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Asymptotic nature of higher Mahler measure

by

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1. Introduction

DEFINITION 1.1. Given a nonzero Laurent polynomial $P(x) \in \mathbb{C}[x^{\pm 1}]$ and $k \in \mathbb{N}$, the *k-higher Mahler measure* of P (see [4]) is defined by

$$m_k(P) := \int_0^1 \log^k |P(e^{2\pi i\theta})| d\theta = \frac{1}{2\pi i} \int_{|z|=1} \log^k |P(z)| \frac{dz}{z}.$$

These m_k's are multiples of the coefficients in the Taylor expansion of Akatsuka's zeta Mahler measure (see [2])

$$Z(s,P) := \int_0^1 |P(e^{2\pi i})|^s d\theta, \quad \text{that is,} \quad Z(s,P) = \sum_{k=0}^{\infty} \frac{m_k(P)}{k!} s^k.$$

For $k = 0, 1, 2, \ldots$, let $a_k(P) = m_k(P)/k!$, so that

$$Z(s,P) = \sum_{k=0}^{\infty} a_k(P) s^k.$$

In this paper we only consider polynomials of type $P(x) = x - r$ with $|r| = 1$. Therefore, from now on, we write $m_k(x-r) = m_k$ and $a_k(x-r) = a_k$ for simplicity.

2. Asymptotic nature of higher Mahler measure of $r - x$ when $|r| = 1$. We will prove

THEOREM 2.1. *Let m_k and a_k be as above. Then*

(a) $\dfrac{m_{k+1}}{(k+1)!} + \dfrac{m_k}{k!} = a_{k+1} + a_k = \mathcal{O}(1/k)$,

(b) $\displaystyle\lim_{k\to\infty} \left|\frac{m_k}{k!}\right| = \lim_{k\to\infty} |a_k| = \frac{1}{\pi}$,

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(c) $\dfrac{m_{k+1}}{(k+1)!} + \dfrac{m_k}{k!} = a_{k+1} + a_k = o(1/k)$,

(d) $\lim\limits_{k\to\infty} \dfrac{1}{k+1} \cdot \dfrac{m_{k+1}}{m_k} = \lim\limits_{k\to\infty} \dfrac{a_{k+1}}{a_k} = -1$.

From [4] we know that for $|s| < 1$,

$$Z(s, r-x) = \exp\Bigg(\sum_{k=2}^{\infty} \frac{(-1)^k(1-2^{1-k})\zeta(k)}{k} s^k\Bigg). \tag{2.1}$$

Differentiating both sides of (2.1) with respect to s we obtain

$$\begin{aligned}\sum_{k=1}^{\infty} k a_k s^{k-1} &= \frac{\partial}{\partial s} Z(s, r-x) \\ &= Z(s, r-x) \sum_{k=2}^{\infty} (-1)^k (1-2^{1-k})\zeta(k) s^{k-1} \\ &= \Big(\sum_{k=0}^{\infty} a_k s^k\Big)\Big(\sum_{k=1}^{\infty} b_k s^k\Big) = \sum_{k=1}^{\infty}\Big(a_0 b_k + \sum_{j=1}^{k-1} a_j b_{k-j}\Big) s^k,\end{aligned}$$

where $b_{k-1} := (-1)^k(1-2^{1-k})\zeta(k)$. From the power series expansion of (2.1) we already know that $a_0 = 1$. Now comparing coefficients on both sides of the last expression we get $a_1 = 0$, $a_2 = \frac{1}{2} a_0 b_1 = \frac{1}{4}\zeta(2)$ and for $k \ge 3$,

$$a_k = \frac{1}{k} \sum_{j=0}^{k-2} a_j b_{k-1-j}, \tag{2.2}$$

where

$$b_k := (-1)^{k+1}(1-2^{-k})\zeta(k+1). \tag{2.3}$$

3. A few remarks and lemmas

Remark 3.1. It can be easily shown by induction that $a_{2k} > 0$ and $a_{2k+1} < 0$ for all $k \ge 1$. It is also easy to see that

$$a_k = \frac{(-1)^k}{k} \sum_{j=0}^{k-2} |a_j b_{k-1-j}| \quad \text{for } k > 1.$$

Remark 3.2. Let $B_k := |b_k|$. Then $B_k \le 1$ for all $k \ge 1$, B_k is increasing and $B_k \to 1$ as $k \to \infty$.

Notice $B_k = \eta(k+1)$ where $\eta(k)$ is Dirichlet's eta function. Since $\eta(k) \to 1$ as $k \to \infty$ and $\eta(k)$ is an increasing function of k by [1], $B(k) \le 1$ for all $k \ge 1$, B_k is increasing and $B_k \to 1$ as $k \to \infty$.

Lemma 3.3. *$|a_k| \le 1$ for all $k \ge 1$.*

Proof. We use induction. First we see that $|a_0| = 1 \le 1$, $|a_1| = 0 \le 1$, and $|a_2| = \zeta(2)/4 = \pi^2/24 \le 1$. Now assume $|a_j| \le 1$ for all $2 < j < k$. Using this along with Remark 3.2 we get

$$|a_k| = \frac{1}{k}\Big|\sum_{j=0}^{k-2} a_j b_{k-1-j}\Big| \le \frac{1}{k}\sum_{j=0}^{k-2} |a_j b_{k-1-j}| \le \frac{1}{k}\sum_{j=0}^{k-2} 1 = \frac{k-1}{k} < 1. \ \blacksquare$$

Lemma 3.4. *For $k \ge 4$, $\zeta(k) - \zeta(k+1) \le 1/k^2$.*

Proof. We use induction. First notice that for all $k \ge 4$ and $n \ge 2$ we have $0 < \sqrt{n}/(\sqrt{n}-1) < 4 \le k$, from which it follows that $n(1-1/k)^2 \ge 1$. For $k = 4$ we see that $\zeta(4) - \zeta(5) \approx 0.045 < 0.0625 = 1/4^2$. Assume the conclusion of the lemma is true for all $4 < j < k$, in particular for $j = k-1$. Since for all $k \ge 4$ and $n \ge 2$ we have $n(1-1/k)^2 \ge 1$, it follows that

$$\begin{aligned}
\frac{1}{k^2} &= \left(\frac{k-1}{k}\right)^2 \cdot \frac{1}{(k-1)^2} \ge \left(1 - \frac{1}{k}\right)^2 (\zeta(k-1) - \zeta(k)) \\
&= \sum_{n=2}^{\infty} n\left(1 - \frac{1}{k}\right)^2 \left(\frac{1}{n^k} - \frac{1}{n^{k+1}}\right) \\
&\ge \sum_{n=2}^{\infty} \left(\frac{1}{n^k} - \frac{1}{n^{k+1}}\right) = \zeta(k) - \zeta(k+1). \ \blacksquare
\end{aligned}$$

Lemma 3.5. *Recall $B_k = |b_k|$. For $k > 1$,*

$$B_k - B_{k-1} \le 1/k^2.$$

Proof. Indeed,

$$\begin{aligned}
\frac{1}{k^2} - (B_k - B_{k-1}) &= \frac{1}{k^2} - B_k + B_{k-1} \\
&= \frac{1}{k^2} - \left(1 - \frac{1}{2^k}\right)\zeta(k+1) + \left(1 - \frac{1}{2^{k-1}}\right)\zeta(k) \\
&= \frac{1}{k^2} - \left(1 - \frac{1}{2^{k+1}} + \frac{1}{3^{k+1}} - \frac{1}{4^{k+1}} + \cdots\right) + \left(1 - \frac{1}{2^k} + \frac{1}{3^k} - \frac{1}{4^k} + \cdots\right) \\
&= \frac{1}{k^2} - \frac{1}{2^k}\left(1 - \frac{1}{2}\right) + \frac{1}{3^k}\left(1 - \frac{1}{3}\right) - \frac{1}{4^k}\left(1 - \frac{1}{4}\right) + \cdots \\
&> \frac{1}{k^2} - \frac{1}{2^k}\left(1 - \frac{1}{2}\right) > 0 \quad \text{for all } k > 1. \ \blacksquare
\end{aligned}$$

4. Proofs of the results of Section 2

Proof of Theorem 2.1(a). Using (2.3) and Lemma 3.4, notice that for $k - j \ge 4$,

$$
\begin{aligned}
&\left|\frac{b_{k-j}}{k+1}+\frac{b_{k-1-j}}{k}\right| \\
&=\left|\frac{(-1)^{k-j+1}(1-2^{-k+j})\zeta(k-j+1)}{k+1}+\frac{(-1)^{k-j}(1-2^{-k+1+j})\zeta(k-j)}{k}\right| \\
&=\left|\frac{(1-2^{-k+1+j})\zeta(k-j)}{k}-\frac{(1-2^{-k+j})\zeta(k-j+1)}{k+1}\right| \\
&=\frac{1}{k(k+1)}\left|(k+1)\left(1-\frac{1}{2^{k-1-j}}\right)\zeta(k-j)-k\left(1-\frac{1}{2^{k-j}}\right)\zeta(k-j+1)\right| \\
&=\frac{1}{k(k+1)}\left|k(\zeta(k-j)-\zeta(k-j+1))-\frac{k}{2^{k-j}}(2\zeta(k-j)-\zeta(k-j+1))\right. \\
&\qquad\qquad\left.+\left(1-\frac{1}{2^{k-1-j}}\right)\zeta(k-j)\right| \\
&\le\frac{1}{k(k+1)}\left[k(\zeta(k-j)-\zeta(k-j+1))\right. \\
&\quad\left.+\frac{k}{2^{k-j}}\{(\zeta(k-j)-\zeta(k-j+1))+\zeta(k-j)\}+\left(1-\frac{1}{2^{k-1-j}}\right)\zeta(k-j)\right] \\
&\le\frac{1}{k(k+1)}\left[\frac{k}{(k-j)^2}+\frac{k}{2^{k-j}}\left\{\frac{1}{(k-j)^2}+\zeta(2)\right\}+\zeta(2)\right] \\
&=\frac{1}{(k+1)(k-j)^2}+\frac{1}{2^{k-j}(k+1)(k-j)^2}+\frac{\zeta(2)}{2^{k-j}(k+1)}+\frac{\zeta(2)}{k(k+1)} \\
&\le\frac{1}{(k+1)(k-j)^2}+\frac{1}{(k+1)(k-j)^2}+\frac{\zeta(2)}{2^{k-j}(k+1)}+\frac{\zeta(2)}{k(k+1)} \\
&=\frac{2}{(k+1)(k-j)^2}+\frac{\zeta(2)}{2^{k-j}(k+1)}+\frac{\zeta(2)}{k(k+1)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&|a_{k+1}+a_k| \\
&\quad=\left|\frac{1}{k+1}\sum_{j=0}^{k-1}a_jb_{k-j}+\frac{1}{k}\sum_{j=0}^{k-2}a_jb_{k-1-j}\right| \\
&\quad=\left|\frac{a_{k-1}b_1}{k+1}+\sum_{j=0}^{k-2}a_j\left(\frac{b_{k-j}}{k+1}+\frac{b_{k-1-j}}{k}\right)\right| \\
&\quad\le\frac{1}{k+1}+\sum_{j=0}^{k-2}\left|\frac{b_{k-j}}{k+1}+\frac{b_{k-1-j}}{k}\right| \qquad \text{by Remark (3.2) and Lemma (3.3)} \\
&\quad\le\frac{1}{k+1}+\sum_{j=0}^{k-4}\left|\frac{b_{k-j}}{k+1}+\frac{b_{k-1-j}}{k}\right|+2\cdot\frac{\max\{|b_3|,|b_2|\}}{k}+2\cdot\frac{\max\{|b_2|,|b_1|\}}{k}
\end{aligned}
$$

$$\le \frac{1}{k+1} + \sum_{j=0}^{k-4}\left[\frac{2}{(k+1)}\cdot\frac{1}{(k-j)^2} + \frac{\zeta(2)}{(k+1)}\cdot\frac{1}{2^{k-j}} + \frac{\zeta(2)}{k(k+1)}\right] + \frac{4}{k}$$

$$\le \frac{5}{k} + \frac{2}{k+1}\sum_{j=0}^{k-4}\frac{1}{(k-j)^2} + \frac{\zeta(2)}{k+1}\sum_{j=0}^{k-4}\frac{1}{2^{k-j}} + \frac{\zeta(2)}{k(k+1)}\sum_{j=0}^{k-4}1$$

$$= \frac{5}{k} + \frac{2}{k+1}\left(\frac{1}{4^2}+\frac{1}{5^2}+\cdots+\frac{1}{k^2}\right) + \frac{\zeta(2)}{k+1}\left(\frac{1}{2^4}+\frac{1}{2^5}+\cdots+\frac{1}{2^k}\right)$$
$$+ \frac{\zeta(2)(k-3)}{k(k+1)}$$

$$\le \frac{5}{k} + \frac{2}{k+1}\cdot\zeta(2) + \frac{\zeta(2)}{k+1}\cdot\frac{1}{1-1/2} + \frac{\zeta(2)}{k+1}$$

$$= \frac{5}{k} + \frac{5\zeta(2)}{k+1} \le \frac{5}{k}(1+\zeta(2)).$$

Therefore for $k \ge 4$,

$$|a_{k+1}+a_k| \le \frac{5}{k}(1+\zeta(2)),$$

and so $a_{k+1}+a_k = \mathcal{O}(1/k)$. ∎

Proof of Theorem 2.1(b). By definition of the Akatsuka zeta Mahler measure (see [2]), the generating function $f(s)$ of a_k's is precisely $Z(s, x-r)$ with $|r|=1$. From [4] we know that for $|r|=1$ and $|s|<1$,

$$f(s) := \sum_{k=0}^{\infty} a_k s^k = Z(s, x-r) = \frac{\Gamma(s+1)}{\Gamma^2(s/2+1)} = \frac{4}{s}\,\frac{\Gamma(s)}{\Gamma^2(s/2)}.$$

Define

$$F(s) := 1 + \sum_{k=1}^{\infty}(-1)^k(a_{k-1}+a_k)s^k.$$

So, $F(s) = (1-s)f(-s)$. Notice that

$$\lim_{s\to 1^-} F(s) = \frac{-4}{\Gamma^2(-1/2)}\lim_{s\to 1^-}(1-s)\Gamma(-s) = \frac{-4}{\Gamma^2(-1/2)}\lim_{s\to -1}(1+s)\Gamma(s) = \frac{1}{\pi},$$

since $\lim_{s\to -1}(1+s)\Gamma(s) = -1$ and $\sqrt{\pi} = \Gamma(1/2) = (-1/2)\Gamma(-1/2)$.

Now $\{k(-1)^k(a_k+a_{k+1})\}$ is a bounded sequence by Theorem 2.1(a). Therefore applying Littlewood's extension of Tauber's Theorem (see [3]) to the sequence $\{(-1)^k(a_k+a_{k+1})\}$ and its generating function $F(s)-1$ we see that

$$\lim_{k\to\infty}|a_k| = 1 - \sum_{k=0}^{\infty}\{(-1)^k(a_k+a_{k+1})\} = 1 + \lim_{s\to 1^-}(F(s)-1) = \frac{1}{\pi}.\ ∎$$

Proof of Theorem 2.1(c). Recall $B_k = |b_k|$ from Lemma 3.5. Now define a new sequence $\{A_k\}$ by setting $A_0 = 1$, $A_1 = 0$ and

$$A_k = \frac{1}{k}\sum_{j=0}^{k-2} A_j B_{k-1-j}$$

for all $k \geq 2$. A careful observation of the individual terms inside a_k and A_k easily shows that $A_k = |a_k|$. Clearly $A_k = |a_k| \leq 1$ by Lemma 3.3. Let $m := \lfloor (k-2)/2 \rfloor$ and $A := 1/\pi$. Since $\lim_{k\to\infty} A_k = 1/\pi = A$, using Remark 3.2 and Lemma 3.5 we see that for each $\epsilon > 0$ there is a sufficiently large integer $N > 0$ such that $k > N$ implies

$$\begin{aligned} |(k+1)(a_{k+1}+a_k)| &= |(k+1)(A_{k+1}-A_k)| \\ &= \Big|\sum_{j=0}^{k-1} A_j B_{k-j} - \sum_{j=0}^{k-2} A_j B_{k-1-j} - A_k\Big| \\ &\leq \Big|A_{k-1}B_1 - A_k + \sum_{j=m+1}^{k-2} A_j(B_{k-j} - B_{k-1-j})\Big| \qquad (4.1) \\ &\quad + \sum_{j=0}^{m} A_j(B_{k-j} - B_{k-1-j}). \end{aligned}$$

Now if the term within the absolute value signs in (4.1) is positive, then

$$\begin{aligned} &|(k+1)(a_{k+1}+a_k)| \\ &\leq \Big|(A+\epsilon)B_1 - (A-\epsilon) + (A+\epsilon)\sum_{j=m+1}^{k-2}(B_{k-j} - B_{k-1-j})\Big| \qquad (4.2) \\ &\quad + \sum_{j=0}^{m} \frac{A_j}{(k-j)^2} \\ &\leq |(A+\epsilon)B_1 - (A-\epsilon) + (A+\epsilon)(B_{k-m-1} - B_1)| \\ &\quad + \frac{1}{(k-m)^2}(m+1) \end{aligned}$$

Notice that $B_{k-m-1} \to 1$ and $(m+1)/(k-m)^2 \to 0$ as $k \to \infty$. Therefore

$$\lim_{k\to\infty} |(k+1)(a_{k+1}+a_k)| \leq |(A+\epsilon)B_1 - (A-\epsilon) + (A+\epsilon)(1-B_1)|.$$

Since this inequality holds for each fixed $\epsilon > 0$, it also holds for $\epsilon = 0$. Hence $|(k+1)(a_{k+1}+a_k)| \to 0$ as $k \to \infty$. Therefore, $a_{k+1} + a_k = o(1/k)$.

If the term within the absolute value signs in (4.1) is negative, then a similar argument gives the same conclusion just by replacing $+\epsilon$ by $-\epsilon$ in (4.2). ∎

Proof of Theorem 2.1(d). From Theorem 2.1(b) we know that $0 < \lim_{k\to\infty} |a_k| = 1/\pi < \infty$. Now using Remark 3.1 we have

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = -1. \blacksquare$$

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