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Higher dimensional Dedekind sums in function fields

by

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1. Introduction. Given relatively prime integers $c > 0$ and a, the classical Dedekind sum is defined as

$$s(a,c) = \frac{1}{4c}\sum_{k=1}^{c-1}\cot\left(\frac{\pi k}{c}\right)\cot\left(\frac{\pi k a}{c}\right).$$

It satisfies a famous relation called the *reciprocity law*,

$$s(a,c) + s(c,a) = \frac{a^2+c^2+1-3ac}{12ac} \quad (a>0).$$

See Rademacher–Grosswald [8] for details. A generalization of Dedekind sums to higher dimensions was presented by Zagier [9]. Let p be a positive integer, and $a_1,\dots,a_{n-1}$ be integers relatively prime to p. We assume that n is odd. Zagier defines a higher dimensional Dedekind sum as follows:

$$d(p;a_1,\dots,a_{n-1}) := (-1)^{(n-1)/2}\frac{1}{p}\sum_{k=1}^{p-1}\cot\left(\frac{\pi k a_1}{p}\right)\cdots\cot\left(\frac{\pi k a_{n-1}}{p}\right).$$

For pairwise coprime positive integers $a_1,\dots,a_n$ (n odd), this sum satisfies the reciprocity law

$$\sum_{j=1}^{n} d(a_j;a_1,\dots,a_{j-1},a_{j+1},\dots,a_n) = 1 - \frac{l_n(a_1,\dots,a_n)}{a_1\cdots a_n},$$

where $l_n(a_1,\dots,a_n)$ is a polynomial in $a_1,\dots,a_n$ defined as the coefficient of t^n in the power series expansion of

$$\prod_{j=1}^{n}\frac{a_j t}{\tanh(a_j t)} = \prod_{j=1}^{n}\left(1+\frac{1}{3}a_j^2t^2 - \frac{1}{45}a_j^4t^4 + \frac{2}{945}a_j^6t^6 - \cdots\right).$$

We note that Beck [1] generalized Zagier's Dedekind sum.

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It is known that $\pi \cot \pi z$ can be expressed as follows:

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right). \tag{1.1}$$

In function fields, for a given A-lattice we have periodic functions that have expressions analogous to (1.1). Based on this, Okada [7] introduced Dedekind sums in rational function fields, and established reciprocity laws for them. For each A-lattice, we can define Dedekind sums which generalize those of Okada. See [6] for details. It should be noted that we also have Dedekind sums over finite fields ([5], [6]). These are like Apostol–Dedekind sums given by

$$s_n(a,c) = \sum_{k=1}^{c-1} \frac{k}{c} \overline{B}_n\left(\frac{ka}{c}\right),$$

where $\overline{B}_n(x)$ denotes the nth Bernoulli function.

The goal of our paper is to introduce new kinds of Dedekind sums defined over rational function fields. Our Dedekind sums are very similar to ordinary Dedekind sums and to Zagier's higher dimensional Dedekind sums [9]. As the main theorem, we establish the reciprocity law for our Dedekind sums. The rationality and characterization of Dedekind sums are also discussed.

Notation.

$\sum'$: the sum over non-vanishing elements
$\prod'$: the product over non-vanishing elements
$\mathbb{F}_q$: the finite field with q elements
$A = \mathbb{F}_q[T]$: the ring of polynomials in an indeterminate T
$K = \mathbb{F}_q(T)$: the quotient field of A
$|\,|$: the normalized absolute value on K such that $|T| = q$
K_∞ : the completion of K with respect to $|\,|$
$\overline{K_\infty}$: a fixed algebraic extension of K_∞
C : the completion of $\overline{K_\infty}$

2. A-lattices. In this section, we give an overview of A-lattices and related periodic functions. For details, see Goss [4]. A rank r *A-lattice* Λ in C is a finitely generated A-submodule of rank r in C that is discrete in the topology of C. For such an A-lattice Λ, define the Euler product

$$e_\Lambda(z) = z \prod_{\lambda\in\Lambda}{}' \left(1 - \frac{z}{\lambda}\right).$$

The product converges uniformly on bounded sets in C, and defines a map $e_\Lambda : C \to C$. The map e_Λ has the following properties:

- e_Λ is entire in the rigid analytic sense, and surjective;
- e_Λ is $\mathbb{F}_q$-linear and Λ-periodic;
- e_Λ has simple zeros at the points of Λ, and no other zeros;
- $de_\Lambda(z)/dz = e'_\Lambda(z) = 1$. Hence

$$\frac{1}{e_\Lambda(z)} = \frac{e'_\Lambda(z)}{e_\Lambda(z)} = \sum_{\lambda\in\Lambda} \frac{1}{z-\lambda}.$$

Let ϕ be the Drinfeld module corresponding to Λ. For any $a \in A\setminus\{0\}$, we denote by $\phi[a] := \{x \in C \mid \phi_a(x) = 0\}$ the A/aA-module of *a-division points*. It is known that $\Lambda/a\Lambda$ is isomorphic to $\phi[a]$ by $\lambda + a\Lambda \mapsto e_\Lambda(\lambda/a)$. Put

$$E_k(\phi[a]) := {\sum_{x\in\phi[a]}}' \frac{1}{x^k} = {\sum_{\lambda\in\Lambda/a\Lambda}}' \frac{1}{e_\Lambda(\lambda/a)^k}$$

for each positive integer k, and set $E_0(\phi[a]) = -1$. We adopt the convention that $\sum'_{\lambda\in\Lambda/a\Lambda}$ is zero when $\Lambda/a\Lambda = \{0\}$. Then we have

$$\begin{aligned}(2.1)\qquad \frac{az}{\phi_a(z)} &= \frac{\phi'_a(z)}{\phi_a(z)} z = \sum_{\lambda\in\Lambda/a\Lambda} \frac{z}{z - e_\Lambda(\frac{\lambda}{a})} \\ &= 1 - {\sum_{\lambda\in\Lambda/a\Lambda}}' \frac{\frac{z}{e_\Lambda(\lambda/a)}}{1 - \frac{z}{e_\Lambda(\lambda/a)}} = -\sum_{k=0}^{\infty} E_k(\phi[a]) z^k.\end{aligned}$$

If $a \in \mathbb{F}_q\setminus\{0\}$, then $E_k(\phi[a]) = 0$ for any positive integer k, and $az/\phi_a(z) = 1$.

3. Higher dimensional Dedekind sums. Let Λ be an A-lattice. We introduce Dedekind sums for Λ. Assume $n \geq 2$. Let $a_1, \ldots, a_{n-1} \in A\setminus\{0\}$ be relatively prime to $a_n \in A\setminus\{0\}$. In other words, if $i \neq n$, then $Aa_i + Aa_n = A$.

Definition 3.1. The *higher dimensional Dedekind sum* is defined as

$$s_\Lambda(a_n; a_1, \ldots, a_{n-1}) = (-1)^{n-1} \frac{1}{a_n} {\sum_{\lambda\in\Lambda/a_n\Lambda}}' e_\Lambda\left(\frac{a_1\lambda}{a_n}\right)^{-1} \cdots e_\Lambda\left(\frac{a_{n-1}\lambda}{a_n}\right)^{-1}.$$

Remark 3.2. (i) When $\Lambda/a\Lambda = \{0\}$, $\sum'_{\lambda\in\Lambda/a\Lambda}$ is zero.

(ii) In the cases $(n, q) = (2, 2)$, $(3, 3)$, our Dedekind sum coincides with one of the Dedekind sums introduced in [6]. In particular, if $\Lambda = L$ is the A-lattice corresponding to the Carlitz module, then this Dedekind sum is as defined in Okada [7].

The Dedekind sum $s_\Lambda(a_n; a_1, \ldots, a_{n-1})$ has similar properties to those of Zagier's Dedekind sum:

Proposition 3.3.

(i) $s_\Lambda(a_n; a_1, \ldots, a_{n-1})$ *only depends on* $a_i + a_nA$,

(ii) $s_\Lambda(a_n; a_1, \ldots, a_{n-1})$ *is symmetric in* $a_1, \ldots, a_{n-1}$,

(iii) $s_\Lambda(a_n;\zeta a_1,\dots,a_{n-1})=\zeta^{-1}s_\Lambda(a_n;a_1,\dots,a_{n-1})$ *for any* $\zeta\in\mathbb{F}_q\backslash\{0\}$,
(iv) $s_\Lambda(a_n;ba_1,\dots,ba_{n-1})=s_\Lambda(a_n;a_1,\dots,a_{n-1})$ *for any* $b\in A$ *prime to* a_n.

The proof is trivial, so we omit it.

Remark 3.4. By Proposition 3.3(ii)–(iv), we have

$$(-1)^{n-1}s_\Lambda(a_n;a_1,\dots,a_{n-1})=s_\Lambda(a_n;a_1,\dots,a_{n-1}).$$

Hence, if $\operatorname{Char}\mathbb{F}_q\neq 2$ and $2\mid n$, then the sum is equal to zero. Therefore in the case $\operatorname{Char}\mathbb{F}_q\neq 2$, we may suppose in advance that n is odd.

We now state the reciprocity law for our higher dimensional Dedekind sums.

Theorem 3.5 (Reciprocity law). *Choose* $a_1,\dots,a_n\in A\backslash\{0\}$. *If* $a_1,\dots,a_n$ *are coprime, then*

$$\sum_{i=1}^{n} s_\Lambda(a_i;a_1,\dots,a_{i-1},a_{i+1},\dots,a_n) = \frac{1}{a_1\cdots a_n}\sum_{\substack{i_1+\cdots+i_n=n-1\\ i_1\geq 0,\dots,i_n\geq 0}} E_{i_1}(\phi[a_1])\cdots E_{i_n}(\phi[a_n]). \tag{3.1}$$

Remark 3.6. We note that for $a_1=\cdots=a_{n-1}=1$, $a_n\in A\setminus\{0\}$, we have

$$s_\Lambda(a_n;\overbrace{1,\dots,1}^{n-1})=\frac{(-1)^{n-1}}{a_n}E_{n-1}(\phi[a_n]).$$

Let a,c be coprime elements of $A\setminus\{0\}$, and let Λ denote an A-lattice in C. The *inhomogeneous Dedekind sum* $s_\Lambda(a,c)$ is defined as

$$s_\Lambda(a,c)=s_\Lambda(c;a,1)=\frac{1}{c}\sum_{\lambda\in\Lambda/c\Lambda}{}' e_\Lambda\left(\frac{a\lambda}{c}\right)^{-1}e_\Lambda\left(\frac{\lambda}{c}\right)^{-1}.$$

The Dedekind sum $s_\Lambda(a,c)$ has the following reciprocity law:

Theorem 3.7 (Reciprocity law). *If* a,c *are coprime, then*

$$s_\Lambda(a,c)+s_\Lambda(c,a)=\frac{E_2(\phi[a])+E_2(\phi[c])-E_1(\phi[a])E_1(\phi[c])}{ac}. \tag{3.2}$$

4. Example. We compute Dedekind sums for special cases. To do this, let us prepare some results.

4.1. Power sums of a-division points. We recall the Newton formula for the power sums of the zeros of a given polynomial.

PROPOSITION 4.1 (The Newton formula, cf. [2], [3]). *Let*

$$f(X) = X^n + c_1 X^{n-1} + \cdots + c_{n-1} X + c_n$$

be a polynomial over a field L, and $\alpha_1, \ldots, \alpha_n$ be the roots of $f(X)$. For each non-negative integer k, put

$$T_k = \alpha_1^k + \cdots + \alpha_n^k.$$

Then

$$\begin{aligned} &T_k + c_1 T_{k-1} + \cdots + c_{k-1} T_1 + k c_k = 0 && (k \le n), \\ &T_k + c_1 T_{k-1} + \cdots + c_{n-1} T_{k-n+1} + c_n T_{k-n} = 0 && (k \ge n). \end{aligned}$$

PROPOSITION 4.2. *Let ϕ be a Drinfeld module, and a be a fixed element in $A \setminus \{0\}$. If $\phi_a(z)$ is written as*

$$\phi_a(z) = \sum_{i=0}^{m} l_i(a) z^{q^i},$$

then

$$E_k(\phi[a]) = \begin{cases} l_1(a)/a & (k = q-1), \\ 0 & (k = 1, \ldots, q-2 \text{ if } q > 2). \end{cases}$$

Proof. The set $\{1/x \mid x \in \phi[a] \setminus \{0\}\}$ consists of the roots of

$$a^{-1} \phi_a(z^{-1}) z^{q^m} = \sum_{i=0}^{m} a^{-1} l_i(a) z^{q^m - q^i}.$$

Applying the Newton formula to this polynomial, we have

$$E_{q-1}(\phi[a]) + (q-1)\frac{l_1(a)}{a} = 0, \quad E_k(\phi[a]) = 0 \quad (k = 1, \ldots, q-2). \ \blacksquare$$

4.2. Higher dimensional Dedekind sums. Let Λ be an A-lattice, and ϕ be the corresponding Drinfeld module.

We give explicit formulas for certain higher dimensional Dedekind sums.

PROPOSITION 4.3. *If $a, b \in A \setminus \{0\}$ are coprime, then*

$$\begin{aligned} s_\Lambda(b; \overbrace{a, \ldots, a}^{n-1}) &= \frac{(-1)^{n-1}}{b} E_{n-1}(\phi[b]) \\ &= \begin{cases} (-1)^{n-1} l_1(b)/b^2 & (n = q), \\ 0 & (n = 1, \ldots, q-1). \end{cases} \end{aligned}$$

Proof. We have

$$
\begin{aligned}
s_\Lambda(b;\overbrace{a,\dots,a}^{n-1}) &= s_\Lambda(b;\overbrace{1,\dots,1}^{n-1}) \quad \text{(by Proposition 3.3(iv))}\\
&= \frac{(-1)^{n-1}}{b}E_{n-1}(\phi[b]) \quad \text{(by definition of } E_{n-1}(\phi[b]))\\
&= \begin{cases} (-1)^{n-1}l_1(b)/b^2 & (n=q)\\ 0 & (n=1,\dots,q-1)\end{cases} \quad \text{(by Proposition 4.2).} \ \blacksquare
\end{aligned}
$$

As a corollary to Theorem 3.5, we have

PROPOSITION 4.4. *If* $a_1,\dots,a_q \in A\setminus\{0\}$ *are coprime, then*

$$\sum_{i=1}^{q} s_\Lambda(a_i;a_1,\dots,a_{i-1},a_{i+1},\dots,a_q) = \frac{(-1)^{q-1}}{a_1\cdots a_q}\left(\frac{l_1(a_1)}{a_1}+\dots+\frac{l_1(a_q)}{a_q}\right).$$

Proof. By Theorem 3.5 and Proposition 4.2, the left-hand side of the identity is written as

$$\frac{(-1)^{q-1}}{a_1\cdots a_q}(E_{q-1}(\phi[a_1])+\dots+E_{q-1}(\phi[a_q])),$$

which yields the right-hand side by Proposition 4.2. $\blacksquare$

We supply a few examples of the reciprocity law for higher dimensional Dedekind sums.

- $q=2$:

$$s_\Lambda(a_1;a_2)+s_\Lambda(a_2;a_1) = \frac{1}{a_1a_2}\left(\frac{l_1(a_1)}{a_1}+\frac{l_1(a_2)}{a_2}\right).$$

- $q=3$:

$$
\begin{aligned}
&s_\Lambda(a_1;a_2)+s_\Lambda(a_2;a_1)=0,\\
&s_\Lambda(a_3;a_1,a_2)+s_\Lambda(a_2;a_1,a_3)+s_\Lambda(a_1;a_2,a_3)\\
&\qquad = \frac{1}{a_1a_2a_3}\left(\frac{l_1(a_1)}{a_1}+\frac{l_1(a_2)}{a_2}+\frac{l_1(a_3)}{a_3}\right).
\end{aligned}
\tag{4.1}
$$

- $3\le q$, $2\le n<q$:

$$\sum_{i=1}^{n} s_\Lambda(a_i;a_1,\dots,a_{i-1},a_{i+1},\dots,a_n)=0.$$

Let L be the A-lattice corresponding to the Carlitz module ρ defined by $\rho_T(z)=Tz+z^q$. As is mentioned in Goss [4],

$$l_1(a)=(a^q-a)/(T^q-T). \tag{4.2}$$

This yields the following examples, given by Okada in [7].

- $q = 2$:

$$s_L(a_1; a_2) + s_L(a_2; a_1) = \frac{a_1 + a_2}{a_1 a_2 (T^2 - T)}.$$

- $q = 3$:

$$s_L(a_1; a_2) + s_L(a_2; a_1) = 0,$$

$$s_L(a_3; a_1, a_2) + s_L(a_2; a_1, a_3) + s_L(a_1; a_2, a_3) = \frac{a_1^2 + a_2^2 + a_3^2}{a_1 a_2 a_3 (T^3 - T)}. \tag{4.3}$$

- $3 \le q$, $2 \le n < q$:

$$\sum_{i=1}^{n} s_L(a_i; a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) = 0.$$

4.3. Inhomogeneous Dedekind sums. Let a, c be coprime elements of $A \setminus \{0\}$, and Λ be an A-lattice.

In the case $q = 3$, by (4.1), we have

$$s_\Lambda(a, c) + s_\Lambda(c, a) = \frac{1}{ac}\left(\frac{l_1(a)}{a} + \frac{l_1(c)}{c}\right).$$

Moreover assuming that Λ is the A-lattice L associated with the Carlitz module, by (4.3), we obtain

$$s_L(a, c) + s_L(c, a) = \frac{a^2 + c^2 + 1}{ac(T^3 - T)}.$$

5. Proofs of the theorems

Proof of Theorem 3.5. Let ϕ be the Drinfeld module corresponding to Λ. Let us consider the rational function

$$F(z) = \frac{1}{\phi_{a_1}(z) \cdots \phi_{a_n}(z)}.$$

By assumption on $a_1, \ldots, a_n$, we have $\phi[a_i] \cap \phi[a_j] = \{0\}$ if $i \ne j$. This implies that $\bigcup_{i=1}^n \phi[a_i] = \{0\}$ or $F(z)$ has a simple pole at any non-zero element of $\bigcup_{i=1}^n \phi[a_i]$. When a_i is not a unit, for any non-zero element $c \in \phi[a_i]$, there exists a unique element $\lambda + a_i\Lambda \in \Lambda/a_i\Lambda$ such that $x = e_\Lambda(\lambda/a_i)$. Then

$$\mathrm{Res}_x(F(z)dz) = \mathrm{Res}_x\left(\frac{dz}{\phi_{a_i}(z)}\right)\prod_{j\ne i}\frac{1}{\phi_{a_j}(x)} = \frac{1}{a_i}\prod_{j\ne i} e_\Lambda\left(\frac{a_j\lambda}{a_i}\right)^{-1}.$$

This contributes $s_\Lambda(a_i; a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$. When a_i is a unit, $\phi[a_i] = 0$. Hence $s_\Lambda(a_i; a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$ does not appear on the right-hand side of (3.1). In other words, it is zero. To compute the left-hand side of Theorem 3.5, we need the following lemma.

LEMMA 5.1. *Let $G(z)$ be a polynomial over a field L of degree > 1, and R be the set of all roots of $G(z)$. Then*

$$\sum_{a\in R} \mathrm{Res}_a\left(\frac{1}{G(z)}dz\right) = 0.$$

Proof. The partial fraction decomposition of $1/G(z)$ can be expressed as

$$\sum_{a\in R}\sum_{n=1}^{\mathrm{ord}(a)} \frac{C_{a,n}}{(z-a)^n},$$

where $\mathrm{ord}(a)$ is the order of a, and $C_{a,n}$ the coefficient of $(z-a)^{-n}$. Then for any $a \in R$, we have $\mathrm{Res}_a(1/G(z)) = C_{a,1}$. It is easy to see that $1/G(z)$ can be rewritten as

$$\frac{1}{G(z)} = \frac{(\sum_{a\in R} C_{a,1})z^{m-1}}{G(z)} + \frac{\text{a polynomial in } z \text{ with degree less than } m-1}{G(z)},$$

where m is the degree of $G(z)$. Hence,

$$1 = \Big(\sum_{a\in R} C_{a,1}\Big) z^{m-1} + \text{a polynomial in } z \text{ with degree less than } m-1.$$

However since $m-1>0$, we easily obtain $\sum_{a\in R} C_{a,1} = 0$. ∎

The set of all poles of $F(z)$ is $\bigcup_{i=1}^n \phi[a_i]$. By the above lemma, we have

$$(-1)^{n-1}\sum_{i=1}^n s_\Lambda(a_i; a_1,\dots,a_{i-1},a_{i+1},\dots,a_n) + \mathrm{Res}_0(F(z)dz) = \sum_{i=1}^n {\sum_{\lambda\in\Lambda/a_i\Lambda}}' \mathrm{Res}_{e_\Lambda(\lambda/a_i)}(F(z)dz) + \mathrm{Res}_0(F(z)dz) = 0.$$

By (2.1), it follows that

$$\mathrm{Res}_0(F(z)dz) = \frac{(-1)^n}{a_1\cdots a_n} \sum_{i_1+\dots+i_n=n-1} E_{i_1}(\phi[a_1])\cdots E_{i_n}(\phi[a_n]).$$

This completes the proof.

Proof of Theorem 3.7. By the reciprocity law (3.1),

$$s_\Lambda(c;a,1) + s_\Lambda(1;a,c) + s_\Lambda(a;c,1) = \frac{1}{ac}\sum_{i+j+k=2} E_i(\phi[a])E_j(\phi[c])E_k(\phi[1]). \tag{5.1}$$

Since $s_\Lambda(1;a,c) = 0$, $E_0(\phi[a]) = E_0(\phi[c]) = E_0(\phi[1]) = -1$ and $E_1(\phi[1]) = E_2(\phi[1]) = 0$, (5.1) yields the reciprocity law (3.2).

6. Rationality. In this section we suppose that the Drinfeld module ϕ associated with Λ is defined over K.

PROPOSITION 6.1. *The higher dimensional Dedekind sum $s_\Lambda(a_n; a_1, \ldots, a_{n-1})$ is rational, that is, $s_\Lambda(a_n; a_1, \ldots, a_{n-1}) \in K$. In particular, the inhomogeneous Dedekind sum $s_\Lambda(a, c)$ is rational.*

Proof. We know that each $e_\Lambda(\lambda/a_n)$ is a root of $\phi_{a_n}(z)$ defined over K, and $e_\Lambda(a_i\lambda/a_n) = \phi_{a_i}(e_\Lambda(\lambda/a_n))$ for each i. Hence $s_\Lambda(a_n; a_1, \ldots, a_{n-1})$ can be rewritten as

$$s_\Lambda(a_n; a_1, \ldots, a_{n-1}) = (-1)^{n-1}\frac{1}{a_n}\sideset{}{'}\sum_{x\in\phi[a_n]} \frac{1}{\phi_{a_1}(x)\cdots\phi_{a_{n-1}}(x)}. \tag{6.1}$$

It is invariant under the action of all elements of $\mathrm{Gal}(K(\phi[a_n])/K)$. The proposition follows from it. ∎

REMARK 6.2. If $\phi_T(z)$ is given by

$$\phi_T(z) = Tz + l_1(T)z^q + \cdots + l_r(T)z^{q^r},$$

then $\phi_{a_1}(z), \ldots, \phi_{a_n}(z) \in K(l_1(T), \ldots, l_r(T))[z]$. By (6.1), it is easy to verify

$$s_\Lambda(a_n; a_1, \ldots, a_{n-1}) \in K(l_1(T), \ldots, l_r(T)).$$

However, $s_\Lambda(a_n; a_1, \ldots, a_{n-1})$ is not always rational. For instance, when $l_1(T) \notin K$, by Proposition 4.3 we have

$$s_\Lambda(T; \overbrace{1, \ldots, 1}^{q-1}) = \frac{(-1)^{q-1}l_1(T)}{T^2} \notin K.$$

7. Characterization of lower dimensional Dedekind sums. As mentioned in Proposition 3.3 and Theorem 3.5, the higher dimensional Dedekind sum $s_\Lambda(a_n; a_1, \ldots, a_{n-1})$ has the following properties:

(1) $s_\Lambda(a_n; a_1, \ldots, a_{n-1})$ only depends on $a_i + a_nA$,
(2) $s_\Lambda(a_n; a_1, \ldots, a_{n-1})$ is symmetric in $a_1, \ldots, a_{n-1}$,
(3) $s_\Lambda(a_n; \zeta a_1, \ldots, a_{n-1}) = \zeta^{-1}s_\Lambda(a_n; a_1, \ldots, a_{n-1})$ for any $\zeta \in \mathbb{F}_q \setminus \{0\}$,
(4) $s_\Lambda(a_n; ba_1, \ldots, ba_{n-1}) = s_\Lambda(a_n; a_1, \ldots, a_{n-1})$ for any $b \in A$ prime to a_n,
(5) the reciprocity law.

These properties characterize one- and two-dimensional Dedekind sums:

PROPOSITION 7.1.

(i) *The one-dimensional Dedekind sum $s_\Lambda(b; a)$ is determined by the conditions* (1)–(5).
(ii) *The two-dimensional Dedekind sum $s_\Lambda(c; a, b)$ is determined by the conditions* (1)–(5).

Proof. (ii) By (4), we have the form $s_\Lambda(c; a, 1)$ for a certain $b' \in A$ with $b'b \equiv 1 \pmod{a_n}$. One can suppose $\deg a < \deg c$ by (1). The reciprocity law (5) justifies interchanging the roles of a and c to get $s_\Lambda(a; c, 1)$. Using the Euclidean algorithm, finally, we have the form $s_\Lambda(1; a, 1) = 0$.

(i) The proof is similar to case (ii). ∎

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