k} - (q)_\infty),$$

which simplifies to (1.13).

To prove (1.14), we need some results from [9]. By Theorem 1 of [9] with $q \to q^2$, $a \to 0$, $t = q$, we see that

$$\sum_{n=0}^{\infty}((q;q^2)_\infty - (q;q^2)_n) = \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}}{(q;q^2)_n} + (q;q^2)_\infty \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}}. \tag{2.1}$$

By Theorem 2 of [9] with $q \to q^2$, $a = b = c = 0$,

$$\sum_{n=0}^{\infty}\left(\frac{1}{(q^2;q^2)_\infty} - \frac{1}{(q^2;q^2)_n}\right) = \frac{1}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}. \tag{2.2}$$

Hence

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{1}{(q^2;q^2)_n}((q;q)_{2n} - (q;q)_\infty)\\
&= \sum_{n=0}^{\infty}\left((q;q^2)_n - (q;q^2)_\infty + (q;q^2)_\infty - \frac{(q;q)_\infty}{(q^2;q^2)_n}\right)\\
&= \sum_{n=0}^{\infty}((q;q^2)_n - (q;q^2)_\infty) + (q;q)_\infty\sum_{n=0}^{\infty}\left(\frac{1}{(q^2;q^2)_\infty} - \frac{1}{(q^2;q^2)_n}\right)\\
&= -\sum_{n=1}^{\infty}\frac{(-1)^n q^{n^2}}{(q;q^2)_n} - (q;q^2)_\infty\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}} + \frac{(q;q)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\\
&\qquad\qquad\text{(by (2.1) and (2.2))}\\
&= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n^2}}{(q;q^2)_n}. \blacksquare
\end{aligned}$$

2.2. Combinatorial interpretation of Theorem 1.2. We give a combinatorial interpretation of part of Theorem 1.2.

Definition 2.1. Let $\mathscr{B}_e(n)$ (resp. $\mathscr{B}_o(n)$) denote the number of partitions of n with an odd number of smallest parts, and with an even (resp. odd) number of parts.

Definition 2.2. Consider partitions into odd parts without gaps, i.e. if k occurs as a part, all the positive odd numbers less than k also occur. For $j = 1$ or 3, let $\mathscr{C}_j(n)$ denote the number of such partitions of n in which the largest part is congruent to $j \bmod 4$.

Corollary 2.3.

$$\mathscr{B}_e(n) - \mathscr{B}_o(n) = \mathscr{C}_3(n) - \mathscr{C}_1(n).$$

Proof. We have

$$\begin{aligned}
\sum_{n=1}^{\infty}(\mathscr{B}_e(n) - \mathscr{B}_o(n))q^n &= \sum_{n=1}^{\infty}(-q^n - q^{3n} - q^{5n} - \cdots)\frac{1}{(-q^{n+1};q)_\infty}\\
&= -\sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\frac{1}{(-q^{n+1};q)_\infty} = \frac{-1}{(-q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^n(-q;q)_{n-1}}{(1-q^n)}
\end{aligned}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}}{(q;q^2)_n} \qquad \text{(by Theorem 1.2)}$$
$$= \sum_{n=1}^{\infty} (\mathscr{C}_3(n) - \mathscr{C}_1(n)) q^n,$$

as observed in [6, p. 404]. ■

EXAMPLE 2.4. Suppose $n = 7$. Below we list the partitions of 7 with an odd number of smallest parts:

π	$\#(\pi)$
7	1
$6+1$	2
$5+2$	2
$4+3$	2
$4+2+1$	3
$4+1+1+1$	4
$3+3+1$	3
$2+2+2+1$	4
$2+2+1+1+1$	5
$2+1+1+1+1+1$	6
$1+1+1+1+1+1+1$	7

We see that $\mathscr{B}_e(7) = 6$ and $\mathscr{B}_o(7) = 5$. There are three partitions of 7 into odd parts with no gaps:

π	largest part
$3+3+1$	3
$3+1+1+1+1$	3
$1+1+1+1+1+1+1$	1

Hence $\mathscr{C}_3(7) = 2$ and $\mathscr{C}_1(7) = 1$. Thus

$$\mathscr{B}_e(7) - \mathscr{B}_o(7) = 6 - 5 = 1 = 2 - 1 = \mathscr{C}_3(7) - \mathscr{C}_1(7).$$

2.3. Proof of Theorem 1.3. First we need some results from [6]. For $m \equiv 1 \pmod{24}$ let $T(m)$ denote the number of inequivalent solutions of

$$u^2 - 6v^2 = m$$

with $u + 3v \equiv \pm 1 \pmod{12}$ minus the number with $u + 3v \equiv \pm 5 \pmod{12}$.

THEOREM 2.5 ([6]).

$$S(n) = T(24n+1) \quad \textit{for } n \geq 0, \tag{2.3}$$
$$2S^*(n) = T(1-24n) \quad \textit{for } n \geq 1. \tag{2.4}$$

For any integer m (positive or negative) satisfying $m \equiv 1 \pmod 6$ and $m \neq 1$, let

$$m = p_1^{e_1} \cdots p_r^{e_r},$$

be the prime factorisation where for each i, $p_i \equiv 1 \pmod 6$ or p_i is the negative of a prime $\equiv 5 \pmod 6$. Then we have

THEOREM 2.6 ([6]).

$$T(m) = T(p_1^{e_1}) \cdots T(p_r^{e_r}), \tag{2.5}$$

where

$$T(p^e) = \begin{cases} 0 & \text{if } p \not\equiv 1 \pmod{24} \text{ and } e \text{ is odd,} \\ 1 & \text{if } p \equiv 13, 19 \pmod{24} \text{ and } e \text{ is even,} \\ (-1)^{e/2} & \text{if } p \equiv 7 \pmod{24} \text{ and } e \text{ is even,} \\ e+1 & \text{if } p \equiv 1 \pmod{24} \text{ and } T(p) = 2, \\ (-1)^e(e+1) & \text{if } p \equiv 1 \pmod{24} \text{ and } T(p) = -2. \end{cases} \tag{2.6}$$

We are now ready to complete the proof of Theorem 1.3. First we write the prime factorisation

$$24n - 1 = p_1^{a_1} \cdots p_r^{a_r} q_1^{b_1} \cdots q_s^{b_s}, \tag{2.7}$$

where each $p_j \equiv 5 \pmod 6$ and $q_j \equiv 1 \pmod 6$ so that

$$1 - 24n = (-p_1)^{a_1} \cdots (-p_r)^{a_r} q_1^{b_1} \cdots q_s^{b_s},$$

and

$$a_1 + \cdots + a_r \equiv 1 \pmod 2. \tag{2.8}$$

From (2.5) we have

$$T(1-24n) = T((-p_1)^{a_1}) \cdots T((-p_r)^{a_r}) T(q_1^{b_1}) \cdots T(q_s^{b_s}). \tag{2.9}$$

By (1.12), Theorem 1.2, (1.16) and (2.4) we have

$$N_{\mathrm{SC}}(n) = -S^*(n) = -\tfrac{1}{2} T(1-24n). \tag{2.10}$$

Part (i) of Theorem 1.3 now follows immediately from Theorem 2.6 and (2.9).

We prove part (ii). We observe from (1.18), (2.4) and (2.10) that $T(1-24n)$ is even and

$$\mathrm{spt}(n) \equiv \tfrac{1}{2} T(1-24n) \pmod 2. \tag{2.11}$$

Now suppose $\mathrm{spt}(n)$ is odd so that $T(1-24n) \neq 0$. From (2.8) we see that at least one of the a_j is odd, say a_1. Since $T(1-24n) \neq 0$ we deduce that $p_1 \equiv 23 \pmod{24}$, and the factor $T((-p_1)^{a_1}) = \pm(a_1+1)$ is even. If $j \neq 1$, a_j is odd and $p_j \equiv 23 \pmod{24}$ then the factor $T((-p_j)^{a_j})$ would also be even and (2.9), (2.6) and (2.11) would imply that $\mathrm{spt}(n)$ is even, which is a contradiction. Therefore each a_j is even for $j \neq 1$. Similarly each b_j is even. Hence each exponent in the factorisation (2.7) is even except a_1. So

$$\tfrac{1}{2}T((-p_1)^{a_1}) = \pm\tfrac{1}{2}(a_1+1) \equiv 1 \pmod{2},$$

$a_1 \equiv 1 \pmod{4}$ and

$$24n-1 = p^{4a+1}m^2,$$

where $p \equiv 23 \pmod{24}$ is prime and $(m,p)=1$. Conversely, if

$$24n-1 = p^{4a+1}m^2,$$

where $p \equiv 23 \pmod{24}$ is prime and $(m,p)=1$, then it easily follows that $\frac{1}{2}T(1-24n)$ is odd, and $\mathrm{spt}(n)$ is odd. ∎

2.4. Examples, and Folsom and Ono's results. We illustrate part (i) of Theorem 1.3 with an example. Below is a table of the six self-conjugate S-partitions of 5:

	weight
$\vec{\pi}_1 = (1, 1+1, 1+1)$	$+1$
$\vec{\pi}_2 = (1, 2, 2)$	$+1$
$\vec{\pi}_3 = (2+1, 1, 1)$	-1
$\vec{\pi}_4 = (3+2, -, -)$	-1
$\vec{\pi}_5 = (4+1, -, -)$	-1
$\vec{\pi}_6 = (5, -, -)$	$+1$

Thus

$$N_{\mathrm{SC}}(5) = \sum_{j=1}^{6} \omega_1(\vec{\pi}_j) = 1+1-1-1-1+1 = 0,$$

as predicted by the theorem, since

$$24 \cdot 5 - 1 = 119 = 7 \cdot 17.$$

In [11], Folsom and Ono incorrectly stated that "$\mathrm{spt}(n)$ is odd if and only if $24n-1 = pm^2$ where $p \equiv 23 \pmod{24}$ is prime and m is an integer." We give some examples illustrating the difference between their statement and ours. We also make some comments on their proof.

Example 2.7. Suppose $n = 507$. Then $24n-1 = 23^3$. Calculation gives

$$\begin{aligned}\mathrm{spt}(507) &= 6047032773755628522506 4\\ &= 2^3 \cdot 3 \cdot 251 \cdot 236699 \cdot 1703123 \cdot 24900893,\end{aligned}$$

which is clearly even as predicted by our theorem. In fact, if $p \equiv 23 \pmod{24}$ is prime and

$$n = \frac{1}{24}(p^3 \cdot m^2 + 1),$$

where $(m, 6p) = 1$, then $24n - 1 = p^3 \cdot m^2$ and

$$\mathrm{spt}(n) \equiv 0 \pmod{24}.$$

This congruence is a special case of [13, Theorem 1.3(i)].

EXAMPLE 2.8. Suppose $n = 268181$. Then $24n - 1 = 23^5$, and

$$\begin{aligned}\mathrm{spt}(268181) &= 17367\cdots 2073 \qquad \text{(a number with 574 decimal digits)}\\ &\equiv 1 \pmod 2,\end{aligned}$$

as predicted by our theorem. Again, using [13, Theorem 1.3(i)] we have

$$\mathrm{spt}\left(\frac{1}{24}(23^{4a+1} + 1)\right) \equiv 1 \pmod 8.$$

We clarify Folsom and Ono's proof. We let $\mathcal{L}(z)$, $\mathcal{S}(z)$ be defined as in equations (1.1) and (1.4) of [11]. We proceed as in Section 4 of [11] to obtain

$$\begin{aligned}\mathcal{L}(z) \equiv &\sum_{n\ge 1}\sum_{m\ge 0}(q^{(12n-1)(12n+24m+1)} + q^{(12n-5)(12n+24m+5)})\\ &+\sum_{n\ge 1}\sum_{m\ge 0}(q^{(12n+1)(12n+24m-1)} + q^{(12n+5)(12n+24m-5)}) \pmod 2.\end{aligned}$$

From [11, Lemma 4.1] we have

$$q^{-1}\mathcal{S}(24z) = \sum_{n\ge 1}\mathrm{spt}(n)q^{24n-1} \equiv \mathcal{L}(24z) \pmod 2,$$

so that

$$\mathrm{spt}(n) \equiv \sum_{\substack{1\le d_1<d_2\\ d_1d_2=24n-1}} 1 \pmod 2,$$

and

$$\mathrm{spt}(n) \equiv \frac{1}{2}\mathrm{d}(24n-1) \pmod 2, \tag{2.12}$$

where $\mathrm{d}(m)$ is the number of positive divisors of m. The spt-parity result in Theorem 1.3(ii) follows in a straightforward manner.

2.5. Proof of Theorem 1.5. We begin with some preliminary facts:

$$\frac{1}{(1-q^n)^2} = \frac{1}{(1+q^n)^2} + 4\frac{q^n}{(1-q^{2n})^2}, \tag{2.13}$$

$$f(q) + 4\Psi(-q) = (q;q^2)_\infty \vartheta_4(0,q) \quad \text{(by [19, p. 63])}, \tag{2.14}$$

where $f(q)$ is the third order mock theta function

$$f(q) = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n^2},$$

$\vartheta_4(z,q)$ is the theta function

$$\vartheta_4(z,q) = \sum_{n=-\infty}^{\infty} (-1)^n e^{2\pi i n z}\, q^{n^2} = (e^{2\pi i z} q; q^2)_\infty (e^{-2\pi i z} q; q^2)_\infty (q^2;q^2)_\infty,$$

and

$$f(q) = \frac{1}{(q;q)_\infty}\left(1 + 4\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^n}\right), \tag{2.15}$$

by [19, p. 64]. We restate Theorem 4 from [5] as

$$\sum_{n=1}^{\infty} \mathrm{spt}(n) q^n = \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} + \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1+q^n)}{(1-q^n)^2}. \tag{2.16}$$

It is well known that

$$\vartheta_4(0,q)^2 = 1 + 4\sum_{m=0}^{\infty} \frac{(-1)^{m+1} q^{2m+1}}{1+q^{2m+1}}. \tag{2.17}$$

See for example [3, eqn. (3.33), p. 462]. By (2.15), we have

$$\begin{aligned} \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^n} &= \frac{1}{4(q;q)_\infty}((q;q)_\infty f(q) - 1) \\ &= -\Psi(-q) + \frac{1}{4}(q;q^2)_\infty \vartheta_4(0,q) - \frac{1}{4(q;q)_\infty} \qquad \text{(by (2.14))} \\ &= -\Psi(-q) + \frac{1}{4(q;q)_\infty}(\vartheta_4(0,q)^2 - 1) \qquad \text{(by [4, p. 23, (2.2.12)])} \\ &= -\Psi(-q) + \frac{1}{(q;q)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+1} q^{2m+1}}{1+q^{2m+1}}, \end{aligned} \tag{2.18}$$

by (2.17). Therefore, by (2.14) and (2.16) we have

$$\begin{aligned} &\sum_{n=1}^{\infty} \mathrm{spt}(n) q^n \\ &\equiv \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} + \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^n} \pmod 4 \\ &\equiv \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - \Psi(-q) + \frac{1}{(q;q)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+1} q^{2m+1}}{1+q^{2m+1}} \qquad \text{(by (2.18))} \end{aligned} \tag{2.19}$$

$$\equiv \frac{1}{(q;q)_\infty}\Big(\sum_{n=0}^{\infty}\frac{q^{4n+1}}{1-q^{4n+1}} - \sum_{n=1}^{\infty}\frac{q^{4n-1}}{1-q^{4n-1}} + 2\sum_{n=0}^{\infty}\frac{q^{4n+2}}{1-q^{4n+2}}\Big)$$

$$- \Psi(-q) + \frac{1}{(q;q)_\infty}\Big(-\sum_{m=0}^{\infty}\frac{q^{4m+1}}{1+q^{4m+1}} + \sum_{m=1}^{\infty}\frac{q^{4m-1}}{1+q^{4m-1}}\Big)$$

$$= \frac{1}{(q;q)_\infty}\Big(2\sum_{n=0}^{\infty}\frac{q^{8n+2}}{1-q^{8n+2}} - 2\sum_{n=1}^{\infty}\frac{q^{8n-2}}{1-q^{8n-2}} + 2\sum_{n=0}^{\infty}\frac{q^{4n+2}}{1-q^{4n+2}}\Big) - \Psi(-q)$$

$$= \frac{4}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{8n+2}}{1-q^{8n+2}} - \Psi(-q) \equiv -\Psi(-q) \pmod 4.$$

Consequently,

$$\mathrm{spt}(n) \equiv (-1)^{n-1}\psi(n) \pmod 4,$$

as desired. ∎

3. Other results. In this section, we give some results that we discovered along the way in our quest to prove Theorem 1.2 and the following

THEOREM 3.1 ([8]).

$$N_S(m,n) \ge 0 \quad \textit{for all } m,\, n. \tag{3.1}$$

For example, before considering the result (3.1) for general m, one might first consider the special case $m = 0$. In Theorem 3.4, we express the generating function of $N_S(0,n)$ in terms of a series involving tails of infinite products. The theorem also contains some natural variations. We first need to extend a result from [9].

PROPOSITION 3.2 ([9, Prop. 2.1, p. 403]). *Suppose that* $f_\alpha(z) = \sum_{n=0}^{\infty}\alpha_n z^n$ *is analytic for* $|z| < 1$. *If* α *is a complex number such that*

(i) $\sum_{n=0}^{\infty}(\alpha - \alpha_n) < \infty$, *and*
(ii) $\lim_{n\to\infty} n(\alpha - \alpha_n) = 0$,

then

$$\lim_{z\to 1^-}\frac{d}{dz}(1-z)f_\alpha(z) = \sum_{n=0}^{\infty}(\alpha - \alpha_n).$$

The extension we need is

LEMMA 3.3. *Suppose that* $f_\alpha(z) = \sum_{n=0}^{\infty}\alpha_n z^n$, $f_\beta(z) = \sum_{n=0}^{\infty}\beta_n z^n$, *and* $f_{\alpha\beta}(z) = \sum_{n=0}^{\infty}\alpha_n\beta_n z^n$ *are analytic for* $|z| < 1$. *And suppose that* (i), (ii) *hold for each of the three sequences* α_n, β_n, $\alpha_n\beta_n$ (*with respective limits* α, β *and* $\alpha\beta$). *Then*

$$\sum_{n=0}^{\infty}\beta_n(\alpha_n - \alpha) = \lim_{z\to 1^-}\frac{d}{dz}(1-z)(\alpha f_\beta(z) - f_{\alpha\beta}(z)).$$

Proof. We compute

$$\sum_{n=0}^{\infty}\beta_n(\alpha_n-\alpha)=\sum_{n=0}^{\infty}(\alpha\beta-\alpha\beta_n+\alpha_n\beta_n-\alpha\beta)$$
$$=\alpha\sum_{n=0}^{\infty}(\beta-\beta_n)-\sum_{n=0}^{\infty}(\alpha\beta-\alpha_n\beta_n).$$

The result follows easily from Proposition 3.2. ∎

THEOREM 3.4. *We have*

$$\sum_{n=0}^{\infty}\frac{1}{(q)_n^2}((q)_{2n}-(q)_\infty)=\sum_{n=1}^{\infty}N_S(0,n)q^n, \tag{3.2}$$

$$\sum_{n=0}^{\infty}\frac{1}{(q)_n^2}((q)_n-(q)_\infty)=\sum_{n=1}^{\infty}\frac{nq^{n^2}}{(q)_n^2}, \tag{3.3}$$

$$\sum_{n=0}^{\infty}\frac{1}{(q^2;q^2)_n}((q)_{2n}-(q)_\infty)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n^2}}{(q;q^2)_n}, \tag{3.4}$$

$$\sum_{n=0}^{\infty}\frac{1}{(q)_n}((q)_n-(q)_\infty)=\sum_{n=1}^{\infty}q^{n^2}\frac{1+q^n}{1-q^n}=\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}. \tag{3.5}$$

Proof. From (1.5) we deduce

$$\sum_{n=1}^{\infty}\sum_{m}N_S(m,n)z^mq^n=\sum_{n=1}^{\infty}\frac{q^n(q^{n+1};q)_\infty}{(zq^n;q)_\infty(z^{-1}q^n;q)_\infty}$$
$$=(q)_\infty\sum_{n=1}^{\infty}\frac{q^n}{(q)_n}\frac{1}{(zq^n;q)_\infty(z^{-1}q^n;q)_\infty}$$
$$=(q)_\infty\sum_{n=1}^{\infty}\frac{q^n}{(q)_n}\sum_{k=0}^{\infty}\frac{(zq^n)^k}{(q)_k}\sum_{m=0}^{\infty}\frac{(z^{-1}q^n)^m}{(q)_m},$$

by [4, p. 19, (2.2.5)]. Picking out the coefficient of z^0 we have

$$\sum_{n=1}^{\infty}N_S(0,n)q^n=(q)_\infty\sum_{n=1}^{\infty}\frac{q^n}{(q)_n}\sum_{k=0}^{\infty}\frac{q^{2nk}}{(q)_k^2}=(q)_\infty\sum_{k=0}^{\infty}\frac{1}{(q)_k^2}\sum_{n=1}^{\infty}\frac{q^{n(2k+1)}}{(q)_n}$$
$$=(q)_\infty\sum_{k=0}^{\infty}\frac{1}{(q)_k^2}\left(-1+\frac{1}{(q^{2k+1};q)_\infty}\right)\quad\text{(by [4, p. 19, (2.2.5)])}$$
$$=\sum_{k=0}^{\infty}\frac{1}{(q)_k^2}((q)_{2k}-(q)_\infty),$$

which gives (3.2).

To prove (3.3) we apply Lemma 3.3 with

$$\alpha_n = (q;q)_n, \qquad \alpha = (q;q)_\infty,$$
$$\beta_n = \frac{1}{(q;q)_n^2}, \qquad \beta = \frac{1}{(q;q)_\infty^2}.$$

This leads to

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}(z;q)_n}{(q;q)_n} = \lim_{\tau\to 0} {}_2\phi_1\begin{pmatrix} \tau^{-1}q, & z; & q, & \tau \\ & 0 & & \end{pmatrix} \tag{3.6}$$

$$= \lim_{\tau\to 0} \frac{(z;q)_\infty (q;q)_\infty}{(t;q)_\infty} {}_2\phi_1\begin{pmatrix} 0, & \tau; & q, & z \\ & q & & \end{pmatrix} \quad \text{(by [14, p. 241, (III.1)])} \tag{3.7}$$

$$= (z;q)_\infty (q;q)_\infty \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n^2}, \tag{3.8}$$

and we have the identity

$$\sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n^2} = \frac{1}{(q;q)_\infty (z;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}(z;q)_n}{(q;q)_n}. \tag{3.9}$$

Thus

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{1}{(q)_n^2}((q)_n - (q)_\infty) \\
&= (q;q)_\infty \lim_{z\to 1^-} \frac{d}{dz}(1-z)\sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n^2} - \lim_{z\to 1^-} \frac{d}{dz}(1-z)\sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} \\
&= \lim_{z\to 1^-} \frac{d}{dz}(1-z)\left(\frac{1}{(z;q)_\infty}\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}(z;q)_n}{(q;q)_n}\right) \\
&\quad - \lim_{z\to 1^-} \frac{d}{dz}\frac{1}{(zq;q)_\infty} \qquad \text{(by (3.9) and [4, p. 19, (2.2.5)])} \\
&= \lim_{z\to 1^-} \frac{d}{dz}\frac{1-z}{(z;q)_\infty}\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}(z;q)_n}{(q;q)_n} \\
&= -\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1-q^n},
\end{aligned}$$

because

$$\lim_{z\to 1^-} \frac{d}{dz}(1-z)F(z) = -F(1) \tag{3.10}$$

when $F(z)$ is analytic at $z=1$. On the other hand,

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{nq^{n^2}}{(q;q)_n^2} &= \left(\frac{d}{dz}\sum_{n=0}^{\infty}\frac{z^nq^{n^2}}{(q;q)_n^2}\right)_{z=1}\\
&= \left(\frac{d}{dz}\lim_{\tau\to 0}{}_2\phi_1\left(\begin{matrix}\tau^{-1}, & q\tau^{-1}; & q, & z\tau^2\\ & q & & \end{matrix}\right)\right)_{z=1}\\
&= \left(\frac{d}{dz}\lim_{\tau\to 0}\frac{(q\tau;q)_\infty(z\tau;q)_\infty}{(q;q)_\infty(z\tau^2;q)_\infty}{}_2\phi_1\left(\begin{matrix}z, & \tau^{-1}; & q, & q\tau\\ & z\tau & & \end{matrix}\right)\right)_{z=1}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(by [14, p. 241, (III.2)]))}\\
&= \left(\frac{d}{dz}\frac{1}{(q;q)_\infty}+\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n(z;q)_nq^{n(n+1)/2}}{(q;q)_n}\right)_{z=1}\\
&= -\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^nq^{n(n+1)/2}}{1-q^n},
\end{aligned}$$

again by (3.10). Thus both sides of (3.3) are equal to the same thing and therefore equal to each other.

Equation (3.4) is contained in Theorem 1.2: it is (1.14). The proof is given in Section 2.1.

Finally we turn to (3.5). The identity

$$\sum_{n=0}^{\infty}\frac{1}{(q)_n}((q)_n-(q)_\infty)=\sum_{n=1}^{\infty}\frac{q^n}{1-q^n} \tag{3.11}$$

is well known; see for example [7, p. 146, (13)]. Moreover

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{q^n}{1-q^n} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q^{mn}=\sum_{n=1}^{\infty}\sum_{m=n}^{\infty}q^{mn}+\sum_{n=1}^{\infty}\sum_{m=1}^{n-1}q^{mn}\\
&= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}q^{n(n+m)}+\sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty}q^{mn}\\
&= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}q^{n(n+m)}+\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}q^{m(m+n)}=\sum_{n=1}^{\infty}q^{n^2}\frac{1+q^n}{1-q^n},
\end{aligned}$$

which completes the proof of (3.5). ■

Some remarks on Theorem 3.4. As mentioned before, we originally wanted to obtain identities for $N_S(m,n)$ in order to approach the result (3.1). The first identity we obtained was (3.2). The series on the left side of (3.3) is a natural tweak. To our suprise this series seemed to also have nonnnegative coefficients and the identity (3.3) was discovered empirically.

A quick search in Neil Sloane's *On-Line Encyclopedia of Integer Sequences* [16] reveals that

$$\sum_{n=0}^{\infty} \frac{1}{(q)_n^2}((q)_n - (q)_\infty) = \sum_{n=1}^{\infty} \frac{nq^{n^2}}{(q)_n^2} = q + \sum_{n=2}^{\infty}\sum_{m=1}^{n} mM(m,n)q^n, \tag{3.12}$$

where $M(m,n)$ is the number of partitions of n with crank m. See sequence `A115995` of [10]. It is clear that the left sides of equations (3.2) and (3.4) are congruent mod 2. The right hand side of (3.4) was found empirically. The coefficients of this series appear to grow very slowly and many of the coefficients are zero. Such q-series are quite rare. These properties led us to quickly identify this series with the special mock theta function, $\sigma^*(q)$, which was studied previously by the first author, Dyson and Hickerson [6]. We note that these coefficients also appear in Sloane's *On-Line Encyclopedia*. See sequence `A003475` of [17]. It was only later we discovered the connection with self-conjugate S-partitions.

The FFW-function. The initial study of the spt-function [5] was inspired by a result of Fokkink, Fokkink and Wang [18]. Recall that $\mathscr{D}$ denotes the set of partitions into distinct parts. Define

$$\mathrm{FFW}(n) := \sum_{\substack{\pi \in \mathscr{D} \\ |\pi| = n}} (-1)^{\#(\pi)-1}\, s(\pi), \tag{3.13}$$

where as before $s(\pi)$ denotes the smallest part in the partition π. Fokkink, Fokkink and Wang [18] proved that

$$\mathrm{FFW}(n) = d(n), \tag{3.14}$$

the number of positive divisors of n. In [5] a q-series proof of this result was given, using the identity

$$\sum_{n=1}^{\infty} \mathrm{FFW}(n) q^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{n(n+1)/2}}{(q)_n (1-q^n)}. \tag{3.15}$$

We extend the FFW-function and obtain similar expressions for the spt-function and spt-crank generating functions. We define

$$\mathrm{FFW}(z,n) := \sum_{\substack{\pi \in \mathscr{D} \\ |\pi| = n}} (-1)^{\#(\pi)-1}\, (1 + z + \cdots + z^{s(\pi)-1}), \tag{3.16}$$

so that

$$\mathrm{FFW}(1,n) = \mathrm{FFW}(n).$$

Theorem 3.5.

$$\sum_{n=1}^{\infty} \mathrm{FFW}(z,n)q^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}q^{n(n+1)/2}}{(1-zq^n)(q)_n} \tag{3.17}$$

$$= \frac{1}{1-z}\left(1-\frac{(q)_\infty}{(zq)_\infty}\right) \tag{3.18}$$

$$= \sum_{k=0}^{\infty} \frac{z^k}{(q)_k}((q)_k-(q)_\infty). \tag{3.19}$$

Proof. Given a partition into n distinct parts and smallest part k we may subtract k from the smallest part, $k+1$ from the next smallest part, $\ldots$, $k+(n-1)$ from the largest part to obtain an unrestricted partition into at most $n-1$ parts. This process can be reversed and we see that

$$q^{n(n-1)/2} \cdot q^{nk} \cdot \frac{1}{(q)_{n-1}}$$

is the generating function for partitions into n distinct parts with smallest part k. Thus

$$\begin{aligned}
&\sum_{n=1}^{\infty} \mathrm{FFW}(z,n)q^n \\
&= \sum_{n=1}^{\infty} (q^n + (1+z)q^{2n} + \cdots + (1+z+\cdots+z^{k-1})q^{kn} + \cdots) \frac{(-1)^{n-1}q^{n(n-1)/2}}{(q)_{n-1}} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}q^{n(n+1)/2}}{(1-zq^n)(q)_n},
\end{aligned}$$

since

$$\sum_{k=1}^{\infty} \frac{z^k-1}{z-1} x^k = \frac{x}{(1-zx)(1-x)}.$$

Now

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}q^{n(n+1)/2}}{(1-zq^n)(q)_n} &= \frac{1}{1-z}\left(1-\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}(z)_n}{(q)_n (zq)_n}\right) \\
&= \frac{1}{1-z}\left(1-\frac{(q)_\infty}{(zq)_\infty}\right),
\end{aligned}$$

arguing as in [5, p. 134]. Lastly we show that

$$\sum_{n=1}^{\infty} \mathrm{FFW}(z,n)q^n = \sum_{k=0}^{\infty} z^k (1-(q^{k+1};q)_\infty). \tag{3.20}$$

We see that the coefficient of $z^k q^n$ in RHS(3.20) is

$$\sum_{\substack{\pi\in\mathscr{D},\,|\pi|=n\\ k+1\le s(\pi)}} (-1)^{\#(\pi)-1},$$

which is also the coefficient of $z^k q^n$ in LHS(3.20). We note that the right side of (3.20) coincides with the right side of (3.19). This completes the proof of (3.17)–(3.19). ∎

Corollary 3.6.

$$\text{FFW}(-1,n) = \sum_{\substack{\pi\in\mathscr{D},\,|\pi|=n\\ s(\pi)\ odd}} (-1)^{\#(\pi)-1} = \begin{cases} 0 & \text{if } n \ne j^2,\\ (-1)^{j-1} & \text{if } n = j^2,\end{cases} \tag{3.21}$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(q)_k}\left((q)_k - (q)_\infty\right) = \sum_{j=1}^{\infty} (-1)^{j-1} q^{j^2}. \tag{3.22}$$

Proof. Equations (3.21)–(3.22) follow from setting $z = -1$ in Theorem 3.5 and using Gauss's result [4, p. 23] that

$$\frac{(q)_\infty}{(-q)_\infty} = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2}. \ ∎$$

Remark 3.7. The result (3.21) is due to Alladi [1, Thm. 2]. Equation (3.22) appears to be new. Alladi [2] has found an extension of (3.21) that is a combinatorial interpretation of a partial theta-function identity [2, (1.1)] that appears in Ramanujan's Lost Notebook [15, p. 38].

Theorem 3.8.

$$S(z,q) = \frac{1}{(zq)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{n(n+1)/2}}{(q)_n (1 - z^{-1} q^n)} \, \frac{z^n - 1}{z - 1}, \tag{3.23}$$

$$\sum_{n=1}^{\infty} \text{spt}(n) q^n = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n q^{n(n+1)/2}}{(q)_n (1 - q^n)}, \tag{3.24}$$

$$\sum_{n=1}^{\infty} N_{\text{SC}}(n) q^n = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n q^{n(n+1)/2}}{(q)_n (1 + q^n)}. \tag{3.25}$$

Proof. In [14, p. 241, (III.2)] we replace z by q, and let $a = z$, $b = z^{-1}$ and $c \to 0$ to obtain

$$\sum_{n=0}^{\infty} \frac{(z)_n (z^{-1})_n}{(q)_n} q^n = \frac{(z^{-1}q)_\infty}{(q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2} (1 - z^{-1}) z^n}{(1 - z^{-1} q^n)(q)_n}. \tag{3.26}$$

From (1.5) we have

$$S(z,q)=\sum_{n=1}^{\infty}\frac{q^n(q^{n+1};q)_\infty}{(zq^n;q)_\infty(z^{-1}q^n;q)_\infty}$$
$$=\frac{(q)_\infty}{(z)_\infty(z^{-1})_\infty}\sum_{n=0}^{\infty}\frac{(z)_n(z^{-1})_n}{(q)_n}q^n-\frac{(q)_\infty}{(z)_\infty(z^{-1})_\infty}$$
$$=\frac{1}{(1-z^{-1})(z)_\infty}\bigg(1+\sum_{n=1}^{\infty}\frac{(-1)^nq^{n(n+1)/2}(1-z^{-1})z^n}{(1-z^{-1}q^n)(q)_n}-\frac{(q)_\infty}{(z^{-1}q)_\infty}\bigg).$$

From (3.17)–(3.18) we have

$$\frac{(q)_\infty}{(z^{-1}q)_\infty}=1+\sum_{n=1}^{\infty}\frac{(-1)^nq^{n(n+1)/2}(1-z^{-1})}{(1-z^{-1}q^n)(q)_n}.$$

Hence

$$S(z,q)=\frac{1}{(1-z^{-1})(z)_\infty}$$
$$\times\bigg(\sum_{n=1}^{\infty}\frac{(-1)^nq^{n(n+1)/2}(1-z^{-1})z^n}{(1-z^{-1}q^n)(q)_n}-\sum_{n=1}^{\infty}\frac{(-1)^nq^{n(n+1)/2}(1-z^{-1})}{(1-z^{-1}q^n)(q)_n}\bigg)$$
$$=\frac{1}{(zq)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n(n+1)/2}}{(q)_n(1-z^{-1}q^n)}\,\frac{z^n-1}{z-1},$$

which is (3.23).

Equation (3.24) follows from (1.6) by letting $z\to 1$ in (3.23).

Before proving (3.25) we need to correct a result in [9]. By Theorem 2 in [9] with $a=q$, $b=0$ and $c=-q$ we have

$$\sum_{n=0}^{\infty}\bigg(\frac{1}{(-q;q)_\infty}-\frac{1}{(-q;q)_n}\bigg)=\frac{-1}{(-q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^{n(n+1)/2}}{(q;q)_n(1-q^n)}\tag{3.27}$$
$$=\frac{-1}{(-q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^{n(n+1)/2}(1+(-1)^{n-1}-(-1)^{n-1})}{(q;q)_n(1-q^n)}$$
$$=\frac{-2}{(-q;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{2n^2+3n+1}}{(q;q)_{2n+1}(1-q^{2n+1})}+\frac{1}{(-q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}$$

(by (3.14)–(3.15))

$$=\frac{-2q}{(-q;q)_\infty(1-q)^2}\lim_{\tau\to 0}{}_3\phi_2\begin{pmatrix}\tau^{-1}q, & q, & \tau^{-1}q; & q^2, & \tau^2q^3\\ & q^3, & q^3 & & \end{pmatrix}$$
$$+\frac{1}{(-q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}$$

$$= -2q\sum_{n=0}^{\infty}(q^2;q^2)_n q^n + \frac{1}{(-q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^n}{1-q^n} \qquad \text{(by [14, p. 241, (III.10)])}$$

$$= -2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n^2}}{(q;q^2)_n} + \frac{1}{(-q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}$$

by [4, p. 29, Ex. 6] with $x = -q^2$ and $y = q$. We have corrected the proof of Case 6 in [9, pp. 405–406]. From (3.27) we obtain

$$\text{(3.28)} \qquad \frac{1}{(-q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^{n(n+1)/2}}{(q;q)_n(1-q^n)} = 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n^2}}{(q;q^2)_n} - \frac{1}{(-q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}.$$

Now

$$\text{(3.29)} \qquad \sum_{n=0}^{\infty}\frac{(-1)^n z^n q^{n(n+1)/2}}{(q;q)_n(1+q^n)} = \frac{1}{2}\lim_{\tau\to 0}{}_2\phi_1\begin{pmatrix}-1, & \tau^{-1}q; & q, & z\tau \\ & -q & & \end{pmatrix}$$

$$= \frac{1}{2}\lim_{\tau\to 0}\frac{(-\tau;q)_\infty(qz;q)_\infty}{(-q;q)_\infty(z\tau;q)_\infty}\,{}_2\phi_1\begin{pmatrix}z, & \tau^{-1}q; & q, & -\tau \\ & qz & & \end{pmatrix}$$

(by [14, p. 241, (III.2)])

$$= \frac{1}{2}\left(\frac{(qz;q)_\infty}{(-q;q)_\infty} + \frac{(qz;q)_\infty}{(-q;q)_\infty}\sum_{n=1}^{\infty}\frac{(z;q)_n q^{n(n+1)/2}}{(q;q)_n(qz;q)_n}\right).$$

After dividing both sides of (3.29) by $(q)_\infty$, applying $\frac{d}{dz}$, and letting $z \to 1$ we find that

$$\text{(3.30)} \qquad \frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\, n\, q^{n(n+1)/2}}{(q)_n(1+q^n)}$$

$$= \frac{1}{2(-q)_\infty}\sum_{n=1}^{\infty}\frac{q^n}{1-q^n} + \frac{1}{2(-q)_\infty}\sum_{n=1}^{\infty}\frac{q^{n(n+1)/2}}{(q)_n(1-q^n)} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n^2}}{(q;q^2)_n},$$

by (3.28). The result (3.25) follows from (1.17) and (3.30). ∎

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