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# Lacunary formal power series and the Stern–Brocot sequence

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*À la mémoire de Philippe Flajolet*

## 1. Introduction

**1.1. Lacunary power series and continued fraction expansions.** Let $\Lambda = (\lambda_n)_{n\ge 0}$ be a sequence of integers with $0 < \lambda_0 < \lambda_1 < \cdots$ satisfying $\lambda_{n+1}/\lambda_n > 2$ for all $n \ge 0$. Consider the formal power series $F(X) := \sum_{n\ge 0}(-1)^{\varepsilon_n} X^{-\lambda_n}$, where $\varepsilon_n = 0, 1$. As is well known, a power series in $X^{-1}$ can be represented by a continued fraction $[A_0(X), A_1(X), A_2(X), \ldots]$, where the $A_j$'s are polynomials in $X$, and for all $i > 0$, $A_i(X)$ is a non-constant polynomial. Quite obviously, in the case of the above $F(X)$, one has $A_0(X) = 0$.

Let $P_n(X)/Q_n(X) = [0, A_1(X), A_2(X), \ldots, A_n(X)]$ be the $n$th convergent of $F(X)$. As was already discovered in [3] and [28], the denominators $Q_n(X)$ are particularly interesting to study: their coefficients are $0, \pm 1$.

**1.2. A sequence of polynomials and a sequence of integers.** The denominators $Q_n(X)$ introduced above can be quite explicitly expressed (see [28]):

$$Q_n(X) = \sum_{k\ge 0} \sigma(k,\varepsilon) \binom{(n+k)/2}{k}_2 X^{\mu(k,\Lambda)}.$$

The exponent of $X$ is given by $\mu(k,\Lambda) = \sum_{q\ge 0} e_q(k)(\lambda_q - \lambda_{q-1})$, with $\lambda_{-1} = 0$, where $e_q(k)$ is the $q$th binary digit of $k = \sum_{q\ge 0} e_q(k) 2^q$. The sign of the monomials is given by $\sigma(k,\varepsilon) = (-1)^{\nu(k)+\bar{\mu}(k,\varepsilon)}$ where $\nu(k)$ is the number of occurrences of the block 10 in the usual left-to-right reading of the binary

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expansion of $k$ (e.g., $\nu(\text{twelve}) = 1$), and where $\bar{\mu}(k, \varepsilon) = \sum_{q \geq 0} e_q(k)(\varepsilon_{q-1} - \varepsilon_{q-2})$, with $\varepsilon_{-1} = \varepsilon_{-2} = 0$. The symbol $\binom{a}{b}_2$ is an *integer* equal to 0 or 1, according to the value modulo 2 of the binomial coefficient $\binom{a}{b}$, with the following convention: if $a$ is not an integer, or if $a$ is a positive integer and $a < b$, then $\binom{a}{b} := 0$. For example, as soon as $n$ and $k$ have opposite parities, $\binom{(n+k)/2}{k}_2 = 0$. In [3] it was observed that the number of nonzero monomials in $Q_n(X)$ is $u_n$, the $n$th term of the celebrated *Stern–Brocot sequence* defined by $u_0 = u_1 = 1$ and the recursive relations $u_{2n} = u_n + u_{n-1}$, $u_{2n+1} = u_n$ for all $n \geq 1$. This sequence is also called the *Stern diatomic series* (see sequence A002487 in [30]). It was studied by several authors: see, e.g., [17] and its list of references (including the historical references [9, 31]), see also [33, 29], or see [23] for a relation between the Stern sequence and the Towers of Hanoi. (Note that some authors have the slightly different definition: $v_0 = 0$, $v_{2n} = v_n$, $v_{2n+1} = v_n + v_{n+1}$; clearly $u_n = v_{n+1}$ for all $n \geq 0$.)

Our purpose here is to pursue our previous discussions on the sequence of polynomials $Q_n(X)$ in relationship with the Stern-Brocot sequence.

Remark 1.1. The sequence $(\nu(n))_{n \geq 0}$ happens to be related to the paperfolding sequence. Indeed, define $v(n) := (-1)^{\nu(n)}$ and $w(n) := v(n)v(n+1)$. From the definition of $\nu$, we have for every $n \geq 0$ the relations $v(2n+1) = v(n)$, $v(4n) = v(2n)$, and $v(4n+2) = -v(n)$. Equivalently, for every $n \geq 0$, we have $v(2n+1) = v(n)$, and $v(2n) = (-1)^n v(n)$. Hence, for every $n \geq 0$, we have $w(n) = v(2n)v(2n+1) = (-1)^n (v(n)^2) = (-1)^n$, and $w(2n+1) = v(2n+1)v(2n+2) = (-1)^{n+1} v(n)v(n+1) = (-1)^{n+1} w(n)$. It it then clear that, if $z(n) := w(2n+1)$, then $z(2n) = -w(2n) = -(-1)^n$ and $z(2n+1) = z(n)$. In other words the sequence $(z(n))_{n \geq 0}$ is the classical paperfolding sequence, and the sequence $(w(n))_{n \geq 0}$ itself is a paperfolding sequence (see e.g. [26, p. 125] where the sequences are indexed by $n \geq 1$ instead of $n \geq 0$).

**1.3. A partial order on the integers.** Let $m = e_0(m)e_1(m)\ldots$ and $k = e_0(k)e_1(k)\ldots$ be two nonnegative integers together with their binary expansion, which of course terminates with a tail of 0's. Lucas [25] observed that

$$\binom{m}{k} \equiv \prod_{i \geq 0} \binom{e_i(m)}{e_i(k)} \bmod 2.$$

This implies the following relation (in $\mathbb{Z}$):

$$\binom{m}{k}_2 = \prod_{i \geq 0} \binom{e_i(m)}{e_i(k)},$$

so that we have $\binom{m}{k}_2 = 1$ if and only if $e_i(k) \leq e_i(m)$ for all $i \geq 0$.

We will say that $m$ *dominates* $k$, and we write $k \ll m$, if $e_i(k) \leq e_i(m)$ for all $i \geq 0$. In other words the sequence $k \to \binom{m}{k}_2$ is the characteristic function of the $k$'s dominated by $m$. (This order was used in, e.g., [2].)

As a consequence of our remarks, the Stern–Brocot sequence has the following representation:

$$u_n = \sum_{k \ll (k+n)/2} 1.$$

REMARK 1.2. This last relation can be easily deduced from a result of Carlitz [11, 12] (Carlitz calls $\theta_0(n)$ what we call $u_n$):

$$u_n = \sum_{0 \leq 2r \leq n} \binom{n-r}{r}_2.$$

Indeed, we have

$$\begin{aligned}
\sum_{k \ll (k+n)/2} 1 &= \sum_{\substack{0 \leq k \leq n \\ k \equiv n \bmod 2}} \binom{(k+n)/2}{k}_2 \\
&= \sum_{\substack{0 \leq k' \leq n \\ k' \equiv 0 \bmod 2}} \binom{n-k'/2}{n-k'}_2 && \text{(let } k' = n-k\text{)} \\
&= \sum_{0 \leq 2r \leq n} \binom{n-r}{n-2r}_2 = \sum_{0 \leq 2r \leq n} \binom{n-r}{r}_2 && \text{(use } \tbinom{a}{b} = \tbinom{a}{a-b}\text{)}.
\end{aligned}$$

Also note that in [12] the range $0 \leq 2r < n$ should be replaced by $0 \leq 2r \leq n$ as in [11] (see also [17, Corollary 6.2] where the index $n$ should be adjusted). Let us finally indicate that this remark is also Corollary 13 in [3].

REMARK 1.3. The relation $u_n = \sum_{0 \leq 2r \leq n} \binom{n-r}{r}_2$ can give the idea (inspired by the classical *binomial transform*) of introducing a map on sequences $(a_n)_{n \geq 0} \mapsto (b_n)_{n \geq 0}$ with $b_n := \sum_{0 \leq 2r \leq n} \binom{n-r}{r}_2 a_r$, so that in particular the image of the constant sequence 1 is the Stern–Brocot sequence. One can also go a step further by defining a map $\mathcal{C}$ which associates with two sequences $\mathbf{a} = (a_n)_{n \geq 0}$ and $\mathbf{b} = (b_n)_{n \geq 0}$ the sequence

$$\mathcal{C}(\mathbf{a}, \mathbf{b}) := \left( \sum_{0 \leq 2r \leq n} \binom{n-r}{r}_2 a_r b_{n-r} \right)_{n \geq 0}.$$

It is unexpected that some variations on the Stern–Brocot sequences (different from but in the spirit of the twisted Stern sequence of [8]) are related to the celebrated Thue–Morse sequence (see, e.g., [5]). In fact, recall that the $\pm 1$ Thue–Morse sequence $\mathbf{t} = (t_n)_{n \geq 0}$ can be defined by $t_0 = 1$ and, for all $n \geq 0$, $t_{2n} = t_n$ and $t_{2n+1} = -t_n$. Now define the sequences $\alpha = (\alpha_n)_{n \geq 0}$,

$\beta = (\beta_n)_{n\ge 0}$, $\gamma = (\gamma_n)_{n\ge 0}$ by

$$\alpha := \mathcal{C}(\mathbf{t}, \mathbf{1}), \quad \beta := \mathcal{C}(\mathbf{1}, \mathbf{t}), \quad \gamma := \mathcal{C}(\mathbf{t}, \mathbf{t}).$$

Then the reader can check that these sequences satisfy

$$\begin{array}{llll}
\alpha(0) = 1, & \alpha(1) = 1, & \forall n \ge 1, & \alpha_{2n} = \alpha_n - \alpha_{n-1}, \quad \alpha_{2n+1} = \alpha_n, \\
\beta(0) = 1, & \beta(1) = -1, & \forall n \ge 1, & \beta_{2n} = \beta_n - \beta_{n-1}, \quad \beta_{2n+1} = -\beta_n, \\
\gamma(0) = 1, & \gamma(1) = -1, & \forall n \ge 1, & \gamma_{2n} = \gamma_n + \gamma_{n-1}, \quad \gamma_{2n+1} = -\gamma_n,
\end{array}$$

so that, with the notation of [30],

$$\begin{aligned}
(\alpha_n)_{n\ge 0} &= (A005590(n+1))_{n\ge 0}, \\
(\beta_n)_{n\ge 0} &= (A177219(n+1))_{n\ge 0}, \\
(\gamma_n)_{n\ge 0} &= (A049347(n))_{n\ge 0}.
\end{aligned}$$

The last sequence $(\gamma_n)_{n\ge 0}$ is the 3-periodic sequence with period $(1, -1, 0)$ (hint: prove by induction on $n$ that $(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+2}) = (1, -1, 0)$ for all $j \le n$).

**2. More on the sequence $Q_n(X)$ and a note on $P_n(X)$ for a special $\Lambda$.** We now specialize to the case $\lambda_n = 2^{n+1} - 1$. In that case, $\mu(k, \Lambda) = k$. Also note that $\sigma(k, \varepsilon) \equiv 1 \bmod 2$. Let $P_n(X)/Q_n(X)$ denote as previously the $n$th convergent of the continued fraction of the formal power series $\sum_{i\ge 1} (-1)^{\varepsilon_i} X^{1-2^i}$. We begin with a short subsection on $P_n$. The rest of the section will be devoted to the "simpler" polynomials $Q_n$.

**2.1. The sequence $P_n$ modulo 2**

Theorem 2.1. *We have $P_n(X) \equiv Q_{n-1}(X) \bmod 2$ for $n \ge 1$.*

*Proof.* Let $F(X) = \sum_{i\ge 1} (-1)^{\varepsilon_i} X^{1-2^i}$. Define the formal power series $\Phi(X)$ by its continued fraction expansion $\Phi(X) = [0, X, X, \ldots]$. Its $n$th convergent is given by $\pi_n(X)/\kappa_n(X) = [0, X, \ldots, X]$ ($n$ partial quotients equal to $X$). An immediate induction shows that $\pi_n(X) = \kappa_{n-1}(X)$ for $n \ge 1$. Reducing $F(X)$ modulo 2, we see that $F^2(X) + XF(X) + 1 \equiv 0 \bmod 2$. On the other hand $\Phi(X) = 1/(X + \Phi(X))$, hence $\Phi^2(X) + X\Phi(X) + 1 \equiv 0 \bmod 2$. This implies that $F(X) \equiv \Phi(X) \bmod 2$. Hence $P_n(X) \equiv \pi_n(X) \bmod 2$ and $Q_n(X) \equiv \kappa_n(X) \bmod 2$: to be sure that the convergents of the reduction modulo 2 of $F$ are equal to the reduction modulo 2 of the convergents of $F(X)$, the reader can look at, e.g., [34]. Thus $P_n(X) \equiv \pi_n(X) = \kappa_{n-1}(X) \equiv Q_{n-1}(X) \bmod 2$. ∎

Corollary 2.2. *The following congruence is satisfied by $Q_n(X)$ for $n \ge 1$:*

$$Q_n^2(X) - Q_{n+1}(X)Q_{n-1}(X) \equiv 1 \bmod 2.$$

*Proof.* Use the classical identity $P_{n+1}(X)Q_n(X) - P_n(X)Q_{n+1}(X) = (-1)^n$ for the convergents of a continued fraction. ∎

**2.2. The sequence $Q_n$ and the Chebyshev polynomials.** We have the formula

$$Q_n(X) \equiv \sum_{k\ge 0} \binom{(n+k)/2}{k}_2 X^k \equiv \sum_{\substack{0\le k\le n\\ k\equiv n \bmod 2}} \binom{(k+n)/2}{k}_2 X^k \mod 2.$$

The Chebyshev polynomials of the second kind (see, e.g., [20, pp. 184–185]) are defined by

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}\cdot$$

They have the well-known explicit expansion

$$U_n(X) = \sum_{0\le k\le n/2} (-1)^k \binom{n-k}{k} (2X)^{n-2k}.$$

We thus get a relationship between $Q_n$ and $U_n$ (compare with the related but not identical result [17, Proposition 6.1]).

THEOREM 2.3. *The reductions modulo* 2 *of* $Q_n(X)$ *and of* $U_n(X/2)$ *are equal.*

*Proof.* We can write modulo 2

$$\begin{aligned} Q_n(X) &\equiv \sum_{\substack{0\le k'\le n\\ k'\equiv 0 \bmod 2}} \binom{n-k'/2}{n-k'}_2 X^{n-k'} && (\text{by letting } k' = n-k)\\ &\equiv \sum_{0\le 2r\le n} \binom{n-r}{n-2r}_2 X^{n-2r}\\ &\equiv \sum_{0\le 2r\le n} \binom{n-r}{r}_2 X^{n-2r} && \left(\text{by using } \tbinom{a}{b} = \tbinom{a}{a-b}\right). \end{aligned}$$

Hence $Q_n(X) \equiv U_n(X/2) \bmod 2$. ∎

As an immediate application of Theorem 2.3 (and of Remark 1.2) we have the following results.

COROLLARY 2.4. *The number of odd coefficients in the* (*scaled*) *Chebyshev polynomial of the second kind* $U_n(X/2)$ *is equal to the Stern–Brocot sequence* $u_n$.

REMARK 2.5. Corollary 2.2 above can also be deduced from Theorem 2.3 using a classical relation for Chebyshev polynomials implied by their expression using sines.

Remark 2.6. The polynomials $Q_n(X)$ are also related to the Fibonacci polynomials (see, e.g., [19]) and to Morgan-Voyce polynomials, which are a variation on the Chebyshev polynomials (for more on Morgan-Voyce polynomials, introduced by Morgan-Voyce in dealing with electrical networks, see e.g. [32, 7, 22] and the references therein). Indeed, the Fibonacci polynomials satisfy

$$F_{n+1}(X) = \sum_{2j \le n} \binom{n-j}{j} X^{n-2j}$$

(compare with the proof of Theorem 2.3), while the Morgan-Voyce polynomials satisfy

$$b_n(X) = \sum_{k \le n} \binom{n+k}{n-k} X^k \quad \text{and} \quad B_n(X) = \sum_{k \le n} \binom{n+k+1}{n-k} X^k$$

(note that $\binom{n+k}{n-k} = \binom{n+k}{2k}$, that $\binom{n+k+1}{n-k} = \binom{n+k+1}{2k+1}$, and see Lemmas 3.1 and 3.3 below).

Remark 2.7. The polynomials that we have defined are related to the Stern–Brocot sequence, but they differ from Stern polynomials occurring in the literature, in particular they are not the same as those introduced in [24]. They also differ from the polynomials studied in [17, 18].

### 2.3. Extension of $Q_n(X)$ to $Q_\omega(X)$ with $\omega \in \mathbb{Z}_2$

Definition 2.8. Let $\omega = \sum_{i \ge 0} \omega_i 2^i = \omega_0\omega_1\omega_2 \ldots \in \mathbb{Z}_2$ be a 2-adic integer, or equivalently an infinite sequence of 0's and 1's. For a nonnegative integer $k$ whose binary expansion is given by $k = \sum_{i \ge 0} k_i 2^i$, we define

$$\binom{\omega}{k}_2 = \prod_{i \ge 0} \binom{\omega_i}{k_i}.$$

The infinite product $\binom{\omega}{k}_2$ is well defined since, for large $i$, $\binom{\omega_i}{k_i}$ reduces to $\binom{\omega_i}{0} = 1$. It is equal to 0 or 1. The above product extends Lucas' observation to all 2-adic integers $\omega$. In particular, since $-1 = \sum_{i \ge 0} 2^i = 1^\infty$, we see that $-1$ dominates all $k \in \mathbb{N}$ (where the order introduced in Section 1.3 is generalized in the obvious way). A similar definition (of binomials and order) occurs in [27].

Definition 2.9. In the general case for $\Lambda$, with $\lambda_{n+1}/\lambda_n > 2$, and $\varepsilon = 0, 1$, the polynomials $Q_n(X)$ above naturally extend to formal power series $Q_\omega(X)$ defined for $\omega = \omega_0\omega_1\omega_2 \ldots \in \mathbb{Z}_2$ by

$$Q_\omega(X) = \sum_{k \ge 0} \sigma(k,\varepsilon) \binom{(\omega+k)/2}{k}_2 X^{\mu(k,\Lambda)} = \sum_{\substack{k \equiv \omega \bmod 2 \\ k \ll (\omega+k)/2}} \sigma(k,\varepsilon) X^{\mu(k,\Lambda)}.$$

Remark 2.10. The reader can check (e.g., by using integer truncations of $\omega$ tending to $\omega$) that

$$\binom{\omega}{k} \equiv \binom{\omega}{k}_2 \bmod 2$$

where the binomial coefficient $\binom{\omega}{k}$ is defined by

$$\binom{\omega}{k} = \frac{\omega(\omega-1)\ldots(\omega-k+1)}{k!} \in \mathbb{Z}_2.$$

In particular, we see that for any 2-adic integer $\ell$,

$$\binom{-\ell}{k} = (-1)^k \binom{\ell+k-1}{k}, \quad \text{hence} \quad \binom{-\ell}{k}_2 = \binom{\ell+k-1}{k}_2.$$

Now for $n \in \mathbb{N}$ we have

$$\begin{aligned} Q_{-n}(X) &= \sum_{k\geq 0} \sigma(k,\varepsilon) \binom{(-n+k)/2}{k}_2 X^{\mu(k,\Lambda)} \\ &= \sum_{k\geq 0} \sigma(k,\varepsilon) \binom{-(n-k)/2}{k}_2 X^{\mu(k,\Lambda)}, \end{aligned}$$

thus

$$\begin{aligned} Q_{-n}(X) &= \sum_{k\geq 0} \sigma(k,\varepsilon) \binom{(n-k)/2+k-1}{k}_2 X^{\mu(k,\Lambda)} \\ &= \sum_{k\geq 0} \sigma(k,\varepsilon) \binom{(n-2+k)/2}{k}_2 X^{\mu(k,\Lambda)} = Q_{n-2}(X). \end{aligned}$$

In particular $Q_{-n}$ and $Q_{n-2}$ have same degree. Also note that the definition of $Q_{-n}$ for $n \in \mathbb{N}$ yields

$$Q_{-1}(X) = \sum_{k\geq 0} \sigma(k,\varepsilon) \binom{(k-1)/2}{k}_2 X^{\mu(k,\Lambda)} = 0.$$

Remark 2.11. If $\lambda_n = 2^{n+1} - 1$, Corollary 2.2 can be extended to 2-adic integers: using again truncations of $\omega$ tending to $\omega$ yields, for any 2-adic integer $\omega$,

$$Q_\omega^2(X) - Q_{\omega+1}(X) Q_{\omega-1}(X) \equiv 1 \bmod 2.$$

**2.4. Extension of the sequence $(u_n)_{n\geq 0}$ to negative indices.** What precedes suggests two ways of extending the sequence $(u_n)_{n\geq 0}$ to negative integer indices. First, we noted the relation $u_n = \sum_{k \ll (n+k)/2} 1$, i.e., $u_n$ is the number of monomials with nonzero coefficients in $Q_n(X)$. But from the previous section, we can define $Q_{-n}(X)$ for $n \in \mathbb{N}$, and we have $Q_{-n}(X) = Q_{n-2}(X)$. This suggests the definition

$$u_{-n} := u_{n-2} \quad \text{for all } n \geq 2.$$

Strictly speaking, this definition leaves the value $u_{-1}$ indeterminate, but, since $u_n$ is the number of monomials with nonzero coefficients in $Q_n$, the remark above that $Q_{-1} = 0$ implies $u_{-1} = 0$.

Another way of generalizing $u_n$ to negative indices would be to use the recursion

$$u_{2n} = u_n + u_{n-1}, \quad u_{2n+1} = u_n, \quad \text{for all } n \geq 1,$$

allowing nonpositive values for $n$. Letting first $n = 0$ leads to $u_0 = u_0 + u_{-1}$, hence $u_{-1} = 0$. On the other hand we claim that the relation $u_{-n} := u_{n-2}$ for all $n \geq 2$ leads to the same recursion formulas for $u_{2n}$ and $u_{2n+1}$ with nonpositive $n$. Indeed, let $m = -n$ with $n \geq 2$. Then

$$u_{2m} = u_{-2n} = u_{2n-2} = u_{2(n-1)} = u_{n-1} + u_{n-2} = u_{-n-1} + u_{-n} = u_{m-1} + u_m$$

and

$$u_{2m+1} = u_{-2n+1} = u_{2n-3} = u_{2(n-2)+1} = u_{n-2} = u_{-n} = u_m.$$

We thus finally have a generalization compatible with both approaches, yielding

$$\ldots, u_{-4} = 2,\ u_{-3} = 1,\ u_{-2} = 1,\ u_{-1} = 0,\ u_0 = 1,\ u_1 = 1,\ u_2 = 2,\ u_3 = 1, \ldots$$

and the following

Definition 2.12. The Stern–Brocot sequence $(u_n)_{n\geq 0}$ can be extended to a sequence $(u_n)_{n\in\mathbb{Z}}$ by letting $u_{-n} = u_{n-2}$ for $n \geq 2$, and $u_{-1} = 0$. This sequence satisfies the same recursive relations as the initial sequence $(u_n)_{n\geq 0}$, namely $u_{2n} = u_n + u_{n-1}$ and $u_{2n+1} = u_n$ for all $n \in \mathbb{Z}$.

**3. The arithmetical nature of the power series $Q_\omega(X)$.** Recall that the formal series $Q_\omega(X)$, where $\omega = \omega_0\omega_1 \ldots$ belongs to $\mathbb{Z}_2$, is given by

$$Q_\omega(X) = \sum_{k\geq 0} \sigma(k,\varepsilon) \binom{(\omega+k)/2}{k}_2 X^{\mu(k,\Lambda)} = \sum_{\substack{k\equiv\omega \bmod 2 \\ k\ll(\omega+k)/2}} \sigma(k,\varepsilon) X^{\mu(k,\Lambda)}.$$

We have seen that $Q_\omega(X)$ reduces to a polynomial if $\omega$ belongs to $\mathbb{Z}$. We will prove that this is a necessary and sufficient condition for this series to be a polynomial. Then we will address the question of the algebraicity of $Q_\omega(X)$, on $\mathbb{Q}(X)$ and on $\mathbb{Z}/2\mathbb{Z}(X)$, in the special case $\lambda_n = 2^{n+1} - 1$. We begin with a lemma.

Lemma 3.1. *Let $\omega = \omega_0\omega_1 \ldots$ belong to $\mathbb{Z}_2$. Then:*

(i) *For every $j \geq 0$,*

$$\binom{\omega + 2^j}{2^{j+1}}_2 \equiv \omega_j + \omega_{j+1} \bmod 2.$$

(ii) *The sequence* $\left(\binom{\omega+2^j}{2^{j+1}}_2\right)_{j\geq 0}$ *is ultimately periodic if and only if* $\omega$ *is rational.*
(iii) *The sequence* $\left(\binom{\omega+2^j}{2^{j+1}}_2\right)_{j\geq 0}$ *is ultimately equal to* 0 *if and only if* $\omega$ *is an integer.*
(iv) *For every* $k \geq 0$,

$$\binom{(\omega+k)/2}{k}_2 = \binom{\omega+k+1}{2k+1}_2.$$

(v) *If* $\omega \neq -1$, *there exist an integer* $\ell \geq 0$ *and a* 2*-adic integer* $\omega'$ *such that* $\omega = 2^\ell - 1 + 2^{\ell+1}\omega'$. *Let* $f_\omega(k) := \binom{(\omega+k)/2}{k}_2 = \binom{\omega+k+1}{2k+1}_2$. *Then for any integer* $k'$ *we have* $f_\omega(2^\ell - 1 + 2^{\ell+1}k') = \binom{\omega'+k'}{2k'}_2$.
(vi) *If there exist* $\ell \geq 0$ *and* $j \geq 0$ *with* $\omega = 2^\ell - 1 + 2^{\ell+1}(2^j(2\omega'+1))$, *then for any integer* $k'$ *we have* $f_\omega(2^\ell - 1 + 2^{\ell+1}(2^j(2k'+1))) = \binom{\omega'+k'+1}{2k'+1}_2$.

*Proof.* In order to prove (i) we write

$$\begin{array}{rcccccccc} \omega + 2^j & = & & \omega_0 & \omega_1 & \dots & \omega_j & \omega_{j+1} & \dots \\ & & + & 0 & 0 & \dots & 1 & 0 & \dots \\ & = & & \omega_0 & \omega_1 & \dots & \alpha_j & \alpha_{j+1} & \dots \end{array}$$

where $\alpha_j$ and $\alpha_{j+1}$ are given by

$$\begin{array}{ll} \text{if } \omega_j = 0 \text{ and } \omega_{j+1} = 0, & \text{then } \alpha_j = 1 \text{ and } \alpha_{j+1} = 0 \\ \text{if } \omega_j = 0 \text{ and } \omega_{j+1} = 1, & \text{then } \alpha_j = 1 \text{ and } \alpha_{j+1} = 1 \\ \text{if } \omega_j = 1 \text{ and } \omega_{j+1} = 0, & \text{then } \alpha_j = 0 \text{ and } \alpha_{j+1} = 1 \\ \text{if } \omega_j = 1 \text{ and } \omega_{j+1} = 1, & \text{then } \alpha_j = 0 \text{ and } \alpha_{j+1} = 0. \end{array}$$

By inspection we see that $\alpha_{j+1} \equiv \omega_j + \omega_{j+1} \bmod 2$. Now we write

$$\begin{aligned} \binom{\omega+2^j}{2^{j+1}}_2 &= \left(\prod_{0\leq k\leq j-1} \binom{\omega_k}{0}_2\right)\binom{\alpha_j}{0}_2\binom{\alpha_{j+1}}{1}_2\left(\prod_{k\geq j+2}\binom{\alpha_k}{0}_2\right) \\ &= \alpha_{j+1} \equiv \omega_j + \omega_{j+1} \bmod 2. \end{aligned}$$

Let us prove (ii). We note that the sequence $((\omega_j + \omega_{j+1}) \bmod 2)_{j\geq 0}$ is ultimately periodic if and only if the sequence $(\omega_j \bmod 2)_{j\geq 0}$ is ultimately periodic (hence if and only if the sequence $(\omega_j)_{j\geq 0}$ itself is ultimately periodic): indeed, $((\omega_j + \omega_{j+1}) \bmod 2)_{j\geq 0}$ is ultimately periodic if and only if the formal power series $G(X) := \sum_{j\geq 0}(\omega_j + \omega_{j+1})X^j$ is rational (as an element of $\mathbb{Z}/2\mathbb{Z}[[X]]$). But, if we let $H(X)$ denote the formal power series $H(X) := \sum_{j\geq 0}\omega_j X^j \in \mathbb{Z}/2\mathbb{Z}[[X]]$, then $XG(X) + \omega_0 = (1+X)H(X)$. So $G(X)$ is rational if and only if $H$ is, if and only if $(\omega_j \bmod 2)_{j\geq 0}$ is ultimately periodic, i.e., if the 2-adic integer $\omega$ is rational.

To prove (iii), we note that $\binom{\omega+2^j}{2^{j+1}}_2 = 0$ for $j$ large enough implies by (i) that $\omega_j + \omega_{j+1} \equiv 0 \bmod 2$ for $j$ large enough. This means that $\omega_j \equiv \omega_{j+1} \bmod 2$ for $j$ large enough, or equivalently $\omega_j = \omega_{j+1}$ for $j$ large enough. But then either $\omega_j = \omega_{j+1} = 0$ for large $j$, hence $\omega$ is a nonnegative integer, or $\omega_j = \omega_{j+1} = 1$ for large $j$, hence $\omega$ is a negative integer. We thus conclude that $\omega$ belongs to $\mathbb{Z}$. The converse is straightforward.

We prove (iv) by considering the parities of $\omega$ and $k$. First note that if $\omega$ and $k$ have opposite parities, then $\binom{(\omega+k)/2}{k}_2 = 0$ while $\binom{\omega+k+1}{2k+1}_2 = 0$ (use Definition 2.8 and look at the last digit of $\omega+k+1$ and of $2k+1$). Now if $\omega = 2\omega'$ and $k = 2k'$, we have $\binom{(\omega+k)/2}{k}_2 = \binom{\omega'+k'}{2k'}_2$ while $\binom{\omega+k+1}{2k+1}_2 = \binom{2(\omega'+k')+1}{4k'+1}_2 = \binom{\omega'+k'}{2k'}_2$ (use Definition 2.8 again). Finally if $\omega = 2\omega'+1$ and $k = 2k'+1$, we have $\binom{(\omega+k)/2}{k}_2 = \binom{\omega'+k'+1}{2k'+1}_2$ while $\binom{\omega+k+1}{2k+1}_2 = \binom{2(\omega'+k'+1)+1}{4k'+3}_2 = \binom{\omega'+k'+1}{2k'+1}_2$ (by Definition 2.8 once more).

Let us prove (v). Since $\omega \neq -1$, its 2-adic expansion contains at least one zero. Write $\omega = 11\ldots10\omega_{\ell+1}\omega_{\ell+2}\ldots$, so that the 2-adic expansion of $\omega$ begins with exactly $\ell \geq 0$ ones. Defining $\omega' := \omega_{\ell+1}\omega_{\ell+2}\ldots$, we thus have $\omega = 2^\ell - 1 + 2^{\ell+1}\omega'$. Now for any integer $k'$ we have, from Definition 2.8,

$$f_\omega(2^\ell - 1 + 2^{\ell+1}k') = \binom{\omega + 2^\ell + 2^{\ell+1}k'}{2^{\ell+1} - 1 + 2^{\ell+1}(2k')}_2 = \binom{2^{\ell+1} - 1 + 2^{\ell+1}(\omega'+k')}{2^{\ell+1} - 1 + 2^{\ell+1}(2k')}_2 = \binom{\omega'+k'}{2k'}_2.$$

We finally prove (vi). Using (v) we see that

$$f_\omega(2^\ell - 1 + 2^{\ell+1}(2^j(2k'+1))) = \binom{2^j(2\omega'+1+2k'+1)+1}{2^{j+1}(2k'+1)+1}_2 = \binom{\omega'+k'+1}{2k'+1}_2. \blacksquare$$

Now we can prove the following result.

THEOREM 3.2. *Let $\omega$ be a 2-adic integer. The formal power series $Q_\omega(X)$ is a polynomial if and only if $\omega$ belongs to $\mathbb{Z}$.*

*Proof.* If $n$ is a nonnegative integer, then $Q_n(X)$ is a polynomial. So is $Q_{-n}(X)$ for $n \neq 1$ because $Q_{-n} = Q_{n-2}$ as we have seen in Remark 2.10. On the other hand $Q_{-1}(X)$ is also a polynomial since $Q_{-1}(X) = 0$. Conversely suppose that $Q_\omega(X)$ is a polynomial for some $\omega = \omega_0\omega_1\ldots$ in $\mathbb{Z}_2$. The coefficients of the monomials $X^{\mu(k,\Lambda)}$ in $Q_\omega(X)$, that is, $\sigma(k,\varepsilon)\binom{(\omega+k)/2}{k}_2$, are equal to zero for $k$ large enough. Thus $f_\omega(k) = \binom{(\omega+k)/2}{k}_2$ is zero for $k$ large enough. We may suppose that $\omega \neq -1$; hence, using the notation in Lemma 3.1(v), we certainly have $f_\omega(2^\ell - 1 + 2^{\ell+1}k') = 0$ for $k'$ large enough.

Using Lemma 3.1(v), we thus have $\binom{\omega'+k'}{2k'}_2 = 0$ for $k'$ large enough. This implies $\binom{\omega'+2^j}{2^j+1}_2 = 0$ for $j$ large enough. Lemma 3.1(iii) shows that $\omega'$, hence $\omega$, belongs to $\mathbb{Z}$. ∎

Before proving our Theorem 3.5 characterizing the algebraicity of the series $Q_\omega(X)$ for a special $\Lambda$, we need a lemma.

Lemma 3.3. *Let $\omega = \omega_0\omega_1\ldots$ be a 2-adic integer. Let $(f_\omega(k))_{k\geq 0}$, $(g_\omega(k))_{k\geq 0}$, $(h_\omega(k))_{k\geq 0}$ denote the sequences*

$$f_\omega(k) := \binom{\omega+k+1}{2k+1}_2, \qquad g_\omega(k) := \binom{\omega+k}{2k}_2, \qquad h_\omega(k) := \binom{\omega+k}{2k+1}_2.$$

*Then we have the following relations:*

$$\begin{array}{lll}
f_{2\omega}(2k) = g_\omega(k), & g_{2\omega}(2k) = g_\omega(k), & h_{2\omega}(2k) = 0, \\
f_{2\omega+1}(2k) = 0, & g_{2\omega+1}(2k) = g_\omega(k), & h_{2\omega+1}(2k) = g_\omega(k), \\
f_{2\omega}(2k+1) = 0, & g_{2\omega}(2k+1) = h_\omega(k), & h_{2\omega}(2k+1) = h_\omega(k), \\
f_{2\omega+1}(2k+1) = f_\omega(k), & g_{2\omega+1}(2k+1) = f_\omega(k), & h_{2\omega+1}(2k+1) = 0.
\end{array}$$

*Proof.* The proof is easy: it uses the definition of $\binom{\omega}{\ell}_2$, which in particular shows for any 2-adic integer $\omega$ and any integer $\ell$ that

$$\binom{2\omega}{2\ell}_2 = \binom{\omega}{\ell}_2\binom{0}{0}_2 = \binom{\omega}{\ell}_2, \qquad \binom{2\omega+1}{2\ell}_2 = \binom{\omega}{\ell}_2\binom{1}{0}_2 = \binom{\omega}{\ell}_2,$$
$$\binom{2\omega}{2\ell+1}_2 = \binom{\omega}{\ell}_2\binom{0}{1}_2 = 0, \qquad \binom{2\omega+1}{2\ell+1}_2 = \binom{\omega}{\ell}_2\binom{1}{1}_2 = \binom{\omega}{\ell}_2. \ ∎$$

Remark 3.4. The sequences above occur in the OEIS [30] when $\omega = n$ is an integer. In particular, $\left(\binom{(n+k)/2}{k}\right)_{n,k} = \left(\binom{n+k+1}{2k+1}\right)_{n,k}$ is equal to A168561; also $\left(\binom{n+k}{2k}\right)_{n,k}$ is equal to A085478; finally, up to shifting $k$, we see that $\left(\binom{n+k}{2k+1}\right)_{n,k}$ is equal to A078812.

We can also note that $f_\omega(k) \equiv g_\omega(k) + h_\omega(k) \bmod 2$, for any integer $k \geq 0$.

Theorem 3.5. *Suppose that $\lambda_n = 2^{n+1} - 1$. Then:*

- *The formal power series $Q_\omega(X)$ is either a polynomial if $\omega \in \mathbb{Z}$ or a transcendental series over $\mathbb{Q}(X)$ if $\omega \in \mathbb{Z}_2 \setminus \mathbb{Z}$.*
- *The formal power series $Q_\omega(X)$ is algebraic over $\mathbb{Z}/2\mathbb{Z}(X)$ if and only if $\omega$ is rational. It is rational if and only if it is a polynomial, which happens if and only if $\omega$ is a rational integer.*

*Proof.* The first assertion is a consequence of a classical theorem of Fatou [21] which states that *a power series $\sum_{n\geq 0} a_n z^n$ with integer coefficients that converges inside the unit disk is either rational or transcendental over $\mathbb{Q}(z)$*. This implies that the formal power series $Q_\omega(X)$ is either rational or transcendental over $\mathbb{Q}(X)$. We then have to prove that if $Q_\omega$ is a rational function, then it is a polynomial, or equivalently that $\omega$ is a rational integer (use Theorem 3.2). Now to say that $Q_\omega$ is rational is to say that the sequence of its coefficients is ultimately periodic, which implies that the sequence of their absolute values $(f_\omega(k))_{k\geq 0} = \left(\binom{\omega+k+1}{2k+1}_2\right)_{k\geq 0}$ is ultimately periodic. Let $\theta$ be its period. We observe, for large $k$, that $\binom{\omega+k+1}{2k+1}_2 = \binom{\omega+k+\theta+1}{2(k+\theta)+1}_2$. If $\theta$ is odd, the left side is zero for $\omega+k$ odd while the right side is zero for $\omega+k$ even. Thus $\binom{\omega+k+1}{2k+1}_2 = 0$ for large $k$, and $Q_\omega$ is a polynomial. So suppose that $\theta$ is even. Suppose further that $\omega$ does not belong to $\mathbb{Z}$. Then its 2-adic expansion contains infinitely many blocks 01. Consider the first such block: there exist $\ell \geq 0$ and $j \geq 0$ such that $\omega = 2^\ell - 1 + 2^{\ell+1}(2^j(2\omega'+1))$. Then for any integer $k'$ we have $f_\omega(2^\ell - 1 + 2^{\ell+1}(2^j(2k'+1))) = \binom{\omega'+k'+1}{2k'+1}_2$. The sequence $(f_\omega(2^\ell - 1 + 2^{\ell+1}(2^j(2k'+1))))_{k'\geq 0}$ is ultimately periodic and $\theta/2$ is a period. But from Lemma 3.1(vi) this sequence is equal to $\left(\binom{\omega'+k'+1}{2k'+1}_2\right)_{k'\geq 0}$. As previously, either $\theta/2$ is odd and this sequence is ultimately equal to zero, or $\theta/2$ is even. In the first case, as above, $\omega'$ belongs to $\mathbb{Z}$, hence so does $\omega$, which is impossible. In the second case, we iterate the reasoning that used Lemma 3.1(vi), with $\omega$ replaced by $\omega'$ and $k$ by $k'$, where the first block 01 occurring in $\omega$ is replaced by the first such block occurring in $\omega'$. The fact that $\theta$ cannot be divisible by arbitrarily large powers of 2 gives the desired contradiction.

In order to prove the second assertion, we first suppose that $Q_\omega(X)$ is algebraic over $\mathbb{Z}/2\mathbb{Z}(X)$. If $\omega = -1$, then $Q_\omega(X) = 0$. Otherwise write $\omega = 2^\ell - 1 + 2^{\ell+1}\omega'$ as in Lemma 3.1(v). The algebraicity of $Q_\omega(X)$ over $\mathbb{Z}/2\mathbb{Z}(X)$ implies that the sequence $\left(\binom{(\omega+k)/2}{k}_2 \bmod 2\right)_{n\geq 0}$ is 2-automatic (from a theorem of Christol, see [15, 16] or [6]). Using Lemma 3.1(iv) we deduce that the sequence $\left(\binom{\omega+k+1}{2k+1}_2\right)_{k\geq 0}$ is 2-automatic. Thus its subsequence obtained for $k = 2^\ell - 1 + 2^{\ell+1}k'$, namely $\left(\binom{\omega+2^\ell+2^{\ell+1}k'}{2^{\ell+1}-1+2^{\ell+1}(2k')}_2\right)_{k'\geq 0}$, is also 2-automatic (see, e.g., [6, Theorem 6.8.1, p. 189]). But this last sequence is equal to $\left(\binom{2^{\ell+1}-1+2^{\ell+1}(\omega'+k')}{2^{\ell+1}-1+2^{\ell+1}(2k')}_2\right)_{k'\geq 0}$, i.e., to $\left(\binom{\omega'+k'}{2k'}_2\right)_{k'\geq 0}$ (look at the 2-adic expansions and use Definition 2.8). But this in turns implies (see, e.g., [6, Corollary 5.5.3, p. 167]) that the subsequence $\left(\binom{\omega'+2^j}{2^{j+1}}_2\right)_{j\geq 0}$ is ultimately periodic. Using Lemma 3.1(ii) this means that $\omega$ is rational.

Now suppose that $\omega$ is rational. Denote by $T\omega$ the 2-adic integer defined by $T\omega = (\omega - \omega_0)/2$ (i.e., $T\omega$ is the 2-adic integer obtained by shifting the sequence of digits of $\omega$). Also denote by $T^j$ the $j$th iteration of $T$. Define (with the notation of Lemma 3.3) the set

$$\mathcal{K} := \bigcup_{j\in\mathbb{N}} \left\{(f_{T^j\omega}(k))_{k\geq 0}, (g_{T^j\omega}(k))_{k\geq 0}, (h_{T^j\omega}(k))_{k\geq 0}\right\}.$$

As a consequence of Lemma 3.3, $\mathcal{K}$ is stable under the maps defined on $\mathcal{K}$ by $(v_k)_{k\geq 0} \mapsto (v_{2k})_{k\geq 0}$ and $(v_k)_{k\geq 0} \mapsto (v_{2k+1})_{k\geq 0}$ (use that for any 2-adic integer $\omega = \omega_0\omega_1\ldots$ one has $\omega = 2T\omega + \omega_0$). On the other hand Lemma 3.1(iv) shows that $\binom{(\omega+k)/2}{k}_2 = f_\omega(k)$. Hence the 2-kernel of the sequence $\left(\binom{(\omega+k)/2}{k}_2\right)_{k\geq 0}$, i.e., the smallest set of sequences containing that sequence and stable under the maps $(v_k)_{k\geq 0} \mapsto (v_{2k})_{k\geq 0}$ and $(v_k)_{k\geq 0} \mapsto (v_{2k+1})_{k\geq 0}$, is a subset of $\mathcal{K}$. Now, since $\omega$ is rational, the set of 2-adic integers $\{T^j\omega : j \in \mathbb{N}\}$ is finite. Hence the 2-kernel of $\left(\binom{(\omega+k)/2}{k}_2\right)_{k\geq 0}$ is finite and this sequence is 2-automatic (see, e.g., [6]). This implies that the formal power series $Q_\omega(X)$ is algebraic over $\mathbb{Z}/2\mathbb{Z}(X)$ (using again Christol's theorem, see [15, 16] or [6]).

Finally, $Q_\omega(X)$ reduced modulo 2 is rational if and only if the sequence of its coefficients $(f_\omega(k))_{k\geq 0} = \left(\binom{\omega+k+1}{2k+1}_2\right)_{k\geq 0}$ modulo 2 is ultimately periodic, which is the same as saying that the sequence $(f_\omega(k))_{k\geq 0} = \left(\binom{\omega+k+1}{2k+1}_2\right)_{k\geq 0}$ itself is ultimately periodic. But from the first part of the proof this implies that $Q_\omega(X)$ (not reduced modulo 2) is a polynomial, hence that $Q_\omega(X)$ modulo 2 is a polynomial. Conversely, if $Q_\omega(X)$ modulo 2 is a polynomial, then the sequence of its coefficients $(f_\omega(k))_{k\geq 0} = \left(\binom{\omega+k+1}{2k+1}_2\right)_{k\geq 0}$ modulo 2 is ultimately 0, and so is $(f_\omega(k))_{k\geq 0}$ not reduced modulo 2. Thus $Q_\omega(X)$ not reduced modulo 2 is a polynomial, so $\omega$ is a rational integer by using Theorem 3.2. ∎

Remark 3.6. • The authors of [4] prove that the formal power series $(1+X)^\omega = \sum_{k\geq 0}\binom{\omega}{k}_2 X^k$ is algebraic over $\mathbb{Z}/2\mathbb{Z}(X)$ if and only if $\omega$ is rational. They do not ask when that series is rational, i.e., belongs to $\mathbb{Z}/2\mathbb{Z}(X)$, but this is clear since for $\omega = a/b$ with integers $a, b > 0$, we have $((1+X)^\omega)^b \equiv (1+X)^a \bmod 2$. Hence if $(1+X)^\omega$ is a rational function $A/B$ with $A$ and $B$ coprime polynomials, then $A^b \equiv (1+X)^a B^b$, hence $B$ is constant, i.e., $(1+X)^\omega$ is a polynomial. Now if $a < 0$ and $b > 0$, we see that $(1+X)^{-\omega}$ is a polynomial, hence $(1+X)^\omega$ is the inverse of a polynomial. Finally $(1+X)^\omega$ is a rational function if and only if $\omega \in \mathbb{Z}$.

• In the same vein, the authors of [4] prove that, if $\omega_1, \ldots, \omega_d$ are 2-adic integers, then the formal power series $(1+X)^{\omega_1}, \ldots, (1+X)^{\omega_d}$ are algebraically independent over $\mathbb{Z}/2\mathbb{Z}(X)$ if and only if $1, \omega_1, \ldots, \omega_d$ are linearly independent over $\mathbb{Z}$. Is a similar statement true for $Q_\omega$?

• Another question is whether a similar study can be done in the $p$-adic case (here $p = 2$). The two papers [13, 14] might prove useful.

• Results of transcendence, hypertranscendence, and algebraic independence of values for the generating function of the Stern–Brocot sequence have been obtained very recently by Bundschuh (see [10], and the references therein).

• A last question is the arithmetic nature of the real numbers $A(\varepsilon, \omega, g)$ defined by $A(\varepsilon, \omega, g) = \sum_{k \ll (k+\omega)/2} \sigma(k, \varepsilon) g^{-k}$ where $g \geq 2$ is an integer, the sequence $(\varepsilon_n)_n$ is ultimately periodic, and $\omega \in \mathbb{Z}_2 \backslash \mathbb{Z}$. Take in particular $\varepsilon = 0$ (thus $\sigma(k, \varepsilon) = (-1)^{\nu(k)}$). We already know that the number $A(0, \omega, g)$ is transcendental for $\omega \in (\mathbb{Q} \cap \mathbb{Z}_2) \setminus \mathbb{Z}$ by using [1], the fact that $((-1)^{\nu(k)})_{k \geq 0}$ is 2-automatic as recalled above, and the fact that $\left(\binom{(k+\omega)/2}{k}_2\right)_{k \geq 0}$ is 2-automatic for $\omega$ rational as seen in the course of the proof of Theorem 3.5 (the fact that $A(0, \omega, g)$ is not rational is a consequence of the non-ultimate periodicity of $\left((-1)^{\nu(k)} \binom{(k+\omega)/2}{k}_2\right)_{k \geq 0}$ for $\omega$ rational but not a rational integer, which has also been seen in the course of the proof of Theorem 3.5).

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