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# Hausdorff dimensions in Engel expansions

by

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**1. Introduction.** Given $x$ in $(0,1]$, let $x=[d_1(x),d_2(x),\ldots]$ denote the *Engel expansion* of $x$, that is,

$$x=\frac{1}{d_1(x)}+\frac{1}{d_1(x)d_2(x)}+\ldots+\frac{1}{d_1(x)d_2(x)\ldots d_n(x)}+\ldots, \tag{1}$$

where $\{d_j(x), j\ge 1\}$ is a sequence of positive integers satisfying $d_1(x)\ge 2$ and $d_{j+1}(x)\ge d_j(x)$ for $j\ge 1$ (see [3]). In [3], János Galambos proved that for almost all $x\in(0,1]$,

$$\lim_{n\to\infty} d_n^{1/n}(x)=e. \tag{2}$$

Also he posed the following questions (see [3], P132):

(i) Find the Hausdorff dimension of the set where (2) fails.

(ii) For any $k\ge 1$, let

$$A_k=\{x\in(0,1]:\log d_n(x)\ge kn \text{ for any } n\ge 1\}.$$

Find the Hausdorff dimension of the set $A_k$.

For (i), the second author [4] has proved that the Hausdorff dimension of the set where (2) fails is 1.

In this paper, we get a stronger result than those in (i) and (ii). We show

Theorem. *For any $\alpha\ge 1$, let*

$$A(\alpha)=\{x\in(0,1]:\lim_{n\to\infty} d_n^{1/n}(x)=\alpha\}.$$

*Then*

$$\dim_{\mathrm H} A(\alpha)=1.$$

As corollaries of the Theorem, both the Hausdorff dimensions in (i) and (ii) are 1.

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We use $|\cdot|$ to denote the diameter of a subset of $(0,1]$, $\dim_{\mathrm{H}}$ to denote the Hausdorff dimension, $[\ \ ]$ the integer part of a real number and cl the closure of a subset of $(0,1]$ respectively.

**2. Proof of the Theorem.** The aim of this section is to prove the main result of this paper.

In what follows we often make use of the code space. Let $\{M_n, n\ge 1\}$ be a sequence of positive numbers such that $M_1>1$, $M_k<M_{k+1}$ for any $k\ge 1$. For any $n\ge 1$, let

$$D_n=\{(\sigma_1,\dots,\sigma_n)\in\mathbb{N}^n : kM_k<\sigma_k\le (k+1)M_k \text{ for all } 1\le k\le n\}.$$

Define

$$D=\bigcup_{n=0}^{\infty} D_n \qquad (D_0=\emptyset).$$

For any $\sigma=(\sigma_1,\dots,\sigma_n)\in D_n$, we use $J_\sigma$ to denote the following closed subinterval of $(0,1]$:

$$J_\sigma=\bigcup_{k=[(n+1)M_{n+1}]+1}^{[(n+2)M_{n+1}]} \mathrm{cl}\{x\in(0,1] : d_1(x)=\sigma_1,\dots,d_n(x)=\sigma_n, d_{n+1}(x)=k\},$$

and call it an *n-order interval*.

Define

$$E=\bigcap_{n=0}^{\infty}\bigcup_{\sigma\in D_n} J_\sigma. \tag{3}$$

It is obvious that

$$E=\{x\in(0,1] : nM_n<d_n(x)\le (n+1)M_n \text{ for all } n\ge 1\}. \tag{4}$$

*Proof of the Theorem.* We divide the proof into two parts:

Part I: $\alpha>1$. For any $n\ge 1$, let $M_n=\alpha^n$. Now we estimate the length of $J_\sigma$ for any $\sigma\in D_n$. Since for any $(n+1)\alpha^{n+1}<k\le (n+2)\alpha^{n+1}$,

$$|\{x\in(0,1] : d_1(x)=\sigma_1,\dots,d_n(x)=\sigma_n,\ d_{n+1}(x)=k\}| = \frac{1}{\sigma_1\dots\sigma_n}\left(\frac{1}{k-1}-\frac{1}{k}\right),$$

we have

$$|J_\sigma|=\sum_{k=[(n+1)M_{n+1}]+1}^{[(n+2)M_{n+1}]}\frac{1}{\sigma_1\dots\sigma_n}\left(\frac{1}{k-1}-\frac{1}{k}\right).$$

Therefore

$$(n+2)^{-(n+2)}\alpha^{-(n+1)(n+2)/2}\alpha^{-(n+1)}\le |J_\sigma|\le \alpha^{-(n+1)(n+2)/2}. \tag{5}$$

Let $\mu$ be a mass distribution supported on $E$ such that for any $n \geq 0$ and $\sigma \in D_n$,

$$\mu(J_\sigma) = \frac{1}{\sharp D_n} \quad (\sharp D_0 = 1). \tag{6}$$

By the definition of $D_n$, it is easy to check that

$$c^{-n}\alpha^{n(n+1)/2} \leq \sharp D_n \leq c^n \alpha^{n(n+1)/2}, \tag{7}$$

where $c$ is a positive constant which does not depend on $n$.

For any $x \in E$, we prove that

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \geq 1, \tag{8}$$

where $B(x,r)$ denotes the open ball with center at $x$ and radius $r$.

For $r < \alpha^{-3}$, choose $n \geq 3$ such that

$$\alpha^{-n(n+1)/2} < r \leq \alpha^{-(n-1)n/2}. \tag{9}$$

By (5), $B(x,r)$ can intersect at most $4n^n\alpha^{n-1}$ $(n-2)$-order intervals, thus by (6) and (7),

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \geq \liminf_{n\to\infty} \frac{\log(c^{n-2}\alpha^{-(n-2)(n-1)/2}4n^n\alpha^{n-1})}{\log \alpha^{-n(n+1)/2}} = 1.$$

By [2], Proposition 2.3, (see also [1], Proposition 4.9) we have $\dim_{\mathrm{H}} E = 1$. Since $E \subset A(\alpha)$, we have $\dim_{\mathrm{H}} A(\alpha) = 1$.

Part II: $\alpha = 1$. The proof of this part is very similar to Part I; we just give an outline.

For any $n \geq 1$, let

$$M_n = \left(1 + \frac{1}{\sqrt{n}}\right)^n.$$

Then as in Part I, we have

$$(n+2)^{-(n+2)}\left(\prod_{k=1}^{n+1}\left(1+\frac{1}{\sqrt{k}}\right)^k\right)^{-1}\left(1+\frac{1}{\sqrt{n+1}}\right)^{-(n+1)} \leq |J_\sigma| \leq \left(\prod_{k=1}^{n+1}\left(1+\frac{1}{\sqrt{k}}\right)^k\right)^{-1}, \tag{10}$$

$$c^{-n}\prod_{k=1}^{n}\left(1+\frac{1}{\sqrt{k}}\right)^k \leq \sharp D_n \leq c^n \prod_{k=1}^{n}\left(1+\frac{1}{\sqrt{k}}\right)^k. \tag{11}$$

For any $x \in E$, $r < (\prod_{k=1}^3 (1+1/\sqrt{k})^k)^{-1}$, choose $n \geq 3$ such that

$$\left(\prod_{k=1}^{n}\left(1+\frac{1}{\sqrt{k}}\right)^k\right)^{-1} < r \leq \left(\prod_{k=1}^{n-1}\left(1+\frac{1}{\sqrt{k}}\right)^k\right)^{-1}. \tag{12}$$

By (10), $B(x,r)$ can intersect at most $4n^n(1+1/\sqrt{n-1})^{n-1}$ $(n-2)$-order intervals, thus by (6) and (11), we have

$$\liminf_{r\to 0}\frac{\log\mu(B(x,r))}{\log r} \tag{13}$$

$$\geq \liminf_{n\to\infty}\frac{\log\left(c^{n-2}\left(\prod_{k=1}^{n-2}\left(1+\frac{1}{\sqrt{k}}\right)^k\right)^{-1}4n^n\left(1+\frac{1}{\sqrt{n-1}}\right)^{n-1}\right)}{\log\left(\prod_{k=1}^{n}\left(1+\frac{1}{\sqrt{k}}\right)^k\right)^{-1}}.$$

Since $\{(1+1/\sqrt{n})^{\sqrt{n}},\ n\geq 1\}$ is an increasing sequence such that for any $n\geq 1$,

$$2\leq\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}\leq e, \tag{14}$$

and

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}\geq\int_1^n x^{-1/2}\,dx=2n^{1/2}-2, \tag{15}$$

we have

$$\liminf_{r\to 0}\frac{\log\mu(B(x,r))}{\log r}\geq 1,$$

completing the proof of the Theorem.

Corollary 1. *For any $k\geq 1$, $\dim_{\mathrm{H}} A_k=1$.*

*Proof.* For any $k\geq 1$, choose $M>e^k$. Let $M_n=M^n$ for any $n\geq 1$. Then $E\subset A_k$. By the proof of the Theorem, we have $\dim_{\mathrm{H}} E=1$, thus $\dim_{\mathrm{H}} A_k=1$.

From the proof of the Theorem, we can also get the following corollaries immediately.

Corollary 2. *For any $n\geq 2$ and $\alpha\geq 1$, let*

$$B(\alpha)=\left\{x\in(0,1]:\lim_{n\to\infty}\frac{d_{n+1}(x)}{d_n(x)-1}=\alpha\right\}.$$

*Then*

$$\dim_{\mathrm{H}} B(\alpha)=1.$$

Corollary 3. *The Hausdorff dimension of the set where* (2) *fails is* 1.

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