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# On quadratic character twists of Hecke $L$-functions attached to cusp forms of varying weights at the central point

by

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**1. Introduction.** Let $2k$ be a positive integer divisible by 4. In [13] the second named author showed that the central critical values of Hecke $L$-functions $L(f,s)\,(s\in\mathbb{C})$ of cuspidal normalized Hecke eigenforms $f$ of weight $2k$ with respect to $\mathrm{SL}_2(\mathbb{Z})$ on average satisfy an analogue of the Lindelöf hypothesis when the weight varies, i.e. one has

$$\sum_{f\in\mathcal{F}_{2k}} L(f,k)\ll_\varepsilon k^{1+\varepsilon}\qquad(k\to\infty)\tag{1}$$

for any $\varepsilon>0$, where $\mathcal{F}_{2k}$ is the set of normalized cuspidal Hecke eigenforms of weight $2k$ and the constant implied in $\ll$ only depends on $\varepsilon$ and is effective.

It is also proved in [13] that if one *assumes* a corresponding Lindelöf hypothesis in weight aspect for each individual $f$, i.e. for any $\varepsilon>0$ one has

$$L(f,k)\ll_\varepsilon k^{\varepsilon}\qquad(k\to\infty,\ f\in\mathcal{F}_{2k})$$

(which being optimistic is suggested by (1)), then

$$\#\{f\in\mathcal{F}_{2k}\mid L(f,k)\neq 0\}\gg_\varepsilon k^{1-\varepsilon}\qquad(k\to\infty)\tag{2}$$

for all $\varepsilon>0$. If the constant implied in $\ll$ is effective, then also the one implied in $\gg$ is effective.

Note that one actually expects that $L(f,k)\neq 0$ for all $f\in\mathcal{F}_{2k}$ and all $k$; for more information on this cf. [2]. The latter has been numerically checked for all even $k\le 250$ [2]. According to [9], (2) is true with $\varepsilon=1/2$.

The main ingredient of the proof of the above two assertions in [13] is an estimate from above and below for the Petersson norm $\|f\|$ in weight aspect due to Iwaniec [8] and Hoffstein–Lockart [7], Goldfeld–Hoffstein–Lieman [6], respectively. The other ingredient is a formula (when specialized to the case

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$s = k$) due to the first named author which expresses

$$\sum_{f\in\mathcal{F}_{2k}} \frac{L(f,s)}{\|f\|^2} \qquad (1 < \mathrm{Re}(s) < 2k-1)$$

in terms of an infinite sum of hypergeometric functions [11].

The purpose of this note is to generalize the above two results to the case of a twist of $L(f,s)$ by a quadratic character $(\frac{D}{\cdot})$ where $D$ is a fundamental discriminant and $k$ is an arbitrary positive integer with $(-1)^k D > 0$.

Except for again exploiting the results of [6–8], the proof we shall give for general $D$ is different from that given in [13] for the case $D = 1$. In fact, we shall use Waldspurger's result relating the twisted central critical values to squares of Fourier coefficients of modular forms of half-integral weight in the more explicit version for level 1 given in [12], together with some simple estimates for Fourier coefficients of Poincaré series of half-integral weight.

Remark. Probably our results can also be proved by properly modifying the methods developed by Duke [4]. However, we are not aware of any workout of this in the literature.

**2. Statement of result.** For $k$ an integer $\geq 6$ we denote by $S_{2k}$ the space of cusp forms of weight $2k$ with respect to $\Gamma_1 := \mathrm{SL}_2(\mathbb{Z})$. If $f \in S_{2k}$ and $D$ is a fundamental discriminant, we denote by $L(f,D,s)$ ($s \in \mathbb{C}$) the $L$-function of $f$ twisted with the quadratic character $\left(\frac{D}{\cdot}\right)$ of the field extension $\mathbb{Q}(\sqrt{D})/\mathbb{Q}$, defined by analytic continuation of the series

$$\sum_{n\geq 1} \left(\frac{D}{n}\right) a(n) n^{-s} \qquad (\mathrm{Re}(s) \gg 0;\ a(n) = n\text{th Fourier coefficient of } f).$$

Recall that $L(f,D,s)$ has an analytic continuation to $\mathbb{C}$ and satisfies a functional equation under $s \mapsto 2k - s$ with root number $\left(\frac{D}{-1}\right)(-1)^k$. In particular, $L(f,D,k) = 0$ for $(-1)^k D < 0$.

As before we let $\mathcal{F}_{2k}$ be the set of normalized Hecke eigenforms in $S_{2k}$.

Theorem 1. *Let $D$ be a fundamental discriminant. Then*

$$\sum_{f\in\mathcal{F}_{2k}} L(f,D,k) \ll_{\varepsilon,D} k^{1+\varepsilon} \qquad (k\to\infty,\ (-1)^k D > 0)$$

*where the constant implied in $\ll$ depends only on $\varepsilon$ and $D$ and is effective.*

Theorem 2. *Let $D$ be a fundamental discriminant. Let $0 < \varepsilon < 1$ be fixed and suppose that*

$$L(f,D,k) \ll_{\varepsilon,D} k^{\varepsilon} \qquad (f \in \mathcal{F}_{2k},\ k\to\infty,\ (-1)^k D > 0)$$

*with an effective constant implied in* $\ll$. *Then*

$$\#\{f \in \mathcal{F}_{2k} \mid L(f,D,k) \neq 0\} \gg_{\varepsilon,D} \frac{k^{1-\varepsilon}}{\log k} \quad (k \to \infty,\ (-1)^k D > 0)$$

*where the constant implied in* $\gg$ *is effective.*

**3. Proofs.** Let $f \in \mathcal{F}_{2k}$. We denote by

$$\|f\|^2 = \int_{\Gamma_1\backslash\mathcal{H}} |f(z)|^2 y^{2k-2}\,dx\,dy \quad (\mathcal{H} = \text{upper half-plane},\ z = x+iy)$$

the *Petersson norm* of $f$.

Let $F$ be the automorphic form on $\mathrm{GL}_3$ which is the adjoint square lift of $f$ and let $L_{\mathrm{St}}(F,s)$ ($s \in \mathbb{C}$) be its standard zeta function, so $L_{\mathrm{St}}(F,s)$ is also the symmetric square $L$-function of $f$ (see [5]; the $L$-functions here are normalized to have functional equations under $s \mapsto 1-s$). One then has

$$\frac{1}{\log(2k+1)} \ll L_{\mathrm{St}}(F,1) \ll_\varepsilon k^\varepsilon \tag{3}$$

for any $\varepsilon > 0$ where the constant implied in the lower bound is absolute and all the constants implied in $\ll$ are effective. The upper bound inequality was proved in [8] and the lower bound inequality in [6, 7]. Note that in the quoted papers the corresponding estimates were given in the context of Maass wave forms (with $2k$ replaced by the corresponding eigenvalue $\lambda$ under the Laplace operator), but that the arguments carry over to the holomorphic case (cf. [7, p. 164] and [3, p. 1183]; cf. also [13]).

Since the symmetric square $L$-function of $f$ up to multiplication with a Riemann zeta function is the Rankin zeta function of $f$ and the latter has a simple pole at $s = 1$ with residue essentially equal to $\|f\|^2$, we see that (3) actually gives bounds for $\|f\|$; working out the constants one finds that

$$\frac{\Gamma(2k)}{(4\pi)^{2k}\log(2k+1)} \ll \|f\|^2 \ll_\varepsilon \frac{\Gamma(2k)}{(4\pi)^{2k}}\,k^\varepsilon \tag{4}$$

for any $\varepsilon > 0$ (cf. [13]). From (4) it follows that

$$\begin{aligned}\frac{\Gamma(2k)}{(4\pi)^{2k}\log(2k+1)} \sum_{f\in\mathcal{F}_{2k}} \frac{L(f,D,k)}{\|f\|^2} &\ll \sum_{f\in\mathcal{F}_{2k}} L(f,D,k)\\ &\ll_\varepsilon \frac{\Gamma(2k)}{(4\pi)^{2k}} k^\varepsilon \sum_{f\in\mathcal{F}_{2k}} \frac{L(f,D,k)}{\|f\|^2}.\end{aligned} \tag{5}$$

Denote by $S^+_{k+1/2}$ the space of cusp forms of weight $k+1/2$ with respect to

$$\Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 \;\middle|\; c \equiv 0 \pmod 4 \right\}$$

with $n$th Fourier coefficients vanishing unless $(-1)^k n \equiv 0, 1 \pmod 4$, equipped with the normalized Petersson scalar product

$$\langle g, h\rangle = \frac{1}{6} \int\limits_{\Gamma_0(4)\backslash\mathcal{H}} g(z)\overline{h(z)} y^{k-3/2}\, dx\, dy \qquad (z = x+iy).$$

Then according to [12] one has

$$\frac{L(f,D,k)}{\|f\|^2} = \frac{\pi^k}{\Gamma(k)} |D|^{1/2-k} \frac{c(|D|)^2}{\|g\|^2} \tag{6}$$

where $g \in S^+_{k+1/2}$ is a Hecke eigenform with real Fourier coefficients corresponding to $f$ under the Shimura correspondence and $c(|D|)$ is the $|D|$th Fourier coefficient of $g$. (This is a more explicit version of Waldspurger's result in the special case of level 1; note that the explicit knowledge of the constant of proportionality is very important for our purposes here.)

Let $P_{k,D}$ be the $|D|$th Poincaré series in $S^+_{k+1/2}$ characterized by

$$\langle h, P_{k,D}\rangle = \frac{1}{6} \cdot \frac{\Gamma(k-1/2)}{(4\pi|D|)^{k-1/2}} c_h(|D|)$$

for all $h \in S^+_{k+1/2}$ where $c_h(|D|)$ denotes the $|D|$th Fourier coefficient of $h$. We write $P_{k,D}$ in terms of a basis $\{g\}$ of Hecke eigenforms corresponding to the basis $\mathcal{F}_{2k}$ and take $|D|$th Fourier coefficients. Using the implied expression for the $|D|$th Fourier coefficient $p_{k,D}(|D|)$ of $P_{k,D}$ we find after inserting (6) into (5) that

$$c_k \cdot \frac{1}{\log(2k+1)} \cdot p_{k,D}(|D|) \ll \sum_{f \in \mathcal{F}_{2k}} L(f,D,k) \ll_\varepsilon c_k \cdot k^\varepsilon \cdot p_{k,D}(|D|) \tag{7}$$

where

$$c_k := \frac{\Gamma(2k)}{\Gamma(k)\Gamma(k-1/2)2^{2k-1}} = \frac{1}{\sqrt{\pi}}\left(k - \frac{1}{2}\right).$$

The Fourier coefficients of $P_{k,D}$ were computed in [10]. In particular, one has

$$\begin{aligned} &p_{k,D}(|D|) \\ &\quad = \frac{2}{3}\left(1 + (-1)^{[(k+1)/2]}\pi\sqrt{2} \sum_{c \ge 1} \frac{1}{4c} H_c(|D|,|D|) J_{k-1/2}\left(\frac{\pi|D|}{c}\right)\right) \end{aligned} \tag{8}$$

where

$$\begin{aligned} &H_c(|D|,|D|) \\ &\quad = (1-(-1)^k i)\left(1 + \left(\frac{4}{c}\right)\right) \sum_{d(4c)^*} \left(\frac{4c}{d}\right)\left(\frac{-4}{d}\right)^{k+1/2} e^{2\pi i |D|(d+d^{-1})/(4c)} \end{aligned} \tag{9}$$

is a generalized Kloosterman sum and $J_{k-1/2}$ is the Bessel function of order $k-1/2$. In (9) the summation is over a primitive residue system modulo $4c$, $d^{-1}$ denotes an integer with $d^{-1}d \equiv 1 \pmod{4c}$, $(\frac{\cdot}{\cdot})$ is the generalized Jacobi–Kronecker symbol and $(\frac{-4}{d})^{1/2}$ is equal to 1 or $i$ according as $d \equiv 1 \pmod 4$ or $d \equiv 3 \pmod 4$, respectively.

The Poisson integral representation

$$J_{k-1/2}(z) = \sqrt{\frac{2}{\pi}} \cdot \frac{z^{k-1/2}}{2^k \Gamma(k)} \int_0^\pi \cos(z\cos\theta)\sin^{2k-1}\theta\, d\theta \qquad (z \neq 0, -\pi/2 < \arg z^{1/2} \leq \pi/2)$$

[1, formula 10.1.13] shows that

$$|J_{k-1/2}(x)| \leq \sqrt{\frac{2}{\pi}} \cdot \frac{x^{k-1/2}}{2^k \Gamma(k)}$$

for positive real $x$.

We split up the sum in (8) into the sum of the finitely many terms with $c < \pi|D|$ and the sum over the terms with $c > \pi|D|$. Using the trivial bound

$$|H_c(|D|,|D|)| \leq \sqrt{2}\cdot 8c$$

we then immediately deduce that

$$p_{k,D}(|D|) \ll_D 1 \qquad (k \to \infty) \tag{10}$$

and

$$p_{k,D}(|D|) \gg_{D,\delta} \frac{2}{3} - \delta \qquad (k \to \infty) \tag{11}$$

for any fixed $\delta > 0$.

Taking into account the value of $c_k$, we find that (10) and the second inequality in (7) imply the assertion of Theorem 1. Likewise (11) and the first inequality in (7) imply Theorem 2.

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