ACTA ARITHMETICA
164.1 (2014)

Arithmetic diophantine approximation for continued fractions-like maps on the interval

by

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1. Introduction

1.1. Motivation and goals. Given an irrational number r and a rational number, written as the unique quotient p/q of the relatively prime integers p and $q > 0$, our fundamental object of interest from diophantine approximation is the *approximation coefficient* $\theta(r, p/q) := q^2|r - p/q|$. Since adding an integer to a fraction does not change its denominator, we have $\theta(r, p/q) = \theta(r - \lfloor r \rfloor, p/q - \lfloor r \rfloor)$, where $\lfloor r \rfloor$ is the integer part of r, allowing us to restrict our attention to the unit interval. We expand the initial seed $x_0 \in (0,1) - \mathbb{Q}$ as a regular continued fraction and obtain the infinite sequence $\{b_n\}_{n=1}^{\infty}$ of partial quotients or 0-*digits* uniquely determined by x_0. The sequence of rational numbers

$$\frac{p_0}{q_0} := \frac{0}{1}, \qquad \frac{p_n}{q_n} := [b_1, \ldots, b_n]_0 = \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cdots + \cfrac{1}{b_n}}}, \qquad n \geq 1,$$

called the *convergents* of x_0 are also uniquely determined (the reason for the subscript $[\cdot]_0$ will become clear later).

Define the approximation coefficient associated with each convergent of x_0 by

$$\theta_n(x_0) = \theta_n := \theta\left(x_0, \frac{p_n}{q_n}\right) = q_n^2 \left| r - \frac{p_n}{q_n} \right|$$

and refer to $\{\theta_n\}_{n=0}^{\infty}$ as the *sequence of approximation coefficients*. Much work has been done with this sequence, from its inception in the classical era, until the more recent excursions [4, 2, 9, 15, 20]. These reveal elegant internal structure as well as simple connections with the sequence of 0-digits. The introduction section of [4] provides a survey of these results, whereas

2010 *Mathematics Subject Classification*: Primary 11J70; Secondary 11Y65.
Key words and phrases: diophantine approximation, continued fractions.

DOI: 10.4064/aa164-1-1

a more thorough treatment can be found in [9]. Furthermore, the essential lower bounds for this sequence determine how well can irrational numbers be approximated using rational numbers and lead to the construction of the Lagrange spectrum [8].

Our goal is to improve the results of Jager and Kraaikamp [14] as well as to extend them to the rather large classes of continued fraction-like expansions, first introduced by Haas and Molnar [12, 13]. While emulating Jager and Kraaikamp's approach, we will diverge from their methods in several key choices regarding the treatment of the dynamical systems at hand. Most notably, we will adopt Nakada's realization for the natural extension map [16], which was used in the proof of the Doeblin–Lenstra conjecture [3].

1.2. The dynamics for regular and backwards continued fractions. From a dynamic point of view, the regular continued fraction expansion, or 0-*expansion*, is a concrete realization of the symbolic representation of irrational numbers in the unit interval under the iterations of the *Gauss map*

$$T_0 : [0,1) \to [0,1), \quad T_0(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \quad T_0(0) := 0.$$

This map is the fractional part of the homeomorphism $A_0 : (0,1) \to (0,\infty)$, $x \mapsto (1-x)/x$, which in turn extends to the Möbius transformation $\hat{A}_0 : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $z \mapsto (1-z)/z$, mapping $[0,1]$ bijectively to $[0,\infty]$ in an orientation reversing manner (since $\hat{A}_0(0) = \infty$ yet $T_0(0) = 0$, we comply with the conventional wisdom that the fractional part of ∞ is 0). The Gauss map is both invariant and ergodic with respect to the probability Gauss measure $\mu_0(E) := (\ln 2)^{-1} \int_E (1-x)^{-1}\, dx$ on the interval.

Another well known continued fraction theory is the backwards continued fraction expansion, or 1-*expansion*, stemming from the *Rényi map*

$$T_1 : [0,1) \to [0,1), \quad T_1(x) := \frac{1}{1-x} - \left\lfloor \frac{1}{1-x} \right\rfloor.$$

This map is the fractional part of the homeomorphism $A_1 : (0,1) \to (0,\infty)$, $x \mapsto x/(1-x)$, which extends to the Möbius transformation $\hat{A}_1 : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $z \mapsto z/(1-z)$, mapping $[0,1]$ bijectively onto $[0,\infty]$ in an orientation preserving manner. The Rényi map is invariant and ergodic with respect to the infinite measure $\mu_1(E) := \int_E x^{-1}\, dx$ on the interval.

Letting $m \in \{0,1\}$, we extract the sequence of digits for the m-expansion of a real number $x_0 \in (0,1)$ using the following iteration process:

(1) Set $n := 1$.
(2) If $x_{n-1} = 0$, write $x_0 = 0$ if $n = 1$ or $x_0 = [b_1, \ldots, b_{n-1}]_m$ if $n > 1$ and exit.

(3) Let $b_n := \lfloor A_m(x_{n-1}) \rfloor + 1 \in \mathbb{N}$ and $x_n := T_m(x_{n-1}) \in [0,1)$. Increase n by 1 and go to step (2).

For instance,

$$[1,1,2]_0 = \frac{1}{1+\frac{1}{1+\frac{1}{2}}} = \frac{1}{1+\frac{2}{3}} = \frac{3}{5} = 1 - \frac{2}{5} = 1 - \frac{1}{2+1-\frac{1}{2}} = [2,2]_1.$$

REMARK 1.1. The fact that T_1 has an indifferent fixed point at the origin forces any absolutely continuous invariant measure to be infinite [19]. This deficiency helps explain why the 1-expansion did not gain nearly as much attention as its 0-expansion cousin, even though it sometimes leads to quicker expansions, as seen in the example above. Refer to [10] for details comparing the speed of approximation between the 0- and 1-expansions. The metrical theory for the backwards continued fraction expansion can be found in [11].

2. Preliminaries. This section is a paraphrased summary of previous work due to Haas and Molnar [12, 13], given for the sake of completeness.

2.1. Gauss-like and Rényi-like continued fractions. In general, the fractional part of Möbius transformations which map $[0,1]$ onto $[0,\infty]$ leads to expansion of real numbers as continued fractions. To characterize all these transformations, we recall that Möbius transformations are uniquely determined by their value on three distinct points. Thus, we will need to introduce a parameter for the image of an additional point besides 0 and 1, which we will naturally take to be ∞. Since our maps fix the real line, the image of ∞, denoted by $-k$, can take any value within the set of all negative real numbers. After letting $m \in \{0,1\}$ equal zero or one for orientation reversing and preserving transformations respectively, we conclude that all such transformations are derived as extensions of the homeomorphisms

$$A_{(m,k)} : (0,1) \to (0,\infty), \quad x \mapsto \frac{k(1-m-x)}{x-m}, \quad k > 0,$$

from the open unit interval to its closure. The maps $T_{(m,k)} : [0,1) \to [0,1)$ such that $0 \mapsto 0$ and

$$\begin{aligned} T_{(m,k)}x &= A_{(m,k)}(x) - \lfloor A_{(m,k)}(x) \rfloor \\ &= \frac{k(1-m-x)}{x-m} - \left\lfloor \frac{k(1-m-x)}{x-m} \right\rfloor, \qquad x > 0, \end{aligned} \tag{2.1}$$

are called *Gauss-like* and *Rényi-like* for $m = 0$ and $m = 1$ respectively.

We expand the initial seed $x_0 \in (0,1)$ as an (m,k)-continued fraction using the following iteration process:

(1) Set $n := 1$.
(2) If $x_{n-1} = 0$, write $x_0 = [a_1, \ldots, a_{n-1}]_{(m,k)}$ and exit.

(3) Set the *remainder* of x_0 at time n to be $r_n := A_{(m,k)}(x_{n-1}) \in (0,\infty)$ and the (m,k)-*expansion* for x_0 at time n to be $x_0 = [r_1]_{(m,k)}$ if $n=1$ or $x_0 = [a_1,\ldots,a_{n-1},r_n]_{(m,k)}$ if $n>1$. Also, set the *digit* and *future* of x_0 at time n to be

$$a_n := \lfloor r_n \rfloor = \lfloor A_{(m,k)}(x_{n-1}) \rfloor = \left\lfloor \frac{k(1-m-x_{n-1})}{x_{n-1}-m} \right\rfloor \in \mathbb{Z}_{\geq 0} \tag{2.2}$$

and $x_n := [r_{n+1}]_{(m,k)} = r_n - a_n \in [0,1)$. Increase n by 1 and go to step (2).

For all $n \geq 0$, we thus have

$$x_{n+1} = T_{(m,k)}(x_n) = \frac{k(1-m-x_n)}{x_n - m} - a_n$$

so that

$$\begin{aligned} x_n = [a_{n+1}, r_{n+2}]_{(m,k)} &= m + \frac{k(1-2m)}{a_{n+1}+k+[r_{n+2}]_{(m,k)}} \\ &= m + \frac{k(1-2m)}{a_{n+1}+k+x_{n+1}}. \end{aligned} \tag{2.3}$$

Therefore, this iteration scheme leads to the expansion of the initial seed x_0 as

$$x_0 = m + \frac{k(1-2m)}{a_1+k+x_1} = m + \cfrac{k(1-2m)}{a_1+k+m+\cfrac{k(1-2m)}{a_2+k+x_2}} = \cdots.$$

Remark 2.1. The $m=0$, $2 \leq k \in \mathbb{N}$ cases have been studied in [1], where they were called *best expansions*. The special case $k=1$ corresponds with the classical Gauss and Rényi maps for $m=0$ and $m=1$ respectively, but with digits that are 1 smaller than their classical representation. For instance,

$$[0,1,2]_{(0,k)} = \cfrac{k}{0+k+\cfrac{k}{1+k+\cfrac{k}{2+k}}} = \frac{k^2+4k+2}{k^2+5k+4}$$

and

$$[0,1,2]_{(1,k)} = 1 - \cfrac{k}{0+k+1-\cfrac{k}{1+k+1-\cfrac{k}{2+k}}} = \frac{k+4}{k^3+3k^2+5k+4}$$

will yield, after plugging $k=1$, the fractions $[1,2,3]_0 = \frac{7}{10}$ and $[1,2,3]_1 = \frac{5}{13}$. We label the digits of the m-expansion b_n and the (m,k)-expansion $a_n = b_n - 1$ to help avoid this confusion.

We denote by $\mathbb{Q}^{(N)}_{(m,k)}$ the set of real numbers in the interval for which this process terminates by the Nth iteration, that is,

$$\mathbb{Q}^{(N)}_{(m,k)} := \{x \in [0,1) : T^n_{(m,k)}(x) = 0 \text{ for some } n \le N\}.$$

Then $x_0 \in \mathbb{Q}_{(m,k)}$ if and only if $x_0 = 0$ or x_0 has a finite (m,k)-expansion, that is, there exists a unique finite sequence of digits $\{a_n\}_{n=1}^N$ such that $x_0 = [a_1, \dots, a_N]_{(m,k)}$. Accordingly, we define the set $\mathbb{Q}_{(m,k)} := \lim_{N\to\infty} \mathbb{Q}^{(N)}_{(m,k)}$ and the set of (m,k)-*irrationals* to be its complement in the interval. We also define the *interval of monotonicity* (or cylinder set) of rank $N \ge 0$ associated with the finite sequence $\{a_1, \dots, a_N\}$ of N non-negative integers to be $\Delta^{(0)} := (0,1)$ and $\Delta^{(N)}_{a_1,\dots,a_N} := \{x_0 \in (0,1) : a_n(x_0) = a_n \text{ for all } 1 \le n \le N\}$. Then the restriction of $T^N_{(m,k)}$ to the interior of any interval of monotonicity of rank N is a homeomorphism onto $(0,1)$ and for all $N \ge 0$ we have

$$(0,1) = \bigcup_{a_1,\dots,a_N \in \mathbb{Z}_{\ge 0}} \Delta^{(N)}_{a_1,\dots,a_N},$$

where this union is disjoint in pairs.

2.2. Approximation coefficients for Gauss-like and Rényi-like maps. Fix $m \in \{0,1\}$ and $k \in [1,\infty)$. The (m,k)-sequence of approximation coefficients $\{\theta_n(x_0)\}_{n=0}^\infty$ for the (m,k)-expansion is defined just like the classical object:

$$\theta_n(x_0) := q_n^2 \left| x_0 - \frac{p_n}{q_n} \right|, \qquad n \ge 1, \tag{2.4}$$

where the (m,k)-rational numbers $p_0/q_0 = 0/1$ and $p_n/q_n = [a_1, \dots, a_n]_{(m,k)}$ are the corresponding convergents for x_0. We further define the *past* of x_0 at time $n \ge 0$ to be $Y_0 := m - k$, $Y_1 := m - k - a_1 \in (-\infty, m-k]$ and

$$Y_n := m - k - a_N - [a_{N-1}, \dots, a_1]_{(m,k)} \in (-\infty, m-k), \qquad n \ge 2. \tag{2.5}$$

The sequence of approximation coefficients relates to the future and past sequences of x_0 using the identity

$$\theta_{n-1}(x_0) = \frac{1}{x_n - Y_n}, \qquad n \ge 1, \tag{2.6}$$

which was first proved for the classical Gauss case $m = 0$, $k = 1$ in 1921 by Perron [17].

When $k > 1$ and $n > 0$, the *Jager pair* $(\theta_{n-1}(x_0), \theta_n(x_0))$ lies within the quadrangle in the Cartesian plane with vertices $(0,0)$, $(1/k, 0)$, $(0, 1/k)$ and $(1/(k+1-2m), 1/(k+1-2m))$. Note that for the classical Gauss case $m = 0$, $k = 1$, this quadrangle degenerates to the triangle with vertices $(0,0)$, $(1,0)$ and $(1,0)$, in accordance with the findings of Jager and Kraaikamp [14]. For the classical Rényi case $m = k = 1$, this quadrangle expands to the infinite region in the first quadrant of the uv-plane bounded between the lines $u-v=1$ and $v-u=1$ (for more details and illustrations refer to [5]). We conclude that for $x_0 \in (0,1) - \mathbb{Q}_{(m,k)}$, $k \geq 1$ and $n > 0$, we have

$$k\theta_{n-1}(x_0) + (1-2m)\theta_n(x_0) \leq 1, \tag{2.7}$$

$$\theta_{n-1}(x_0) + (1-2m)k\theta_n(x_0) \leq 1. \tag{2.8}$$

2.3. The natural extension. For fixed $m \in \{0,1\}$ and $k \in [1,\infty)$, the maps $T_{(m,k)}$ are both invariant and ergodic with respect to the measures whose densities on the interval are

$$\mu_{(m,k)}(x) := \left(\ln\left(\frac{k+1-m}{k-m}\right)(x+k-1)\right)^{-1}.$$

The induced dynamical systems

$$\{(0,1), \mathcal{L}, \mu, T\}_{(m,k)} := \{(0,1) - \mathbb{Q}_{(m,k)}, \mathcal{L}, \mu_{(m,k)}, T_{(m,k)}\},$$

where $\mathcal{L}$ is the Lebesgue σ-algebra, are not invertible since the maps $T_{(m,k)}$ are not bijections. However, there is a canonical way to extend non-invertible dynamical systems to invertible ones [18]. Our realization of the natural extension maps begins by defining the region $\Omega'_{(m,k)} := [0,1) \times (-\infty, m-k]$, the set

$$\mathbb{Q}'_{(m,k)} := \left\{m - k - b - q : b \in \mathbb{Z}_{\geq 0} \text{ and } q \in \mathbb{Q}_{(m,k)}\right\} \subset (-\infty, m-k]$$

and the *space of dynamic pairs*

$$\Omega_{(m,k)} := \Omega'_{(m,k)} - ([0,1) \times \mathbb{Q}'_{(m,k)}) \cup (\mathbb{Q}_{(m,k)} \times (-\infty, m-k]). \tag{2.9}$$

The *natural extension map* $\mathscr{T}_{(m,k)} : \Omega_{(m,k)} \to \Omega_{(m,k)}$ is defined as

$$\begin{aligned}\mathscr{T}_{(m,k)}(x,y) &= (A_{(m,k)}(x) - \lfloor A_{(m,k)}(x) \rfloor, A_{(m,k)}(y) - \lfloor A_{(m,k)}(x) \rfloor) \\ &= (T_{(m,k)}(x), A_{(m,k)}(y) - \lfloor A_{(m,k)}(x) \rfloor).\end{aligned}$$

After using the definition (2.1) of $T_{(m,k)}$, this map is written explicitly as

$$\begin{aligned}&\mathscr{T}_{(m,k)}(x,y) \\ &= \left(\frac{k(1-m-x)}{x-m} - \left\lfloor \frac{k(1-m-x)}{x-m} \right\rfloor, \frac{k(1-m-y)}{y-m} - \left\lfloor \frac{k(1-m-x)}{x-m} \right\rfloor\right).\end{aligned} \tag{2.10}$$

The maps $\mathscr{T}_{(m,k)}$ are both invariant and ergodic with respect to the probability measures $\rho_{(m,k)}(D) := \ln((k+1-m)/(k-m))^{-1} \iint_D (x-y)^{-2}\, dx\, dy$ when $k > m$ and the infinite measure $\rho_{(1,1)}(D) := \iint_D (x-y)^{-2}\, dx\, dy$ for the classical Rényi case $m = k = 1$ (since there is no finite invariant measure for T_1, there is also no finite invariant measure for $\mathscr{T}_{(1,1)}$; see Remark 1.1). Furthermore, the dynamical system $\{(0,1), \mathcal{L}, \mu, T\}_{(m,k)}$ is realized as a left factor to the invertible dynamical system $\{\Omega_{(m,k)}, \mathcal{L}^2, \rho_{(m,k)}, \mathscr{T}_{(m,k)}\}$. From now on, we will require the parameter k to be grater than or equal to one.

This choice is not arbitrary, for the $0 < k < 1$ cases are known to have certain pathologies [6].

For the given parameters $m \in \{0,1\}$ and $k \geq 1$, and an initial seed $(x_0, y_0) \in \Omega_{(m,k)}$, we let $\{a_n\}_{n=1}^{\infty}$ be the unique sequence of non-negative integers and $\{r_n\}_{n=1}^{\infty}$ be the unique sequence of remainders such that

$$x_0 = [r_1]_{(m,k)} = [a_1, r_2]_{(m,k)} = [a_1, a_2, r_3]_{(m,k)} = \cdots .$$

Since $y_0 < m - k$, there exists a unique non-negative integer a_0 such that $m - k - a_0 - y_0 \in (0,1)$. Also, from the definition (2.9) of Ω, we see that this number is an (m,k)-irrational, hence we take $\{a_n\}_{n=-1}^{-\infty}$ to be the unique sequence of non-negative integers and $\{s_n\}_{n=0}^{-\infty}$ to be the unique sequence of remainders such that

$$\begin{aligned} m - k - a_0 - y_0 &= [s_0]_{(m,k)} = [a_{-1}, s_{-1}]_{(m,k)} \\ &= [a_{-1}, a_{-2}, s_{-2}]_{(m,k)} = \cdots . \end{aligned} \tag{2.11}$$

Using (2.2) and the definition (2.10) of $\mathscr{T}$, we see that for all $n \in \mathbb{Z}$ we have

$$\begin{aligned} (x_{n+1}, y_{n+1}) &= \mathscr{T}_{(m,k)}(x_n, y_n) \\ &= \left(\frac{k(1-m-x_n)}{x_n - m} - a_{n+1}, \frac{k(1-m-y_n)}{y_n - m} - a_{n+1} \right). \end{aligned} \tag{2.12}$$

We now apply (2.3) to write an explicit formula for the inverse map $\mathscr{T}^{-1}$ as

$$\begin{aligned} (x_n, y_n) &= \mathscr{T}^{-1}_{(m,k)}(x_{n+1}, y_{n+1}) \\ &:= \left(m + \frac{(1-2m)k}{k + a_{n+1} + x_{n+1}}, m + \frac{(1-2m)k}{k + a_{n+1} + y_{n+1}} \right). \end{aligned} \tag{2.13}$$

Since the quantity x_n is no other than the future of x_0 at time n when $n \geq 1$, we naturally call x_n and y_n the *future* and *past* of (x_0, y_0) at time $n \in \mathbb{Z}$. The pair $(x_n, y_n) := \mathscr{T}^{n}_{(m,k)}(x_0, y_0)$ is called the *dynamic pair* of (x_0, y_0) at time $n \in \mathbb{Z}$, and the bi-sequence $\{a_n\}_{-n=\infty}^{\infty}$ is called the (m,k)-*digit bi-sequence* for (x_0, y_0).

From a heuristic point of view, the map $\mathscr{T}_{(m,k)}$ can be thought of as an invertible left shift operator on the infinite (m,k)-digit bi-sequence

$$\begin{aligned} &[[\ldots, -a_{n-1}, -a_n \mid a_{n+1}, a_{n+2}, \ldots]]_{(m,k)} \\ &\qquad \overset{\mathscr{T}}{\mapsto} [[\ldots, -a_n, a_{n+1} \mid a_{n+2}, a_{n+3}, \ldots]]_{(m,k)}. \end{aligned}$$

The vertical line in this symbolic digit representation stands for the present time and $\mathscr{T}$ acts as one tick of a clock, pushing the present one step forward into the future.

3. Dynamic pairs and approximation pairs. Starting with an initial seed pair $(x_0, y_0) \in \Omega_{(m,k)}$, we take the hint from formula (2.6) and *define* the approximation coefficient for the initial seed at time $n-1 \in \mathbb{Z}$ to be

$$\theta_{n-1}(x_0, y_0) := \frac{1}{x_n - y_n}, \tag{3.1}$$

and refer to $\{\theta_n(x_0, y_0)\}_{n=-\infty}^{\infty}$ as the *bi-sequence of approximation coefficients*. Define the continuous map

$$\Psi_{(m,k)} : \Omega_{(m,k)} \to \mathbb{R}^2, \qquad (x, y) \mapsto \left(\frac{1}{x-y}, \frac{(m-x)(m-y)}{(2m-1)k(x-y)}\right), \tag{3.2}$$

and use formulas (2.12) and (3.1) to obtain

$$\Psi_{(m,k)}(x_n, y_n) = (\theta_{n-1}(x_0, y_0), \theta_n(x_0, y_0)), \qquad n \in \mathbb{Z}. \tag{3.3}$$

We denote the image $\Psi_{(m,k)}(\Omega_{(m,k)})$ by $\Gamma_{(m,k)}$ and, in order to ease the notation, suppress the subscripts (m, k) from now on.

Proposition 3.1. *For all $(u, v) \in \Gamma$, we have*

$$ku + (1-2m)v \le 1, \tag{3.4}$$

$$(1-2m)u + kv \le 1. \tag{3.5}$$

Proof. We first assume that $k - m > 0$. Let $(x_0, y_0) \in \Omega_{(m,k)}$ be any point in the preimage of (u, v) under Ψ and let $\{a_n\}_{n=-\infty}^{\infty}$ be the digit bi-sequence for its (m, k)-expansion. Letting Y_n and y_n be the past of x_0 and (x_0, y_0) at time $n \ge 1$ as in definitions (2.5) and (2.11), we see that for all $n \ge 2$, both $m + k + a_n - y_n$ and $m + k + a_n - Y_n$ belong to the interval of monotonicity $\Delta_{(m,k)}^{a_{n-1}, a_{n-2}, \ldots, a_1}$. Since the length (or Lebesgue measure) of intervals of monotonicity tends to zero as their depth tends to infinity, we have $y_n - Y_n \to 0$ as $n \to \infty$. Since $x_n > 0$ and $y_n < m - k$, we see that the sequence $\{x_n - y_n\}_{n=0}^{\infty}$ is uniformly bounded from below by the positive number $k - m$. Thus, we have

$$|\theta_{n+1}(x_0) - \theta_{n+1}(x_0, y_0)| = \left|\frac{1}{x_n - Y_n} - \frac{1}{x_n - y_n}\right| \to 0 \qquad \text{as } n \to \infty.$$

The fact that $\Psi_{(m,k)}$ is continuous allows us to conclude that the Jager pairs for (x_0, y_0) have the same uniform bounds as those of x_0, as expressed in the inequalities (2.7) and (2.8), which is precisely the result.

When $m = k = 1$, the continuity of Ψ implies

$$\Gamma_{(1,1)} = \Psi_{(1,1)}(\Omega_{(1,1)}) = \lim_{k \to 1+} \Psi_{(1,k)}(\Omega_{(1,k)}) = \lim_{k \to 1+} \Gamma_{(1,k)}.$$

Since the result holds for all $k > 1$, it remains true for the classical Rényi case as well. ∎

Define for all $u, v \ge 0$ the quantity

$$D(u, v) = D_{(m,k)}(u, v) := \sqrt{1 + 4(2m-1)kuv}. \tag{3.6}$$

The following lemma was first proved for the classical Gauss case $m = 0$, $k = 1$ by Jager and Kraaikamp [14].

LEMMA 3.2. *The map $\Psi : \Omega \to \Gamma$ is a homeomorphism with inverse*

$$\Psi^{-1}(u,v) := \left(m + \frac{1 - D(u,v)}{2u}, m - \frac{1 + D(u,v)}{2u}\right). \tag{3.7}$$

Proof. First, we will show that Ψ is a bijection. Since Ψ is surjective onto its image Γ, we need only show injectivity. Let $(x_1, y_1), (x_2, y_2)$ be two points in Ω such that $\Psi(x_1, y_1) = \Psi(x_2, y_2)$, that is,

$$\left(\frac{1}{x_1 - y_1}, \frac{(m - x_1)(m - y_1)}{(2m - 1)k(x_1 - y_1)}\right) = \left(\frac{1}{x_2 - y_2}, \frac{(m - x_2)(m - y_2)}{(2m - 1)k(x_2 - y_2)}\right).$$

Equate the first terms to obtain

$$x_1 - y_1 = x_2 - y_2, \tag{3.8}$$

and then equate the second terms to obtain

$$(m - x_1)(m - y_1) = (m - x_2)(m - y_2).$$

The basic algebraic equality $(\alpha + \beta)^2 - (\alpha - \beta)^2 = 4\alpha\beta$, using $\alpha = m - x_1$, $\beta = m - y_1$, will now yield

$$\begin{aligned}(2m - (x_1 + y_1))^2 - (x_1 - y_1)^2 &= 4(m - x_1)(m - y_1) = 4(m - x_2)(m - y_2) \\ &= (2m - (x_2 + y_2))^2 - (x_2 - y_2)^2.\end{aligned}$$

Another application of (3.8) reduces the last equation to

$$(2m - (x_1 + y_1))^2 = (2m - (x_2 + y_2))^2.$$

We use the definition (2.9) of Ω' and observe that $x + y \le 2m$ for all $(x, y) \in \Omega \subset \Omega'$, so that we may infer the equality $x_1 + y_1 = x_2 + y_2$. Further applications of (3.8) will first prove that

$$x_1 = \tfrac{1}{2}((x_1 + y_1) + (x_1 - y_1)) = \tfrac{1}{2}((x_2 + y_2) + (x_2 - y_2)) = x_2,$$

and then that $y_1 = y_2$ as well. Therefore, Ψ is an injection.

It is left to prove that Ψ^{-1} is well defined and continuous on Γ and that it is the inverse from the left for Ψ on Γ. Given $(u, v) \in \Gamma$, set

$$(x, y) := \Psi^{-1}(u,v) = \left(m + \frac{1 - D(u,v)}{2u}, m - \frac{1 + D(u,v)}{2u}\right).$$

For the Gauss-like $m = 0$ case, we see from inequality (3.4) that Γ lies on or underneath the line $ku + v = 1$ in the uv plane. The only point of intersection for this line and the hyperbola $4kuv = 1$ is $(u, v) = \left(\frac{1}{2k}, \frac{1}{2}\right)$, hence Γ must lie on or underneath this hyperbola as well. We conclude that $4kuv \le 1$ for all $(u, v) \in \Gamma$, hence $D(u, v)$ and then x and y are real. We use the inequality $ku + v \leq 1$ again to obtain

$$D(u,v)^2 = 1 - 4kuv \geq 4u^2k^2 - 4ku + 1 = (2ku - 1)^2,$$

and conclude that $1 + D(u,v) \geq 2ku$ and $y = -(1 + D(u,v))/(2u) \leq -k$. Next, we observe that $D(u,v) = \sqrt{1 - 4kuv} < 1$, so that $1 - D(u,v) > 0$, which proves $x = (1 - D(u,v))/(2u) > 0$. If we further assume by contradiction that $x = (1 - D(u,v))/(2u) \geq 1$ then

$$(1 - D(u,v))(1 + D(u,v)) \geq 2u(1 + D(u,v))$$

so that

$$4kuv = 1 - D(u,v)^2 = (1 - D(u,v))(1 + D(u,v)) \geq 2u(1 + D(u,v)).$$

This implies $2kv - 1 > D(u,v) \geq 0$ so that $(2kv - 1)^2 \geq D(u,v)^2$, hence $4k^2v^2 - 4kv + 1 \geq 1 - 4kv$ and $u + kv \geq 1$, in contradiction to (3.5).

For the Rényi-like $m = 1$ case, we have $D(u,v) = \sqrt{1 + 4kuv} > 1$ so that $x = 1 + (1 - D(u,v))/2u < 1$. The inequality (3.5) yields

$$D(u,v)^2 = 1 + 4ukv < 1 + 4u(1+u) = (2u+1)^2,$$

so that $D(u,v) < 2u + 1$. Then $(1 - D(u,v))/(2u) > -1$ and $x = 1 + (1 - D(u,v))/(2u) > 0$. Also $1 + 4kuv > 4u^2k^2 - 4ku + 1$, which implies $\sqrt{1 + 4kuv} > 2ku - 1$, so that $(1 + \sqrt{1 + 4kuv})/(2u) = 1 - y > k$. Then $y = 1 - (1 + D(u,v))/(2u) < 1 - k$. We conclude that $(x,y) \in \Omega$, hence Ψ^{-1} is well defined. Also, Ψ^{-1} is clearly continuous on Γ.

We complete the proof by showing that Ψ^{-1} is injective, i.e., $\Psi^{-1}\Psi(x,y) = (x,y)$ for all $(x,y) \in \Omega$. We use our definitions (3.2) for Ψ, (3.7) for Ψ^{-1}, and the fact that $(2m-1)^2 = 1$ whenever $m \in \{0,1\}$ to obtain

$$\begin{aligned}
\Psi^{-1}\Psi(x,y) &= \Psi^{-1}\left(\frac{1}{x-y}, \frac{(m-x)(m-y)}{(2m-1)k(x-y)}\right) \\
&= \left(m + \frac{x-y}{2}\left(1 - \sqrt{1 + \frac{4(m-x)(m-y)}{(x-y)^2}}\right),\right. \\
&\qquad\qquad \left. m - \frac{x-y}{2}\left(1 + \sqrt{1 + \frac{4(m-x)(m-y)}{(x-y)^2}}\right)\right) \\
&= \left(m + \frac{x-y}{2}\left(1 - \sqrt{\left(\frac{2m-x-y}{x-y}\right)^2}\right),\right. \\
&\qquad\qquad \left. m - \frac{x-y}{2}\left(1 + \sqrt{\left(\frac{2m-x-y}{x-y}\right)^2}\right)\right).
\end{aligned}$$

But since $2m - x - y \geq x - y > 0$ for all $(x,y) \in \Omega$, this allows us to conclude that the last expression simplifies to

$$\left(m+\frac{x-y}{2}\,\frac{2(x-m)}{x-y},\, m-\frac{x-y}{2}\,\frac{2(m-y)}{x-y}\right)=(x,y),$$

as desired. ∎

4. Symmetries in the bi-sequence of approximation coefficients. In this section, we reveal an elegant symmetrical structure in the bi-sequence of approximation coefficients, allowing us to recover it entirely from a pair of consecutive terms. First, we see that the digit a_{n+1} can be determined from both the pairs of approximation coefficients at times n and $n+1$ in precisely the same fashion. We define

$$D_{(m,k,n)}=D_n:=D(\theta_{n-1},\theta_n)=\sqrt{1+4(2m-1)k\theta_{n-1}\theta_n}, \tag{4.1}$$

as in (3.6).

PROPOSITION 4.1. *Let a_{n+1} be the (m,k)-digit at time $n+1$ and (θ_{n-1},θ_n) be the (m,k)-pair of approximation coefficients at time n for the initial seed pair $(x_0,y_0)\in\Omega$. Then*

$$a_{n+1}=\left\lfloor\frac{D_n+1}{2\theta_n}-k\right\rfloor=\left\lfloor\frac{D_{n+1}+1}{2\theta_n}-k\right\rfloor. \tag{4.2}$$

Proof. Using (3.3), the fact that Ψ is a bijection and the definition (3.7) of Ψ^{-1}, we have

$$(x_n,y_n)=\Psi^{-1}(\theta_{n-1},\theta_n)=\left(m+\frac{1-D_n}{2\theta_{n-1}},\, m-\frac{1+D_n}{2\theta_{n-1}}\right). \tag{4.3}$$

Using (2.3), the first components in the exterior terms of (4.3) equate to

$$a_{n+1}+k+[r_{n+2}]=\frac{(1-2m)k}{x_n-m}=\frac{(1-2m)2k\theta_{n-1}}{1-D_n}.$$

But since $[r_{n+2}]<1$, we obtain

$$\begin{aligned}a_{n+1}&=\lfloor a_{n+1}+[r_{n+2}]\rfloor\\&=\left\lfloor\frac{(1-2m)2k\theta_{n-1}}{1-D_n}-k\right\rfloor=\left\lfloor\frac{(1-2m)2k\theta_{n-1}(D_n+1)}{1-D_n^2}-k\right\rfloor.\end{aligned}$$

After applying the definition (4.1) of D_n, this expression will simplify to the first equality in (4.2). Using (2.11), the second components in the exterior terms of (4.3) equate to

$$k+a_n+[s_n]=m-y_n=\frac{D_n+1}{2\theta_{n-1}}.$$

But since $[s_n]<1$, we have

$$a_n=\lfloor a_n+[s_n]\rfloor=\left\lfloor\frac{D_n+1}{2\theta_{n-1}}-k\right\rfloor.$$

Adding 1 to all indices will establish the equality of the exterior terms in (4.2), completing the proof. ∎

Next, we will derive a formula to extend the bi-sequence of approximation coefficients from a pair of consecutive terms, which applies to either the future or the past tail. Define the function $g_{(m,k,a)} = g_a : \Gamma \to \mathbb{R}$,

$$g_a(u,v) = u + \frac{D(u,v)}{(1-2m)k}(m+k+a) + \frac{v}{(2m-1)k}(m+k+a)^2. \tag{4.4}$$

Theorem 4.2. *Given the initial seed pair $(x_0, y_0) \in \Omega$, let a_{n+1} be the (m,k)-digit at time $n+1$ and let $(\theta_{n-1}, \theta_n, \theta_{n+1})$ be the (m,k)-approximation coefficients at time $n-1$, n and $n+1$. Then*

$$\theta_{n\pm 1} = g_{a_{n+1}}(\theta_{n\mp 1}, \theta_n).$$

Combining this result with Theorem 4.1 and the definition (4.1) of D_n allows us to explicitly write $\theta_{n\pm1}$ in terms of $(\theta_{n\mp1}, \theta_n)$ as

$$\begin{aligned}\theta_{n\pm1} = \theta_{n\mp1} + \frac{\sqrt{1+(2m-1)4k\theta_{n\mp1}\theta_n}}{(1-2m)k} \\ \times\left(m+k+\left\lfloor\frac{1+\sqrt{1+(2m-1)4k\theta_{n\mp1}\theta_n}}{2\theta_n}-k\right\rfloor\right) \\ + \frac{\theta_n}{(2m-1)k}\left(m+k+\left\lfloor\frac{1+\sqrt{1+(2m-1)4k\theta_{n\mp1}\theta_n}}{2\theta_n}-k\right\rfloor\right)^2.\end{aligned}$$

In order to establish this identity, we will first prove that:

Lemma 4.3. *Let $(u,v) \in \Gamma$ and let $a := \lfloor (D(u,v)+1)/(2v) - k \rfloor$. Then*

$$\Psi\mathscr{T}\Psi^{-1}(u,v) = (v, g_a(u,v)) \quad \text{and} \quad u = g_a(g_a(u,v),v).$$

Proof. Given $(u,v) \in \Gamma$, use the definition (3.7) of Ψ^{-1} and define

$$(x_0, y_0) := \Psi^{-1}(u,v) = \left(m + \frac{1-D(u,v)}{2u}, m - \frac{1+D(u,v)}{2u}\right) \in \Omega. \tag{4.5}$$

After applying the definition (3.2) of Ψ, we have

$$(u,v) = \Psi(x_0,y_0) = \left(\frac{1}{x_0-y_0}, \frac{(m-x_0)(m-y_0)}{(2m-1)k(x_0-y_0)}\right). \tag{4.6}$$

We also define (x_1, y_1) to be the image of (x_0, y_0) under $\mathscr{T}$, which after using its definition (2.10) is written as

$$(x_1, y_1) = \left(\frac{k(1-m-x_0)}{x_0-m} - a_1, \frac{k(1-m-y_0)}{y_0-m} - a_1\right),$$

where $a_1 := \lfloor (k(1-m-x_0))/(x_0-m) \rfloor$; hence

$$x_1 - y_1 = \frac{k(1-m-x_0)}{x_0-m} - \frac{k(1-m-y_0)}{y_0-m} = \frac{(2m-1)k(x_0-y_0)}{(x_0-m)(y_0-m)}. \tag{4.7}$$

Applying (4.5) and the definition (3.6) of $D(u,v)$ allows us to rewrite this pair as

$$(4.8)\qquad \begin{aligned}(x_1,y_1) &= \left(\frac{2(1-2m)ku}{1-D(u,v)}-k-a_1, \frac{2(2m-1)ku}{1+D(u,v)}-k-a_1\right)\\ &= \left(\frac{D(u,v)+1}{2v}-k-a_1, \frac{D(u,v)-1}{2v}-k-a_1\right)\end{aligned}$$

where

$$(4.9)\qquad a_1 = a = \left\lfloor \frac{D(u,v)+1}{2v} - k \right\rfloor$$

is as in the hypothesis. Next, use the definition (3.2) of Ψ and set

$$(v',w) := \Psi(x_1,y_1) = \left(\frac{1}{x_1-y_1}, \frac{(m-x_1)(m-y_1)}{(2m-1)k(x_1-y_1)}\right).$$

Together with (4.6) and (4.7), this implies that $v = v' = (x_1-y_1)^{-1}$. Using this identity with (4.8) and the definition (3.6) of D, we find that the second component of $\Psi(x_1,y_1)$ is

$$\begin{aligned} w &= \frac{(m-x_1)(m-y_1)}{(2m-1)k(x_1-y_1)} = \frac{v}{(2m-1)k}(m-x_1)(m-y_1)\\ &= \frac{v}{(2m-1)k}\left((m+k+a)-\frac{D(u,v)+1}{2v}\right)\left((m+k+a)-\frac{D(u,v)-1}{2v}\right)\\ &= \frac{v}{(2m-1)k}\left((m+k+a)^2-\frac{D(u,v)}{v}(m+k+a)+\frac{(2m-1)ku}{v}\right).\end{aligned}$$

We conclude that

$$(4.10)\qquad w = u + \frac{D(u,v)}{(1-2m)k}(m+k+a) + \frac{v}{(2m-1)k}(m+k+a)^2 = g_a(u,v),$$

and since

$$(v,w) = (v',w) = \Psi(x_1,y_1) = \Psi\mathscr{T}(x_0,y_0) = \Psi\mathscr{T}\Psi^{-1}(u,v)$$

this gives the validity of the first equation in the assertion.

To prove the second part, we use the definition (2.13) of $\mathscr{T}^{-1}$ and write

$$(x_0,y_0) = \mathscr{T}^{-1}(x_1,y_1) = \left(m + \frac{(1-2m)k}{k+a+x_1}, m + \frac{(1-2m)k}{k+a+y_1}\right)$$

where a is as in (4.9). We again use (3.7) to write

$$(x_1,y_1) = \Psi^{-1}(v,w) = \left(m + \frac{1-D(v,w)}{2v}, m - \frac{1+D(v,w)}{2v}\right).$$

Combining these observations, we obtain

$$(4.11)\qquad (x_0, y_0) = \bigg(m + \frac{2(1-2m)kv}{2(m+k+a)v + (1 - D(v,w))},\\ m + \frac{2(1-2m)kv}{2(m+k+a)v - (1 + D(v,w))}\bigg).$$

Using (3.2), we rewrite $(u, v) = \Psi(x_0, y_0)$ as

$$(u, v) = \bigg(\frac{1}{x_0 - y_0}, \frac{(2m-1)(m-x_0)(m-y_0)}{k(x_0 - y_0)}\bigg) = \bigg(\frac{(2m-1)kv}{(m-x_0)(m-y_0)}, v\bigg).$$

Together with (4.11) and the definition (3.6) of D, we obtain

$$\begin{aligned}
u &= (2m-1)kv\bigg(\frac{2(2m-1)kv}{2(m+k+a)v + (1 - D(v,w))}\bigg)^{-1} \\
&\quad \times \bigg(\frac{2(2m-1)kv}{2(m+k+a)v - (1 + D(v,w))}\bigg)^{-1} \\
&= \frac{2m-1}{4kv}\big(4(m+k+a)^2v^2 - 4(m+k+a)vD(v,w) - (1 - D(v,w)^2)\big) \\
&= w + \frac{D(v,w)}{(1-2m)k}(m+k+a) + \frac{(2m-1)v}{k}(m+k+a)^2 \\
&= g_a(w, v).
\end{aligned}$$

But from (4.10) we have $w = g_a(u, v)$, so that this last observation shows the second equation in the assertion, completing the proof. ∎

Proof of Theorem 4.2. We have

$$\begin{aligned}
(\theta_n, \theta_{n+1}) &= \Psi(x_{n+1}, y_{n+1}) = \Psi\mathscr{T}(x_n, y_n) = \Psi\mathscr{T}\Psi^{-1}(\theta_{n-1}, \theta_n), \\
(\theta_{n-1}, \theta_n) &= \Psi(x_n, y_n) = \Psi\mathscr{T}^{-1}(x_{n+1}, y_{n+1}) = \Psi\mathscr{T}^{-1}\Psi^{-1}(\theta_n, \theta_{n+1}) \\
&= (\Psi\mathscr{T}\Psi^{-1})^{-1}(\theta_n, \theta_{n+1}).
\end{aligned}$$

After setting $(u, v) := (\theta_{n\pm1}, \theta_n)$, the result is obtained at once from Lemma 4.3 and Proposition 4.1. ∎

Corollary 4.4. *Under the assumptions of the previous theorem, we have*

$$(4.12)\qquad m + k + a_{n+1} = \frac{D_n + D_{n+1}}{2(1-2m)\theta_n}.$$

Proof. Using the definition (4.4) of g_a and the result of the theorem, we write

$$
\begin{aligned}
\theta_{n-1} &= g_{a_{n+1}}(\theta_{n+1},\theta_n)\\
&= \theta_{n+1} + \frac{D_{n+1}}{k}(m+k+a_{n+1}) + \frac{(2m-1)\theta_n}{k}(m+k+a_{n+1})^2\\
&= g_{a_{n+1}}(\theta_{n-1},\theta_n) + \frac{D_{n+1}}{k}(m+k+a_{n+1})\\
&\quad + \frac{(2m-1)\theta_n}{k}(m+k+a_{n+1})^2\\
&= \theta_{n-1} + \frac{D_n+D_{n+1}}{k}(m+k+a_{n+1}) + \frac{2(2m-1)\theta_n}{k}(m+k+a_{n+1})^2,
\end{aligned}
$$

which yields the desired result after the appropriate cancellations and rearrangements. ∎

5. The constant bi-sequence of approximation coefficients. For all $m \in \{0,1\}$, $k \in [1,\infty)$ and $a \in \mathbb{Z}^+$, define the constants

$$\xi_{(m,k,a)} = \xi_a := [\overline{a}]_{(m,k)} = [a,a,\dots]_{(m,k)}$$

and

$$C_{(m,k,a)} = C_a := \frac{1}{\sqrt{(m+k+a)^2+4(1-2m)k}}, \tag{5.1}$$

where we take $C_{(1,1,0)}$ to be ∞. Given two non-negative integers a and b, it is clear that

$$a \le b \quad \text{if and only if} \quad C_b \le C_a \tag{5.2}$$

and that this inequality remains true if we allow a or b to equal ∞.

THEOREM 5.1. *Let $(x_0,y_0) \in \Omega_{(m,k)}$ and write $a_n := a_n(x_0,y_0)$ and $\theta_n := \theta_n(x_0,y_0)$ for all $n \in \mathbb{Z}$. Let a be a non-negative integer. Then the following are equivalent:*

(i) $a_n = a$ *for all* $n \in \mathbb{Z}$.
(ii) $(x_0,y_0) = (\xi_a, m-a-k-\xi_a)$.
(iii) $\theta_{-1} = \theta_0 = C_{(m,k,a)}$.
(iv) $\theta_n = C_{(m,k,a)}$ *for all* $n \in \mathbb{Z}$.

Proof. (i)⇒(ii) follows directly from (2.3), (2.11) and the definition of ξ_a.

(ii)⇒(iii). When $x = \xi_a = [\overline{a}]$, we have $a_1(x,y) = a_1(x) = a$. Furthermore, T acts as a left shift operator on the digits of expansion, and hence it fixes ξ_a. From the definition (2.1) of T, we have

$$\xi_a = [\overline{a}] = T([\overline{a}]) = T(\xi_a) = \frac{k(1-m-\xi_a)}{\xi_a - m} - a,$$

so that

$$\xi_a^2 - (m-k-a)\xi_a + (mk-k-ma) = 0. \tag{5.3}$$

Using the quadratic formula, we obtain the roots

$$\tfrac{1}{2}\big(m-k-a\pm\sqrt{(a+k-m)^2-4(mk-k-ma)}\,\big)$$
$$=\tfrac{1}{2}\big(m-k-a\pm\sqrt{(m+k+a)^2+4k(1-2m)}\,\big).$$

Since the smaller root is clearly negative, we have

$$\xi_a=\tfrac{1}{2}\big(\sqrt{(m+k+a)^2+4(1-2m)k}+(m-k-a)\big).$$

In tandem with formula (5.1), this provides the relationship

$$C_a=\frac{1}{2\xi_a-(m-k-a)}. \tag{5.4}$$

The starting assumption and the definition (3.2) of Ψ will now yield

$$\begin{aligned}\Psi(x_0,y_0)&=\Psi(\xi_a,m-k-a-\xi_a)\\&=\left(\frac{1}{2\xi_a-(m-k-a)},\frac{(2m-1)(m-\xi_a)(a+k+\xi_a)}{k(2\xi_a-(m-k-a))}\right)\\&=\left(C_a,\frac{2m-1}{k}C_a(ma+mk+(m-k-a)\xi_a-\xi_a^2)\right)=(C_a,C_a),\end{aligned}$$

where the last equality is obtained from (5.3). Combining this last observation with (3.3) yields

$$(\theta_{-1},\theta_0)=\Psi(x_0,y_0)=(C_a,C_a), \tag{5.5}$$

which is the desired result.

(iii)⇒(iv). From the definition (5.1) of C_a, we have

$$1+4(2m-1)kC_a^2=1-\frac{4(1-2m)k}{(m+k+a)^2+4(1-2m)k}=(m+k+a)^2{C_a}^2.$$

We use this observation and the definition (4.4) of g_a to conclude that

$$\begin{aligned}&g_a(C_a,C_a)\\&=C_a+\frac{m+k+a}{(1-2m)k}\sqrt{1+4(2m-1)kC_a^2}+\frac{(2m-1)C_a}{k}(m+k+a)^2=C_a.\end{aligned} \tag{5.6}$$

If $\theta_{-1}=\theta_0=C_a$ then, since Ψ is a bijection, formula (5.5) implies that $(x_0,y_0)=(\xi_a,-a-k-\xi_a)$ and $a_n(x_0,y_0)=a$ for all $n\in\mathbb{Z}$. Theorem 4.2 and (5.6) now prove the equalities

$$\theta_1=g_{a_1}(\theta_{-1},\theta_0)=g_a(C_a,C_a)=C_a,$$
$$\theta_{-2}=g_{a_1}(\theta_0,\theta_{-1})=g_a(C_a,C_a)=C_a.$$

The proof that $\{\theta_n\}_{n=-\infty}^{\infty}=\{C_a\}$ is the indefinite extension of this argument to all $n\in\mathbb{Z}$.

(iv)⇒(i). If $\theta_n = C_a$ for all $n \in \mathbb{Z}$, then $D_n = \sqrt{1 + 4(2m-1)kC_a{}^2}$ for all $n \in \mathbb{Z}$. Corollary 4.4 now yields

$$(m+k+a_{n+1})^2 = \frac{(D_n+D_{n+1})^2}{4\theta_n^2} = \frac{4(1+4(2m-1)kC_a{}^2)}{4C_a^2} = \frac{1}{C_a^2}+4(2m-1)k.$$

But from (5.1), we know that $(m+k+a_{n+1})^2 = 1/C_{a_{n+1}}^2 + 4(2m-1)k$, so that we must have $a_{n+1} = a$ for all $n \in \mathbb{Z}$. ∎

Corollary 5.2. *Let* $(x_0, y_0) \in \Omega$ *and write* $a_n := a_n(x_0, y_0)$ *and* $\theta_n := \theta_n(x_0, y_0)$ *for all* $n \in \mathbb{Z}$. *Then* $\{a_n\}_{n=-\infty}^{\infty}$ *is constant if and only if* $\{\theta_n\}_{n=-\infty}^{\infty}$ *is constant.*

Proof. Necessity follows immediately from the previous theorem. Suppose $\{\theta_n\} = \{\theta\}$ is constant and write $D := \sqrt{1+4(2m-1)k\theta^2}$, so that $D = D_0 = D_1$ is as in (4.1). Then Corollary 4.4 yields

$$(m+k+a_1)^2 = \left(\frac{D_0+D_1}{2\theta_0}\right)^2 = \frac{D^2}{\theta^2} = \frac{1}{\theta^2} + 4(2m-1)k.$$

But from (5.1), we know that $(m+k+a_1)^2 = C_{a_1}^{-2} + 4(2m-1)k$, and since $\theta_n > 0$, we conclude that $\theta = C_{a_1}$. The previous theorem now shows that $a_n(x_0, y_0) = a_1$ for all $n \in \mathbb{Z}$. ∎

6. Essential bounds. Thus far, our treatment of the Gauss-like and Rényi-like cases ran along the same lines. However, these bi-sequences can be no further apart when it comes to their essential bounds.

6.1. The Gauss-like case. In this subsection, we focus on the Gauss-like case $m = 0$. We fix $k \in [1, \infty)$ and omit the subscript $(0, k)$ throughout.

Theorem 6.1. *Suppose that* $(x_0, y_0) \in \Omega$. *For all* $n \in \mathbb{Z}$ *write* $a_{n+1} := a_{n+1}(x_0, y_0)$ *and* $\theta_n := \theta_n(x_0, y_0)$. *Then the inequalities*

$$\min\{\theta_{n-1}, \theta_n, \theta_{n+1}\} \le C_{a_{n+1}} \quad \text{and} \quad \max\{\theta_{n-1}, \theta_n, \theta_{n+1}\} \ge C_{a_{n+1}}$$

are sharp, with equality occurring precisely when $\theta_{n-1} = \theta_n = \theta_{n+1} = C_{a_{n+1}}$.

Proof. Assume, for contradiction, that $\min\{\theta_{n-1}, \theta_n, \theta_{n+1}\} > C_{a_{n+1}}$. Then, using the definition (5.1) of C_a, we obtain

$$\min\{\theta_{n-1}\theta_n, \theta_n\theta_{n+1}\} > C_{a_{n+1}}^2 = \frac{1}{(a_{n+1}+k)^2 + 4k},$$

hence

$$\begin{aligned} D_n D_{n+1} &\le \max\{D_n^2, D_{n+1}^2\} = \max\{1 - 4k\theta_{n-1}\theta_n, 1 - 4k\theta_n\theta_{n+1}\} \\ &< 1 - \frac{4k}{(a_{n+1}+k)^2 + 4k}. \end{aligned}$$

We conclude

$$\text{(6.1)} \qquad \max\{D_nD_{n+1}, D_n^2, D_{n+1}^2\} < 1 - \frac{4k}{(a_{n+1}+k)^2+4k} = \frac{(a_{n+1}+k)^2}{(a_{n+1}+k)^2+4k}.$$

Also, since we are assuming $\theta_n > C_{a_{n+1}}$, we have

$$\frac{1}{4\theta_n^2} < \frac{1}{4C_{a_{n+1}}^2} = \frac{(a_{n+1}+k)^2+4k}{4}.$$

Using this last observation together with Corollary 4.4 and (6.1), we obtain the contradiction

$$\begin{aligned}(a_{n+1}+k)^2 &= \frac{1}{4\theta_n^2}(D_n^2 + D_{n+1}^2 + 2D_nD_{n+1}) \\ &< \frac{(a_{n+1}+k)^2+4k}{4}\,\frac{4(a_{n+1}+k)^2}{(a_{n+1}+k)^2+4k} = (a_{n+1}+k)^2,\end{aligned}$$

which proves the first inequality in the theorem. The proof of the second is the same *mutatis mutandis*. Finally, if $(x,y) = (\xi_a, -(a+k+\xi_a)) \in \Omega$, then we conclude from Theorem 5.1 that $a_{n+1} = a$ and $\theta_n = C_a$ for all $n \in \mathbb{Z}$. Thus these inequalities are sharp and equality is realized precisely when $\{\theta_n\}_{n=-\infty}^{\infty} = \{C_a\}$. ■

From (5.2), we have $C_a \leq C_0$ for all $a \geq 0$ and, as a direct result, we conclude:

Corollary 6.2. *Under the assumptions of the previous theorem, the inequality*

$$\liminf \theta_n(x_0, y_0) \leq C_0 = \frac{1}{\sqrt{k^2+4k}} \leq \frac{1}{\sqrt{5}}$$

is sharp.

A classical theorem due to Legendre states that if p/q is a rational number with $\theta(x_0, p/q) < .5$, then p/q is an RCF convergent for x_0 [7, Theorem 5.12]. Since $k \in \mathbb{Q}$ implies that every $(0,k)$-convergent is also in $\mathbb{Q}$ and since $C_{a_{n+1}} \leq 5^{-.5} < .5$, the following result follows at once from Legendre's theorem and the previous corollary.

Corollary 6.3. *When $k \in [1,\infty) \cap \mathbb{Q}$, at least one in every three consecutive members of the $(0,k)$-convergents for x_0 is, in fact, a classical regular continued fraction convergent!*

6.2. The Rényi-like case. In this subsection, we focus on the Rényi-like case $m = 1$. We fix $k \in [1,\infty)$ and omit the subscript $(1,k)$ throughout. The main result for this section is:

THEOREM 6.4. *Suppose* $(x_0, y_0) \in \Omega$. *For all* $n \in \mathbb{Z}$ *write* $a_n := a_n(x_0, y_0)$ *and* $\theta_n := \theta_n(x_0, y_0)$. *Let* $0 \le l \le L \le \infty$ *be such that*

$$l = \liminf_{n\in\mathbb{Z}}\{a_n\} \le \limsup_{n\in\mathbb{Z}}\{a_n\} = L.$$

Then the inequalities

$$C_L \le \liminf_{n\in\mathbb{Z}}\{\theta_n\} \le \limsup_{n\in\mathbb{Z}}\{\theta_n\} \le C_l$$

are sharp.

In order to prove this theorem, we first prove:

LEMMA 6.5. *Suppose* $(x_0, y_0) \in \Omega$. *For all* $n \in \mathbb{Z}$ *write* $a_n := a_n(x_0, y_0)$ *and* $\theta_n := \theta_n(x_0, y_0)$. *If* $\theta_n = \max\{\theta_{n-1}, \theta_n, \theta_{n+1}\}$ *then* $\theta_n \le C_{a_{n+1}}$ *with equality precisely when* $\{\theta_n\}_{n=-\infty}^{\infty}$ *is constant. Similarly, if* $\theta_n = \min\{\theta_{n-1}, \theta_n, \theta_{n+1}\}$ *then* $\theta_n \ge C_{a_{n+1}}$ *with equality precisely when* $\{\theta_n\}_{n=-\infty}^{\infty}$ *is constant.*

Proof. We will only prove the first claim; the proof for the second one is the same *mutatis mutandis.* If $\theta_n = \max\{\theta_{n-1}, \theta_n, \theta_{n+1}\}$ then

$$D_n = \sqrt{1 + 4k\theta_{n-1}\theta_n} \le \sqrt{1 + 4k\theta_n^2}$$

with equality precisely when $\theta_{n-1} = \theta_n$, and

$$D_{n+1} = \sqrt{1 + 4k\theta_n\theta_{n+1}} \le \sqrt{1 + 4k\theta_n^2}$$

with equality precisely when $\theta_{n+1} = \theta_n$. We conclude that the weak inequality

$$\frac{1}{4\theta_n^2}(D_n^2 + D_{n+1}^2 + 2D_nD_{n+1}) \le \frac{1}{4\theta_n^2}4(1 + 4k\theta_n^2)$$

must hold and that equality is obtained if and only if $\theta_{n-1} = \theta_n = \theta_{n+1}$. In this case, we deduce from Theorem 5.1 that $\{\theta_n\}_{n=-\infty}^{\infty}$ is constant. Otherwise, we may replace the weak inequality with a strict one. If we further assume that $\theta_n \ge C_{a_{n+1}}$ then Corollary 4.4 and the definition (5.1) of C_a with $a = a_{n+1}$, yield the contradiction

$$\begin{aligned}(a_{n+1} + k + 1)^2 &= \frac{1}{4\theta_n^2}(D_{n-1}^2 + D_n^2 + 2D_{n-1}D_n) < \frac{1}{4\theta_n^2}4(1 + 4k\theta_n^2) \\ &= \frac{1}{\theta_n^2} + 4k \le \frac{1}{C_{a_{n+1}}^2} + 4k = (a_{n+1} + k + 1)^2.\end{aligned}$$

We conclude that θ_n must be strictly smaller than $C_{a_{n+1}}$, as desired. ∎

Proof of Theorem 6.4. From our assumption, there exists $N_0 \ge 1$ such that $l \le a_{n+1}(x_0, y_0) \le L$ for all $n \ge N_0$ and for all $n \le 1 - N_0$. After using the inequality (5.2), we conclude that

$$C_L \le C_{a_{n+1}} \le C_l \quad \text{for all } n \ge N_0 \text{ and for all } n \le 1 - N_0. \tag{6.2}$$

We will first prove the theorem when at least one of the sequences $\{\theta_n\}_{n=N_0}^{\infty}$ and $\{\theta_n\}_{n=1-N_0}^{-\infty}$ is eventually monotone. Then we will show that this inequality holds in general, after proving its validity when neither sequence is eventually monotone. Finally, we will prove that the constants C_l and C_L are the best possible by giving specific examples for which they are obtained.

First, suppose $\{\theta_n\}_{n=N_0}^{\infty}$ is eventually monotone in the broader sense. Then there exists $N_1 \geq N_0$ for which $\{\theta_n\}_{n=N_1}^{\infty}$ is monotone. By Proposition 3.1, this sequence is bounded in $[0, C_0]$, so it must converge to some real number $C \in [0, C_0]$. Thus

$$\lim_{n\to\infty} D_n := \lim_{n\to\infty} \sqrt{1 + 4k\theta_{n-1}\theta_n} = \sqrt{1 + 4kC^2}.$$

Using (5.1) and Corollary 4.4, we obtain

$$\frac{1}{C_{a_{n+1}}^2} + 4k = (a_{n+1} + k + 1)^2 = \frac{1}{4\theta_n^2}(D_n^2 + D_{n+1}^2 + 2D_nD_{n+1}),$$

so that

$$\lim_{n\to\infty} \frac{1}{C_{a_{n+1}}^2} + 4k = \frac{1}{C^2}(1 + 4kC^2) = \frac{1}{C^2} + 4k$$

and $\lim_{n\to\infty} C_{a_{n+1}} = C$. When $k = 1$ is the classical Rényi case, both C and $\lim_{n\to\infty} D_n$ might equal infinity, implying that $1/C^2 + 4k = 0$. Since $\{C_{a_n}\}_{n\in\mathbb{Z}}$ is a discrete set, there must exist a non-negative integer a and a positive integer $N_2 \geq N_1$ such that $\theta_n = C_{a_{n+1}} = C_a = C$ for all $n \geq N_2$. But this implies from Theorem 5.1 that $\theta_n = C_a$ for all $n \in \mathbb{Z}$. Since $N_2 \geq N_1 \geq N_0$, we use the inequality (6.2) to conclude that $C_L \leq \theta_n = C_a \leq C_l$ for all $n \in \mathbb{Z}$, which gives the validity of the assertion for this scenario. Proving that the case when $\{\theta_n\}_{n=1-N_0}^{-\infty}$ is eventually monotone reduces to the constant case is the same *mutatis mutandis*.

Now suppose that neither $\{\theta_n\}_{n=N_0}^{\infty}$ nor $\{\theta_n\}_{n=1-N_0}^{-\infty}$ is eventually monotone, in the broader sense. In particular, $\{\theta_n\}_{n=-\infty}^{\infty}$ is not constant, so that an application of Theorem 5.1 yields $\theta_{n-1} \neq \theta_n$ for all $n \in \mathbb{Z}$. Let $N_1 \geq N_0$ be the first time the sequence $\{\theta_n\}_{n=N_0}^{\infty}$ changes direction, that is, we have either $\theta_{N_1} = \min\{\theta_{N_1-1}, \theta_{N_1}, \theta_{N_1+1}\}$, or $\theta_{N_1} = \max\{\theta_{N_1-1}, \theta_{N_1}, \theta_{N_1+1}\}$. We now show that $C_L < \theta_n < C_l$ for all $n \geq N_1$.

Fixing $N \geq N_1$, take N', N'' such that $\theta_{N'}$ and $\theta_{N''}$ are the closest local extrema to θ_N in the sequence $\{\theta_n\}_{n=N_1}^{\infty}$ from the left and right. That is, $N_1 \leq N' < N < N''$ and we have either

$$\theta_{N'} < \theta_{N'+1} < \cdots < \theta_N < \theta_{N+1} < \cdots < \theta_{N''}$$

and $\theta_{N'} < \theta_{N'-1}$, $\theta_{N''} > \theta_{N''+1}$, or

$$\theta_{N'} > \theta_{N'+1} > \cdots > \theta_N > \theta_{N+1} > \cdots > \theta_{N''}$$

and $\theta_{N'} > \theta_{N'-1}$, $\theta_{N''} < \theta_{N''+1}$. In the first case, applying the previous lemma to $\theta_{N'} = \min\{\theta_{N'-1}, \theta_{N'}, \theta_{N'+1}\}$ implies $\theta_{N'} > C_{a_{N'+1}}$, and applying the previous lemma to $\theta_{N''} = \max\{\theta_{N''-1}, \theta_{N''}, \theta_{N''+1}\}$ implies $\theta_{N''} < C_{a_{N''+1}}$. In the second case, applying the previous lemma to $\theta_{N'} = \max\{\theta_{N'-1}, \theta_{N'}, \theta_{N'+1}\}$ implies $\theta_{N'} < C_{a_{N'+1}}$, and applying the previous lemma to $\theta_{N''} = \min\{\theta_{N''-1}, \theta_{N''}, \theta_{N''+1}\}$ implies $\theta_{N''} > C_{a_{N''+1}}$. But $N'' > N' \geq N_1 \geq N_0$, so that $l \leq a_{N'+1}, a_{N''+1} \leq L$. We conclude that

$$C_L \leq C_{a_{N'+1}} < \theta_{N'} < \theta_N < \theta_{N''} < C_{a_{N''+1}} \leq C_l$$

in the first case, and

$$C_L \leq C_{a_{N''+1}} < \theta_{N''} < \theta_N < \theta_{N'} < C_{a_{N'+1}} \leq C_l$$

in the second. In either case $C_L < \theta_N < C_l$, as desired. Similarly, we let $N_2 \geq N_0$ be the first time the sequence $\{\theta_n\}_{n=1-N_0}^{-\infty}$ changes direction. The proof that $C_L < \theta_n < C_l$ for all $n \leq 1 - N_2$ is the same *mutatis mutandis.* After setting $N_3 := \max\{N_1, N_2\}$, we conclude that $C_L < \theta_n < C_l$ for all $|n| > N_3$, which shows the validity of the assertion for this scenario as well.

Finally, we prove that C_L and C_l are the best possible bounds. Clearly, $C_L = 0$ is the best bound when $L = \infty$, and similarly $C_l = 0$ is the best bound when $l = L = \infty$. To prove that C_l is the best possible upper bound when $l < \infty$, fix $l \leq L < \infty$, define $x_0 = [a_1, a_2, \ldots]_{(1,k)}$ by

$$a_n := \begin{cases} L & \text{if } \log_2 n \text{ is a positive integer,} \\ l & \text{otherwise,} \end{cases}$$

and let y_0 be its *reflection*, that is, $y_n := 1 - k - a_1 - [a_2, a_3, \ldots]_{(1,k)}$. Then $l \leq a_n(x_0, y_0) \leq L$ for all $n \in \mathbb{Z}$ and both digits appear infinitely often. If $N \geq 1$ is such that $a_N(x_0, y_0) = L$ then

$$x_{(N+\log_2 N+1)} = [\overbrace{l, l, l, l, l, \ldots, l}^{\log_2 N \text{ times}}, r_{(\log_2 N+1)}],$$

$$y_{(N+\log_2 N+1)} = 1 - k - l - [\overbrace{l, l, l, l, l, \ldots, l}^{\log_2 N - 1 \text{ times}}, s_{(\log_2 N-1)}].$$

Since this occurs for infinitely many N, there is a subsequence $\{n_j\} \subset \mathbb{Z}$ such that $(x_{n_j}, y_{n_j}) \to (\xi_l, 1-k-l-\xi_l)$. Then Theorem 3.1 and formula (5.4) prove $\theta_{(n_j+1)}(x_0, y_0) = (x_{n_j} - y_{n_j})^{-1} \to (2\xi_l + k + l - 1)^{-1} = C_l$ as $j \to \infty$. Therefore, C_l cannot be replaced with a smaller constant. The proof that C_L cannot be replaced with a larger constant is the same *mutatis mutandis.* ■

7. Back to one-sided sequences. We finish by quoting those results which apply to the one-sided sequence of approximation coefficients as well. Fix $m \in \{0, 1\}$, $k \geq 1$ and an initial seed $x_0 \in (0, 1) - \mathbb{Q}_{(m,k)}$. Write $a_n := a_n(x_0)$ and $\theta_n = \theta_n(x_0) = (x_n - Y_n)^{-1}$ for all $n \geq 1$, where x_n and Y_n are the future and past of x_0 at time n as in (2.3) and (2.5), and $\theta_n(x_0)$ is as in (2.4). Using (2.10) and (2.13), we see that the maps $\mathscr{T}$ and $\mathscr{T}^{-1}$ are well defined on (x_n, Y_n), and that for all $n \geq 1$, we have $(x_n, Y_n) = \mathscr{T}^n(x_0, Y_0)$. Using Haas' result (2.6) and the definition (3.2) of the map Ψ, we also see that $\Psi(x_n, Y_n) = (\theta_{n-1}, \theta_n)$. Thus, the proofs of Proposition 4.1 as well as Theorems 4.2, 6.1 and 6.4 remain true, after we restrict $n \geq 1$ and replace y_n with Y_n. Consequently, these results apply to the one-sided sequences $\{a_n\}_{n=1}^{\infty}$ and $\{\theta_n\}_{n=1}^{\infty}$ for all parameters $k \geq 1$ in both the Gauss-like and Rényi-like cases.

The proof of Proposition 4.1 for the classical one-sided Gauss case $m = 0$, $k = 1$ was recently published by the author [4]. The first part of the classical Gauss map, one-sided version of Theorem 6.1 was first proved by Bagemihl and McLaughlin [2], as an improvement on a previous result due to Borel [9, Theorem 5.1.5], where the symmetric second part is due to Tong [20]. As expected, the constant $C_{(0,1,0)}$ is no other than the Hurwitz constant $5^{-.5}$.

Acknowledgements. This work is a development of part of the author's Ph.D. dissertation at the University of Connecticut. The author has benefited tremendously from the patience and rigor of his advisor Andrew Haas and would like to thank him for all his efforts. He would also like to extend his gratitude to Alvaro Lozano-Robledo for his suggestions and corrections.

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Received on 2.7.2012
and in revised form on 17.4.2014 (7114)