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Multiple p-adic log-gamma functions and their characterization theorem

by

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1. Introduction and main results. Morita's p-adic gamma function $\Gamma_p(x)$ is the unique continuous function on the ring of p-adic integers $\mathbb{Z}_p$ satisfying

$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -x & (x \in \mathbb{Z}_p^\times), \\ -1 & (x \in p\mathbb{Z}_p), \end{cases} \tag{1.1}$$

and the initial condition $\Gamma_p(0) = 1$. By (1.1), we have

$$\log_p \Gamma_p(x+1) - \log_p \Gamma_p(x) = \begin{cases} \log_p x & (x \in \mathbb{Z}_p^\times), \\ 0 & (x \in p\mathbb{Z}_p), \end{cases}$$

where $\log_p$ is the Iwasawa p-adic logarithm. It is easy to see that there is no continuous function ψ on $\mathbb{Z}_p$ such that $\psi(x+1) - \psi(x) = \log_p x$ for all $x \in \mathbb{Z}_p \setminus \{0\}$ (see [10, p. 182]). However, there exists a continuous function ψ on $\mathbb{C}_p \setminus \mathbb{Z}_p$ satisfying $\psi(x+1) - \psi(x) = \log_p x$ for all $x \in \mathbb{C}_p \setminus \mathbb{Z}_p$ where $\mathbb{C}_p$ is the completion of the algebraic closure of the p-adic field $\mathbb{Q}_p$. An example of such a function is Diamond's p-adic log-gamma function $\operatorname{Log} \Gamma_{\mathrm{D}}(x)$, which is defined by

$$\operatorname{Log} \Gamma_{\mathrm{D}}(x) = \int_{\mathbb{Z}_p} ((x+t)\log_p(x+t) - (x+t))\, dt \qquad (x \in \mathbb{C}_p \setminus \mathbb{Z}_p),$$

where $\int_{\mathbb{Z}_p} f(t)\, dt$ is the Volkenborn integral of f defined by

$$\int_{\mathbb{Z}_p} f(t)\, dt = \lim_{N\to\infty} p^{-N} \sum_{a=0}^{p^N-1} f(a)$$

(see Section 2). This function satisfies the expected difference equation

$$\operatorname{Log} \Gamma_{\mathrm{D}}(x+1) - \operatorname{Log} \Gamma_{\mathrm{D}}(x) = \log_p x \tag{1.2}$$

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for all $x \in \mathbb{C}_p \backslash \mathbb{Z}_p$. Although the difference equation (1.2) cannot characterize the function $\operatorname{Log} \Gamma_{\mathrm{D}}(x)$, Cohen and Friedman [3] proved that (1.2) and a certain integro-differential equation characterize it.

Theorem 1.1 (Cohen–Friedman [3, Section 1]). *Diamond's p-adic log-gamma function* $\operatorname{Log} \Gamma_{\mathrm{D}}(x)$ *satisfies*

$$\int_{\mathbb{Z}_p} \operatorname{Log} \Gamma_{\mathrm{D}}(x+t)\, dt = (x-1)(\operatorname{Log} \Gamma_{\mathrm{D}})'(x) - x + \frac{1}{2} \quad (x \in \mathbb{C}_p \setminus \mathbb{Z}_p). \tag{1.3}$$

It is the unique strictly differentiable function $f : \mathbb{C}_p \setminus \mathbb{Z}_p \to \mathbb{C}_p$ satisfying the difference equation

$$f(x+1) - f(x) = \log_p x$$

and the Volkenborn integro-differential equation

$$\int_{\mathbb{Z}_p} f(x+t)\, dt = (x-1) f'(x) - x + \frac{1}{2}.$$

As stated in [3, Section 1], formula (1.3) above can be regarded as a p-adic analogue of Raabe's classical formula:

$$\int_0^1 \log\left(\frac{\Gamma(x+t)}{\sqrt{2\pi}}\right) dt = x \log x - x \quad (x > 0),$$

where $\log x$ is the ordinary logarithm function on $\mathbb{R}$.

There are many studies on multiple analogues of the gamma function and the log-gamma function. In the complex case, around 1900, Barnes studied multiple gamma functions which are defined by using multiple Hurwitz zeta functions (see e.g. [1]). Vignéras [12] redefined these multiple gamma functions to be functions satisfying a Bohr–Mollerup type theorem. These functions have many applications. For example, Shintani [11] used the double gamma function to study Kronecker's limit formula for certain algebraic fields. In the p-adic case, Cassou-Noguès [2] defined multiple p-adic log-gamma functions. Variants of multiple p-adic log-gamma functions have also been investigated by many authors (e.g. Imai [6] and Kashio [7]).

In this paper, we focus on a simple multiple analogue of Diamond's p-adic log-gamma function, denoted by $\operatorname{Log} \Gamma_{\mathrm{D},r}(x)$. For more general forms of these functions $\operatorname{Log} \Gamma_{\mathrm{D},r}(x)$, see Cassou-Noguès [2, p. 53] and Kashio [7, Section 5]. The function $\operatorname{Log} \Gamma_{\mathrm{D},r}(x)$ $(r \geq 1)$ satisfies the difference equation $\operatorname{Log} \Gamma_{\mathrm{D},r}(x+1) - \operatorname{Log} \Gamma_{\mathrm{D},r}(x) = \operatorname{Log} \Gamma_{\mathrm{D},r-1}(x)$ for all $x \in \mathbb{C}_p \setminus \mathbb{Z}_p$ (Proposition 3.4). As a main result, we show that the function $\operatorname{Log} \Gamma_{\mathrm{D},r}(x)$ satisfies a Raabe-type formula and a characterization theorem. This result is a generalization of Theorem 1.1 because $\operatorname{Log} \Gamma_{\mathrm{D},1}(x) = \operatorname{Log} \Gamma_{\mathrm{D}}(x)$ and $\operatorname{Log} \Gamma_{\mathrm{D},0}(x) = \log_p x$.

Main Theorem.

(i) *For a positive integer r and $x \in \mathbb{C}_p \setminus \mathbb{Z}_p$, we have*

$$r \int_{\mathbb{Z}_p} \operatorname{Log} \Gamma_{\mathrm{D},r}(x+t)\, dt = (x-r)(\operatorname{Log} \Gamma_{\mathrm{D},r})'(x) - S_r(x), \tag{1.4}$$

where $S_r(x) \in \mathbb{Q}[x]$ is the multiple Bernoulli polynomial defined by (2.8).

(ii) *For a positive integer r, the multiple p-adic log-gamma function $\operatorname{Log} \Gamma_{\mathrm{D},r}(x)$ is the unique strictly differentiable function $f : \mathbb{C}_p \setminus \mathbb{Z}_p \to \mathbb{C}_p$ satisfying the following conditions:*

(A) $f(x+1) - f(x) = \operatorname{Log} \Gamma_{\mathrm{D},r-1}(x)$.

(B) $r \int_{\mathbb{Z}_p} f(x+t)\, dt = (x-r) f'(x) - S_r(x)$.

The plan of this paper is as follows. In Section 2, we review Volkenborn integrals and multiple Bernoulli polynomials. In Section 3, we define multiple p-adic log-gamma functions $\operatorname{Log} \Gamma_{\mathrm{D},r}(x)$ on $\mathbb{C}_p \setminus \mathbb{Z}_p$ and give some properties of them. In Section 4, we prove our Main Theorem. In the last Section 5, we deal with multiple p-adic log-gamma functions on $\mathbb{Z}_p$. They are generalizations of the logarithm of Morita's p-adic gamma function.

2. Multiple Bernoulli polynomials.

For a positive integer n, we set

$$\nabla^n \mathbb{Z}_p = \{(x_1, \ldots, x_n) \in \mathbb{Z}_p^n \mid x_i \neq x_j \text{ if } i \neq j\}.$$

The nth *(order) difference quotient* $\Phi_n f : \nabla^{n+1} \mathbb{Z}_p \to \mathbb{C}_p$ of a function $f : \mathbb{Z}_p \to \mathbb{C}_p$ is inductively given by $\Phi_0 f = f$ and for $(x_1, \ldots, x_{n+1}) \in \nabla^{n+1} \mathbb{Z}_p$ by

$$\Phi_n f(x_1, \ldots, x_{n+1}) = \frac{\Phi_{n-1} f(x_1, x_3, \ldots, x_{n+1}) - \Phi_{n-1} f(x_2, x_3, \ldots, x_{n+1})}{x_1 - x_2}.$$

A function f is called a C^n*-function* if $\Phi_n f$ can be extended to a continuous function $\bar{\Phi}_n f : \mathbb{Z}_p^{n+1} \to \mathbb{C}_p$. The set of all C^n-functions from $\mathbb{Z}_p$ to $\mathbb{C}_p$ is denoted by $C^n(\mathbb{Z}_p \to \mathbb{C}_p)$. Moreover, we set $C^\infty(\mathbb{Z}_p \to \mathbb{C}_p) = \bigcap_{n=1}^\infty C^n(\mathbb{Z}_p \to \mathbb{C}_p)$ (e.g. [10, Section 29]). We note that C^1-functions and strictly differentiable functions are exactly the same.

For a function $f \in C^1(\mathbb{Z}_p \to \mathbb{C}_p)$, the limit value

$$\lim_{N \to \infty} p^{-N} \sum_{a=0}^{p^N - 1} f(a)$$

exists. It is called the *Volkenborn integral* of f and is denoted by

$$\int_{\mathbb{Z}_p} f(t)\, dt$$

(e.g. [10, p. 167]). For a continuous function $f : \mathbb{Z}_p \to \mathbb{C}_p$, we denote the *indefinite sum* of f by Sf, that is, Sf is the unique continuous function on $\mathbb{Z}_p$ satisfying $Sf(n) = \sum_{j=0}^{n-1} f(j)$ for any positive integer n (e.g. [10, p. 106]). For $f \in C^1(\mathbb{Z}_p \to \mathbb{C}_p)$ and $x \in \mathbb{Z}_p$, the following identities are known (cf. [10, p. 168]):

$$\int_{\mathbb{Z}_p} f(x+t)\,dt = (Sf)'(x), \tag{2.1}$$

$$\int_{\mathbb{Z}_p} f(x+t)\,dt - \int_{\mathbb{Z}_p} f(t)\,dt = (Sf')(x), \tag{2.2}$$

$$\int_{\mathbb{Z}_p} f(x+t+1)\,dt - \int_{\mathbb{Z}_p} f(x+t)\,dt = f'(x), \tag{2.3}$$

$$\int_{\mathbb{Z}_p} f(-t)\,dt = \int_{\mathbb{Z}_p} f(t+1)\,dt. \tag{2.4}$$

Moreover, for $f \in C^2(\mathbb{Z}_p \to \mathbb{C}_p)$, we have

$$\frac{d}{dx}\int_{\mathbb{Z}_p} f(x+t)\,dt = \int_{\mathbb{Z}_p} f'(x+t)\,dt \tag{2.5}$$

for $x \in \mathbb{Z}_p$ ([9, p. 268]).

The Bernoulli polynomials $B_n(x)$ are defined by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n.$$

When $x = 0$, the numbers $B_n(0) = B_n$ are the ordinary Bernoulli numbers. It is known that the Bernoulli polynomials are expressed by using a Volkenborn integral:

$$\int_{\mathbb{Z}_p} (x+t)^n\,dt = B_n(x) \qquad (n \geq 0). \tag{2.6}$$

In particular, we have

$$\int_{\mathbb{Z}_p} t^n\,dt = B_n \qquad (n \geq 0) \tag{2.7}$$

(e.g. [9, p. 271]).

Let r be a positive integer and $x \in \mathbb{C}_p$. As a generalization of (2.6), we define *multiple Bernoulli polynomials* as

$$S_r(x) = \frac{1}{r!}\int_{\mathbb{Z}_p^r} (x+t_1+\cdots+t_r)^r\,dt_1\ldots dt_r, \tag{2.8}$$

where $\int_{\mathbb{Z}_p^r}$ means $\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r}$. From the multinomial expansion and equations

(2.6) and (2.7), we have

$$S_r(x) = \sum \frac{B_{k_1}(x)B_{k_2}\cdots B_{k_r}}{k_1!\cdots k_r!}, \tag{2.9}$$

where the summation is over all non-negative integers $k_1, \ldots, k_r$ with $k_1 + \cdots + k_r = r$. Thus $S_r(x)$ is a polynomial with rational coefficients of degree r. We note that $S_r(x)$ is a special case of Barnes's multiple Bernoulli polynomials (cf. Ota [8, Section 2]).

3. Multiple p-adic log-gamma functions. For an integer $r \geq 0$, Endo [5] introduced the function $\varphi_r : \mathbb{C}_p^\times \to \mathbb{C}_p$ defined by

$$\varphi_r(x) = \begin{cases} \dfrac{x^r}{r!}\Big(\log_p x - \displaystyle\sum_{i=1}^{r} \frac{1}{i}\Big) & (r \geq 1), \\ \log_p x & (r = 0). \end{cases}$$

Using this function, he defined multiple p-adic log-gamma functions on $\mathbb{Z}_p$, which are generalizations of the logarithm of Morita's p-adic gamma function. Endo's multiple p-adic log-gamma functions will be dealt with in the last section.

From the definition, it is easily proved that for $r \geq 1$,

$$x\varphi_{r-1}(x) = r\varphi_r(x) + \frac{x^r}{r!}. \tag{3.1}$$

Since $(\log_p x)' = 1/x$ for $x \in \mathbb{C}_p^\times$, we have $\frac{d}{dx}\varphi_r(x) = \varphi_{r-1}(x)$ for $r \geq 1$. Moreover, since $\log_p(xy) = \log_p x + \log_p y$ for all $x, y \in \mathbb{C}_p^\times$, we deduce that, for integers $r \geq 0$ and $k \geq 1$,

$$\varphi_r(kx) = k^r\varphi_r(x) + \frac{(kx)^r}{r!}\log_p k. \tag{3.2}$$

In particular, since $\log_p p = \log_p(-1) = 0$, we have $\varphi_r(px) = p^r\varphi_r(x)$ and $\varphi_r(-x) = (-1)^r\varphi_r(x)$.

Lemma 3.1. *Let $f \in C^r(\mathbb{Z}_p \to \mathbb{C}_p)$ $(r \geq 2)$. Then $F(x) = \int_{\mathbb{Z}_p} f(x+t)\,dt \in C^{r-1}(\mathbb{Z}_p \to \mathbb{C}_p)$. Therefore, the integral*

$$\int_{\mathbb{Z}_p^r} f(t_1 + \cdots + t_r)\,dt_1 \ldots dt_r$$

can be defined if $f \in C^r(\mathbb{Z}_p \to \mathbb{C}_p)$.

Proof. If $f \in C^r(\mathbb{Z}_p \to \mathbb{C}_p)$, then $Sf \in C^r(\mathbb{Z}_p \to \mathbb{C}_p)$ (e.g. [10, Corollary 54.3]). Moreover, if $Sf \in C^r(\mathbb{Z}_p \to \mathbb{C}_p)$, then $(Sf)' \in C^{r-1}(\mathbb{Z}_p \to \mathbb{C}_p)$ (cf. [10, Theorem 78.2]). By (2.1), we obtain the first part of the lemma. The second part can be proved by induction on r. ∎

Definition 3.2. For any integer $r \geq 0$ and $x \in \mathbb{C}_p \setminus \mathbb{Z}_p$, we define *multiple p-adic log-gamma functions* by

$$\operatorname{Log} \Gamma_{\mathrm{D},r}(x) = \begin{cases} \int_{\mathbb{Z}_p^r} \varphi_r(x + t_1 + \cdots + t_r)\, dt_1 \ldots dt_r & (r \geq 1), \\ \log_p x & (r = 0). \end{cases}$$

Since locally analytic functions are C^∞-functions (e.g. [10, Corollary 29.11]), we have $\varphi_r \in C^\infty(\mathbb{Z}_p \to \mathbb{C}_p)$ for all $r \geq 0$. Therefore, by Lemma 3.1, this definition makes sense and $t \mapsto \operatorname{Log} \Gamma_{\mathrm{D},r}(x + t)$ is also a C^∞-function from $\mathbb{Z}_p$ to $\mathbb{C}_p$ for a fixed $x \in \mathbb{C}_p \setminus \mathbb{Z}_p$. When $r = 1$, we have

$$\operatorname{Log} \Gamma_{\mathrm{D},1}(x) = \int_{\mathbb{Z}_p} ((x + t) \log_p(x + t) - (x + t))\, dt \qquad (x \in \mathbb{C}_p \setminus \mathbb{Z}_p)$$

and this function is nothing but Diamond's p-adic log-gamma function $\operatorname{Log} \Gamma_{\mathrm{D}}(x)$ (it was originally denoted by $G_p(x)$, see [4]). Diamond proved that $\operatorname{Log} \Gamma_{\mathrm{D}}(x + 1) - \operatorname{Log} \Gamma_{\mathrm{D}}(x) = \log_p x$ for all $x \in \mathbb{C}_p \setminus \mathbb{Z}_p$.

We prove the following lemma which is needed to give properties of multiple p-adic log-gamma functions. This identity (3.3) has appeared in [10, p. 170] without proof, and we give its proof here.

Lemma 3.3. *Let k be a positive integer. For $f \in C^1(\mathbb{Z}_p \to \mathbb{C}_p)$, we have*

$$\int_{\mathbb{Z}_p} f(t)\, dt = \frac{1}{k} \sum_{i=0}^{k-1} \int_{\mathbb{Z}_p} f(i + ks)\, ds. \tag{3.3}$$

Proof. From the definition of Volkenborn integrals, we have

$$\begin{aligned} \sum_{i=0}^{k-1} \int_{\mathbb{Z}_p} f(i + ks)\, ds &= \sum_{i=0}^{k-1} \lim_{N\to\infty} \frac{1}{p^N} \sum_{j=0}^{p^N-1} f(i + kj) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{l=0}^{kp^N-1} f(l) \\ &= \sum_{i=0}^{k-1} \lim_{N\to\infty} \frac{1}{p^N} \sum_{j=0}^{p^N-1} f(ip^N + j). \end{aligned}$$

Hence, by using the uniform convergence of the series, we obtain

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{p^N} \sum_{j=0}^{p^N-1} f(ip^N + j) &= \lim_{N\to\infty} \frac{1}{p^N} \sum_{j=0}^{p^N-1} \lim_{M\to\infty} f(ip^M + j) \\ &= \lim_{N\to\infty} \frac{1}{p^N} \sum_{j=0}^{p^N-1} f(j) = \int_{\mathbb{Z}_p} f(t)\, dt \end{aligned}$$

for any integer i. This proves the lemma. ∎

Now we give some properties of multiple p-adic log-gamma functions, which are generalizations of those of Diamond's p-adic log-gamma function (see [10, Theorem 60.2]).

PROPOSITION 3.4. *Let r and k be positive integers. For $x \in \mathbb{C}_p \setminus \mathbb{Z}_p$, the following identities hold:*

(i) $\operatorname{Log} \varGamma_{\mathrm{D},r}(x+1) - \operatorname{Log} \varGamma_{\mathrm{D},r}(x) = \operatorname{Log} \varGamma_{\mathrm{D},r-1}(x)$.

(ii) $\operatorname{Log} \varGamma_{\mathrm{D},r}(-x) = (-1)^r \operatorname{Log} \varGamma_{\mathrm{D},r}(x+r)$.

(iii) $\operatorname{Log} \varGamma_{\mathrm{D},r}(x)$

$$= \sum_{i_1=0}^{k-1} \dots \sum_{i_r=0}^{k-1} \operatorname{Log} \varGamma_{\mathrm{D},r}\left(\frac{x+i_1+\dots+i_r}{k}\right) + (\log_p k) S_r(x).$$

In particular, when $k = p$, we have

$$\operatorname{Log} \varGamma_{\mathrm{D},r}(x) = \sum_{i_1=0}^{p-1} \dots \sum_{i_r=0}^{p-1} \operatorname{Log} \varGamma_{\mathrm{D},r}\left(\frac{x+i_1+\dots+i_r}{p}\right).$$

Proof. Assertion (i) is easily proved by (2.3) and (2.5). By the identity $\varphi_r(-x) = (-1)^r \varphi_r(x)$ and (2.4), we have

$$\begin{aligned} \operatorname{Log} \varGamma_{\mathrm{D},r}(-x) &= \int_{\mathbb{Z}_p^r} \varphi_r(-x+t_1+\dots+t_r)\, dt_1 \dots dt_r \\ &= (-1)^r \int_{\mathbb{Z}_p^r} \varphi_r(x-t_1-\dots-t_r)\, dt_1 \dots dt_r \\ &= (-1)^r \int_{\mathbb{Z}_p^r} \varphi_r(x+(t_1+1)+\dots+(t_r+1))\, dt_1 \dots dt_r \\ &= (-1)^r \operatorname{Log} \varGamma_{\mathrm{D},r}(x+r) \end{aligned}$$

and this proves (ii). Assertion (iii) follows by (3.2) and Lemma 3.3. In fact,

$$\begin{aligned} &\operatorname{Log} \varGamma_{\mathrm{D},r}(x) = \int_{\mathbb{Z}_p^r} \varphi_r(x+t_1+\dots+t_r)\, dt_1 \dots dt_r \\ &= \frac{1}{k^r} \sum_{i_1=0}^{k-1} \dots \sum_{i_r=0}^{k-1} \int_{\mathbb{Z}_p^r} \varphi_r(x+(i_1+ks_1)+\dots+(i_r+ks_r))\, ds_1 \dots ds_r \\ &= \frac{1}{k^r} \sum_{i_1=0}^{k-1} \dots \sum_{i_r=0}^{k-1} \int_{\mathbb{Z}_p^r} \biggl(k^r \varphi_r\left(\frac{x+i_1+\dots+i_r}{k} + s_1+\dots+s_r\right) \\ &\quad + \frac{\log_p k}{r!}(x+i_1+\dots+i_r+ks_1+\dots+ks_r)^r\biggr)\, ds_1 \dots ds_r \\ &= \sum_{i_1=0}^{k-1} \dots \sum_{i_r=0}^{k-1} \operatorname{Log} \varGamma_{\mathrm{D},r}\left(\frac{x+i_1+\dots+i_r}{k}\right) \\ &\quad + \frac{1}{k^r} \sum_{i_1=0}^{k-1} \dots \sum_{i_r=0}^{k-1} \frac{\log_p k}{r!} \int_{\mathbb{Z}_p^r} (x+i_1+\dots+i_r+ks_1+\dots+ks_r)^r\, ds_1 \dots ds_r. \end{aligned}$$

By using Lemma 3.3 again, we have

$$\begin{aligned}\operatorname{Log}\Gamma_{\mathrm{D},r}(x) &= \sum_{i_1=0}^{k-1}\cdots\sum_{i_r=0}^{k-1}\operatorname{Log}\Gamma_{\mathrm{D},r}\left(\frac{x+i_1+\cdots+i_r}{k}\right)\\ &\quad+\frac{\log_p k}{r!}\int_{\mathbb{Z}_p^r}(x+t_1+\cdots+t_r)^r\,dt_1\ldots dt_r\\ &= \sum_{i_1=0}^{k-1}\cdots\sum_{i_r=0}^{k-1}\operatorname{Log}\Gamma_{\mathrm{D},r}\left(\frac{x+i_1+\cdots+i_r}{k}\right)+(\log_p k)S_r(x).\end{aligned}$$

The last formula immediately follows because $\log_p p = 0$. ∎

4. Proof of the Main Theorem. We first give lemmas to prove our Main Theorem.

LEMMA 4.1. *For $f \in C^2(\mathbb{Z}_p \to \mathbb{C}_p)$, we have*

$$\int_{\mathbb{Z}_p}(t+1)f'(t)\,dt = \int_{\mathbb{Z}_p} f(t)\,dt - \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} f(x+t)\,dx\,dt. \tag{4.1}$$

Proof. It is known that

$$\int_{\mathbb{Z}_p}(t+1)f(t)\,dt = -\int_{\mathbb{Z}_p} Sf(t)\,dt \tag{4.2}$$

(cf. [10, p. 170]). By (2.2), we have

$$\int_{\mathbb{Z}_p}(t+1)f'(t)\,dt = -\int_{\mathbb{Z}_p}(Sf')(t)\,dt = \int_{\mathbb{Z}_p} f(t)\,dt - \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} f(x+t)\,dx\,dt, \tag{4.3}$$

and this proves the lemma. ∎

LEMMA 4.2. *Let $f \in C^r(\mathbb{Z}_p \to \mathbb{C}_p)$ $(r \ge 1)$. Then*

$$\int_{\mathbb{Z}_p^r}(t_i+1)f(t_1+\cdots+t_r)\,dt_1\ldots dt_r = \int_{\mathbb{Z}_p^r}(t_j+1)f(t_1+\cdots+t_r)\,dt_1\ldots dt_r$$

for any $1 \le i, j \le r$.

Proof. We only have to prove the case $r = 2$:

$$\int_{\mathbb{Z}_p^2}(t_1+1)f(t_1+t_2)\,dt_1\,dt_2 = \int_{\mathbb{Z}_p^2}(t_2+1)f(t_1+t_2)\,dt_1\,dt_2 \tag{4.4}$$

for $f \in C^2(\mathbb{Z}_p \to \mathbb{C}_p)$. We put $F_{t_2}(t_1) = f(t_1+t_2)$. Then

$$SF_{t_2}(t_1) = Sf(t_1+t_2) - Sf(t_2).$$

By (4.2), the left-hand side of (4.4) is equal to

$$\begin{aligned}\int_{\mathbb{Z}_p^2}(t_1+1)F_{t_2}(t_1)\,dt_1\,dt_2 &= -\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}(SF_{t_2})(t_1)\,dt_1\,dt_2\\ &= -\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}((Sf)(t_1+t_2)-(Sf)(t_2))\,dt_1\,dt_2\\ &= -\int_{\mathbb{Z}_p^2}(Sf)(t_1+t_2)\,dt_1\,dt_2+\int_{\mathbb{Z}_p}(Sf)(t_2)\,dt_2.\end{aligned}$$

On the other hand, the right-hand side of (4.4) is equal to

$$\begin{aligned}\int_{\mathbb{Z}_p}(t_2+1)\int_{\mathbb{Z}_p}f(t_1+t_2)\,dt_1\,dt_2 &= \int_{\mathbb{Z}_p}(t_2+1)(Sf)'(t_2)\,dt_2\\ &= -\int_{\mathbb{Z}_p^2}(Sf)(t_1+t_2)\,dt_1\,dt_2+\int_{\mathbb{Z}_p}(Sf)(t_2)\,dt_2\end{aligned}$$

because of Lemma 4.1. As a consequence, equation (4.4) holds. ∎

We are now in a position to prove our Main Theorem.

Proof of the Main Theorem. First we prove the uniqueness (ii). This actually follows from a more general result in [3, Section 1], but we give a proof to make the paper self-contained. We assume that strictly differentiable functions $f(x)$ and $g(x)$ satisfy conditions (A) and (B). Set $h(x)=f(x)-g(x)$. By (B), we have $r\int_{\mathbb{Z}_p}h(x+t)\,dt=(x-r)h'(x)$. By (A), we have $h(x+1)=h(x)$ for all $x\in\mathbb{Z}_p$. Therefore $\int_{\mathbb{Z}_p}h(x+t)\,dt=h(x)$. Moreover, $h'(x)=0$ because

$$\lim_{n\to\infty}\frac{h(x+p^n)-h(x)}{p^n}=0.$$

As a consequence, $h(x)=0$, and this proves (ii).

Now we prove (i). We calculate the following integral in two ways:

$$\int_{\mathbb{Z}_p^r}(x+t_1+\cdots+t_r)\varphi_{r-1}(x+t_1+\cdots+t_r)\,dt_1\dots dt_r. \tag{4.5}$$

By equation (3.1), we obtain

$$\begin{aligned}&\int_{\mathbb{Z}_p^r}(x+t_1+\cdots+t_r)\varphi_{r-1}(x+t_1+\cdots+t_r)\,dt_1\dots dt_r\\ &\quad=\int_{\mathbb{Z}_p^r}\left(r\varphi_r(x+t_1+\cdots+t_r)+\frac{(x+t_1+\cdots+t_r)^r}{r!}\right)dt_1\dots dt_r\\ &\quad= r\operatorname{Log}\Gamma_{\mathrm{D},r}(x)+S_r(x).\end{aligned} \tag{4.6}$$

On the other hand, by Lemma 4.2,

$$\begin{aligned}\int_{\mathbb{Z}_p^r}&(x+t_1+\cdots+t_r)\varphi_{r-1}(x+t_1+\cdots+t_r)\,dt_1\ldots dt_r\\&=\sum_{i=1}^{r}\int_{\mathbb{Z}_p^r}(t_i+1)\varphi_{r-1}(x+t_1+\cdots+t_r)\,dt_1\ldots dt_r\\&\quad+(x-r)\int_{\mathbb{Z}_p^r}\varphi_{r-1}(x+t_1+\cdots+t_r)\,dt_1\ldots dt_r\\&=r\int_{\mathbb{Z}_p^r}(t_1+1)\varphi_{r-1}(x+t_1+\cdots+t_r)\,dt_1\ldots dt_r\\&\quad+(x-r)\int_{\mathbb{Z}_p^r}\varphi_{r-1}(x+t_1+\cdots+t_r)\,dt_1\ldots dt_r.\end{aligned}$$

By Lemma 4.1 and the relation $\varphi_r'(x)=\varphi_{r-1}(x)$, we have

$$\begin{aligned}\int_{\mathbb{Z}_p^r}&(t_1+1)\varphi_{r-1}(x+t_1+\cdots+t_r)\,dt_1\ldots dt_r\\&=\int_{\mathbb{Z}_p^r}\varphi_r(x+t_1+\cdots+t_r)\,dt_1\ldots dt_r\\&\quad-\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p^r}\varphi_r(x+t_1+\cdots+t_r+t)\,dt_1\ldots dt_r\,dt\\&=\operatorname{Log}\Gamma_{\mathrm{D},r}(x)-\int_{\mathbb{Z}_p}\operatorname{Log}\Gamma_{\mathrm{D},r}(x+t)\,dt.\end{aligned}$$

Moreover, by (2.5),

$$\begin{aligned}\int_{\mathbb{Z}_p^r}\varphi_{r-1}(x+t_1+\cdots+t_r)\,dt_1\ldots dt_r&=\frac{d}{dx}\int_{\mathbb{Z}_p^r}\varphi_r(x+t_1+\cdots+t_r)\,dt_1\ldots dt_r\\&=(\operatorname{Log}\Gamma_{\mathrm{D},r})'(x).\end{aligned}$$

Therefore

$$\begin{aligned}\int_{\mathbb{Z}_p^r}&(x+t_1+\cdots+t_r)\varphi_{r-1}(x+t_1+\cdots+t_r)\,dt_1\ldots dt_r\\&=r\operatorname{Log}\Gamma_{\mathrm{D},r}(x)-r\int_{\mathbb{Z}_p}\operatorname{Log}\Gamma_{\mathrm{D},r}(x+t)\,dt+(x-r)(\operatorname{Log}\Gamma_{\mathrm{D},r})'(x).\end{aligned}\tag{4.7}$$

Combining (4.6) and (4.7), we obtain (1.4). ∎

5. Multiple p-adic log-gamma functions on $\mathbb{Z}_p$. In this last section, we deal with multiple p-adic log-gamma functions defined on $\mathbb{Z}_p$. For a continuous function $f : \mathbb{Z}_p \to \mathbb{C}_p$, we use the notation

$$f^*(x) = \begin{cases} f(x) & (\text{if } x \in \mathbb{Z}_p^\times), \\ 0 & (\text{if } x \in p\mathbb{Z}_p). \end{cases}$$

It is clear that $f^*(x)$ is also continuous on $\mathbb{Z}_p$ and $\frac{d}{dx}(f^*) = \left(\frac{d}{dx}f\right)^*$ if f is differentiable (cf. [5]). For an integer $r \geq 0$ and $x \in \mathbb{Z}_p$, we define *multiple p-adic log-gamma functions* on $\mathbb{Z}_p$ as

$$\text{(5.1)} \qquad \operatorname{Log} \Gamma_{\mathrm{M},r}(x) = \begin{cases} \int_{\mathbb{Z}_p^r} \varphi_r^*(x + t_1 + \dots + t_r)\, dt_1 \dots dt_r & (r \geq 1), \\ \log_p^*(x) & (r = 0). \end{cases}$$

The function $\operatorname{Log} \Gamma_{\mathrm{M},r}(x)$ satisfies the difference equation

$$\text{(5.2)} \qquad \operatorname{Log} \Gamma_{\mathrm{M},r}(x+1) - \operatorname{Log} \Gamma_{\mathrm{M},r}(x) = \operatorname{Log} \Gamma_{\mathrm{M},r-1}(x) \qquad (x \in \mathbb{Z}_p)$$

for all $r \geq 1$. When $r = 1$, the function $\operatorname{Log} \Gamma_{\mathrm{M},1}(x)$ is the logarithm of Morita's p-adic gamma function, i.e. $\operatorname{Log} \Gamma_{\mathrm{M},1}(x) = \log_p \Gamma_p(x)$ (e.g. [3, p. 370]).

REMARK 1. Endo [5, p. 45] introduced multiple p-adic log-gamma functions $G_r(x)$ for $r \geq 1$ and $x \in \mathbb{Z}_p$ as

$$\text{(5.3)} \qquad G_r(x) = \int_{\mathbb{Z}_p^r} \left[\varphi_r^*(x + t_1 + \dots + t_r) - \sum_{k=0}^{r} \binom{x}{r-k} \varphi_k^*(t_1 + \dots + t_k) \right] dt_1 \dots dt_r.$$

He showed that the function G_r satisfies not only the difference equation $G_{r+1}(x+1) - G_{r+1}(x) = G_r(x)$ but the good initial condition $G_r(0) = 0$ for all $r \geq 1$ ([5, Theorem 5]). Therefore the function G_r can be considered as a modification of (5.1), but for the sake of simplicity, we consider (5.1) in this paper.

The following proposition gives a relation between $\operatorname{Log} \Gamma_{\mathrm{M},r}(x)$ and $\operatorname{Log} \Gamma_{\mathrm{D},r}(x)$. This is a generalization of the known formula (e.g. [10, Theorem 60.2]):

$$\text{(5.4)} \qquad \operatorname{Log} \Gamma_{\mathrm{M},1}(x) = \sum_{\substack{i=0 \\ p \nmid (x+i)}}^{p-1} \operatorname{Log} \Gamma_{\mathrm{D},1}\left(\frac{x+i}{p}\right) \qquad (x \in \mathbb{Z}_p).$$

PROPOSITION 5.1. *For a positive integer r and $x \in \mathbb{Z}_p$, we have*

$$\operatorname{Log} \Gamma_{\mathrm{M},r}(x) = \sum_{i_1=0}^{p-1} \dots \sum_{\substack{i_r=0 \\ p \nmid (x+i_1+\dots+i_r)}}^{p-1} \operatorname{Log} \Gamma_{\mathrm{D},r}\left(\frac{x + i_1 + \dots + i_r}{p}\right).$$

Proof. By (3.3), we obtain

$$\begin{aligned}\operatorname{Log}\Gamma_{\mathrm{M},r}(x) &= \int_{\mathbb{Z}_p^r} \varphi_r^*(x+t_1+\cdots+t_r)\,dt_1\ldots dt_r \\ &= \frac{1}{p^r}\sum_{i_1=0}^{p-1}\cdots\sum_{i_r=0}^{p-1}\int_{\mathbb{Z}_p^r}\varphi_r^*(x+(i_1+ps_1)+\cdots+(i_r+ps_r))\,ds_1\ldots ds_r \\ &= \frac{1}{p^r}\sum_{\substack{i_1=0\\ p\nmid(x+i_1+\cdots+i_r)}}^{p-1}\cdots\sum_{i_r=0}^{p-1}\int_{\mathbb{Z}_p^r}\varphi_r(x+i_1+\cdots+i_r+ps_1+\cdots+ps_r)\,ds_1\ldots ds_r.\end{aligned}$$

By the equation $\varphi_r(px)=p^r\varphi_r(x)$, we have

$$\begin{aligned}&\operatorname{Log}\Gamma_{\mathrm{M},r}(x) \\ &\quad= \sum_{\substack{i_1=0\\ p\nmid(x+i_1+\cdots+i_r)}}^{p-1}\cdots\sum_{i_r=0}^{p-1}\int_{\mathbb{Z}_p^r}\varphi_r\left(\frac{x+i_1+\cdots+i_r}{p}+s_1+\cdots+s_r\right)ds_1\ldots ds_r \\ &\quad= \sum_{\substack{i_1=0\\ p\nmid(x+i_1+\cdots+i_r)}}^{p-1}\cdots\sum_{i_r=0}^{p-1}\operatorname{Log}\Gamma_{\mathrm{D},r}\left(\frac{x+i_1+\cdots+i_r}{p}\right). \ \blacksquare\end{aligned}$$

In the last part of this paper, we show that the function $\operatorname{Log}\Gamma_{\mathrm{M},r}(x)$ satisfies the following integro-differential equation similar to (1.4).

Proposition 5.2. *For a positive integer r and $x\in\mathbb{Z}_p$, we have*

$$\begin{aligned}&r\int_{\mathbb{Z}_p}\operatorname{Log}\Gamma_{\mathrm{M},r}(x+t)\,dt \\ &\qquad= (x-r)(\operatorname{Log}\Gamma_{\mathrm{M},r})'(x)-S_r(x)+{\sum}' S_r\left(\frac{x+i_1+\cdots+i_r}{p}\right),\end{aligned}$$

where the summation is over all integers $i_1,\ldots,i_r$ with $0\le i_l\le p-1$ $(1\le l\le r)$ and $p\mid(x+i_1+\cdots+i_r)$.

This proposition is a generalization of the formula in [3, Proposition 2.4]:

$$\int_{\mathbb{Z}_p}\operatorname{Log}\Gamma_{\mathrm{M},1}(x+t)\,dt=(x-1)(\operatorname{Log}\Gamma_{\mathrm{M},1})'(x)-x+\left\lceil\frac{x}{p}\right\rceil \quad (x\in\mathbb{Z}_p), \tag{5.5}$$

where $\lceil x/p\rceil$ $(x\in\mathbb{Z}_p)$ is the p-adic limit of the usual integer ceiling function $\lceil x_n/p\rceil$ as $x_n\to x$ through $x_n\in\mathbb{Z}$. In fact, when $r=1$ in Proposition 5.2, then $S_r(x) = B_1(x) = x - 1/2$ and the last term of (5.2) is equal to

$$\sum_{\substack{0\le i\le p-1\\ p\mid(x+i)}} B_1\left(\frac{x+i}{p}\right) = B_1\left(\left\lceil\frac{x}{p}\right\rceil\right) = \left\lceil\frac{x}{p}\right\rceil - \frac{1}{2}. \tag{5.6}$$

In a way similar to the proof of the Main Theorem, we have

$$\begin{aligned} r\int_{\mathbb{Z}_p} \operatorname{Log}\Gamma_{\mathrm{M},r}(x+t)\,dt &\\ = (x-r)(\operatorname{Log}\Gamma_{\mathrm{M},r})'(x) - \frac{1}{r!}\int_{\mathbb{Z}_p^r} (x+t_1+\cdots+t_r)^{r*}\,dt_1\ldots dt_r. \end{aligned} \tag{5.7}$$

Therefore Proposition 5.2 follows from the next lemma.

Lemma 5.3. *For a positive integer r and $x\in\mathbb{Z}_p$, we have*

$$\int_{\mathbb{Z}_p^r} (x+t_1+\cdots+t_r)^{r*}\,dt_1\ldots dt_r = r!S_r(x) - r!{\sum}' S_r\left(\frac{x+i_1+\cdots+i_r}{p}\right),$$

where the summation is the same as in Proposition 5.2.

Proof. By (3.3), we have

$$\begin{aligned} &\int_{\mathbb{Z}_p^r} (x+t_1+\cdots+t_r)^{r*}\,dt_1\ldots dt_r\\ &= \int_{\mathbb{Z}_p^r} \frac{1}{p^r}\sum_{i_1=0}^{p-1}\cdots\sum_{i_r=0}^{p-1} (x+(i_1+ps_1)+\cdots+(i_r+ps_r))^{r*}\,ds_1\ldots ds_r\\ &= \int_{\mathbb{Z}_p^r} \frac{1}{p^r}\sum_{i_1=0}^{p-1}\cdots\sum_{i_r=0}^{p-1}(x+i_1+\cdots+i_r+ps_1+\cdots+ps_r)^r\,ds_1\ldots ds_r\\ &\quad - \int_{\mathbb{Z}_p^r} \frac{1}{p^r}\sum_{\substack{i_1=0\\ p\mid(x+i_1+\cdots+i_r)}}^{p-1}\cdots\sum_{i_r=0}^{p-1} (x+i_1+\cdots+i_r+ps_1+\cdots+ps_r)^r\,ds_1\ldots ds_r\\ &= \int_{\mathbb{Z}_p^r}(x+t_1+\cdots+t_r)^r\,dt_1\ldots dt_r\\ &\quad - {\sum}'\int_{\mathbb{Z}_p^r}\left(\frac{x+i_1+\cdots+i_r}{p}+s_1+\cdots+s_r\right)^r ds_1\ldots ds_r\\ &= r!S_r(x) - r!{\sum}' S_r\left(\frac{x+i_1+\cdots+i_r}{p}\right). \ \blacksquare \end{aligned}$$

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