ACTA ARITHMETICA
166.1 (2014)

A dynamical Shafarevich theorem for twists of rational morphisms

by

Brian Justin Stout (Annapolis, MD)

1. Introduction. Let K be a number field and S a finite set of places of K which includes all the Archimedian places. For arithmetic objects defined over K one can pose questions about the number of K-isomorphism classes which have good reduction at all places not in S. Initially, Shafarevich asked this question for elliptic curves over K and proved the number of classes to be finite (see [11]). Faltings subsequently proved the same for abelian varieties (see [4]).

The similarity between the arithmetic theory of elliptic curves and the arithmetic theory of endomorphisms of varieties has prompted many questions about dynamical analogues of Shafarevich's theorem for rational morphisms on projective space. A similar finiteness result for rational morphisms can easily be seen to be false: any monic polynomial defined over $\mathcal{O}_K$ on $\mathbb{P}^1$ exhibits everywhere good reduction. For each $d \geq 2$ it is easy to show that there are infinitely many such polynomials which are non-isomorphic.

The notion of a dynamical Shafarevich theorem was studied first by Szpiro and Tucker in [13] for rational maps on $\mathbb{P}^1$ defined over a number field K. Szapiro and Tucker weaken the notion of K-isomorphism of rational maps by allowing pre-composition and post-composition with different elements of PGL_2 and altering the notion of good reduction. They subsequently obtain a finiteness result for rational maps with *critical good reduction*. In [6] Petsche obtains a different finiteness theorem by restricting the families of rational maps on $\mathbb{P}^1$ under consideration, but retains the normal notion of K-isomorphism for rational maps. In a previous paper the author and Petsche consider whether the set of quadratic rational maps of $\mathbb{P}^1$ with good reduction outside S is Zariski dense in the moduli space $\mathcal{M}_2$ of quadratic rational maps (see [8]). On the contrary, quadratic

2010 *Mathematics Subject Classification*: Primary 37P45; Secondary 14G25, 37P15.
Key words and phrases: arithmetic dynamics, twists of rational morphisms, good reduction.

rational maps with everywhere good reduction over $\mathbb{Z}$ are Zariski dense in $\mathcal{M}_2(\mathbb{Q})$. By restricting the class of rational maps to those with two unramified fixed points and strengthening the notion of good reduction, the author and Petsche prove a Zariski non-density result.

In the present paper we consider a Shafarevich question originally posed by Silverman in Chapter 3 of [12] regarding the finiteness of rational morphisms $\psi : \mathbb{P}^n \to \mathbb{P}^n$ defined over K of degree $d \geq 2$ which have good reduction at all places $v \notin S$ and are twists of a given rational morphism ϕ defined over K.

We say that two rational morphisms $\phi, \psi : \mathbb{P}^n \to \mathbb{P}^n$ defined over K are $\bar{K}$-*isomorphic* if $\psi = \phi^f$ for $f \in \mathrm{PGL}_{n+1}(\bar{K})$, and K-*isomorphic* if $\psi = \phi^f$ for $f \in \mathrm{PGL}_{n+1}(K)$. Here ϕ^f denotes the conjugation of ϕ by f (see Section 2). These notions are clearly equivalence relations; we denote the set of rational morphisms which are $\bar{K}$-isomorphic to ϕ by

$$[\phi] = \{\psi \text{ defined over } K \mid \psi = \phi^f \text{ for some } f \in \mathrm{PGL}_{n+1}(\bar{K})\},$$

and the set of rational morphisms which are K-isomorphic to ϕ by

$$[\phi]_K = \{\psi \text{ defined over } K \mid \psi = \phi^f \text{ for some } f \in \mathrm{PGL}_{n+1}(K)\}.$$

We then define the set of *twists* of ϕ as the set of K-isomorphism classes of rational morphisms ψ defined over K which are $\bar{K}$-isomorphic to ϕ:

$$\mathrm{Twist}(\phi/K) = \{[\psi]_K \mid \psi \text{ is defined over } K \text{ and } [\phi] = [\psi]\}.$$

We say that a morphism $\phi : \mathbb{P}^n \to \mathbb{P}^n$ defined over K has *good reduction* at a non-Archimedean place v of K if there exists some conjugate $\psi = \phi^f$ defined over K for $f \in \mathrm{PGL}_{n+1}(\bar{K})$ such that ψ extends to a morphism of the same degree over the ring of v-adic integers $\mathcal{O}_v$. For an equivalent and more precise definition, see Section 2. We remark that the notion of good reduction of a morphism is K-isomorphism invariant, and therefore the notion of good reduction of a twist at a place v is immediate.

The principal theorem of this paper is the following.

THEOREM 1. *Let $\phi : \mathbb{P}^n \to \mathbb{P}^n$ be a rational morphism of degree $d \geq 2$ defined over K and let S be a finite set of places including the Archimedean places. Let*

$$\mathcal{V}(S) = \{[\psi]_K \in \mathrm{Twist}(\phi/K) \mid [\psi]_K \textit{ has good reduction outside } S\}.$$

Then $\mathcal{V}(S)$ is finite.

The following is a sketch of the ideas behind the proof. Assume $[\psi]_K$ is a twist of $[\phi]_K$, both with good reduction outside S. Let $f \in \mathrm{PGL}_{n+1}(\bar{K})$ be the automorphism such that $\psi = \phi^f$. Fix an integer $M > n+1$ and let $\mathrm{PrePer}(\phi, M)$ denote the set of preperiodic points for ϕ with forward orbit of size less than or equal to M. Then the automorphism f defines a bijection between $\mathrm{PrePer}(\phi, M)$ and $\mathrm{PrePer}(\psi, M)$ by $P \mapsto f(P)$. The premise of the proof is that, after letting M become sufficiently large, the sets $\mathrm{PrePer}(\psi, M)$ and $\mathrm{PrePer}(\phi, M)$ can be assumed to be equal, from the Diophantine finiteness theorem which we will prove in Section 3. The main ingredient of this step will follow from a finiteness theorem for forms with unit discriminant due to Evertse–Győry [2]. Consequently, f will be a bijection on a finite subset of $\mathbb{P}^n$ and, after concluding that some subset of $\mathrm{PrePer}(\phi, M)$ is in general position, only finitely many f can exist.

The referee of this paper pointed out a proof for the analagous theorem for abelian varieties over number fields, which does not translate to the dynamical setting. Let A be an abelian variety defined over K, $\mathcal{O} \in A(K)$ be the identity point, and $G_{\bar{K}/K}$ the absolute Galois group for K. It is well known that $\mathrm{Twist}((A, \mathcal{O})/K) \cong H^1(G_{\bar{K}/K}, \mathrm{Aut}(A))$, or equivalently, that every K-twist of A can be obtained via a character $\chi : G_{\bar{K}/K} \to \mathrm{Aut}(A)$. We will denote by A^χ the twist of A by χ.

Let v be a non-Archimedean place of K and k_v the corresponding residue field. Furthermore, for $m \geq 1$ let $A[m]$ denote the m-torsion points of A. The Néron–Ogg–Shafarevich criterion states that A has good reduction at v if and only if $A[m]$ is unramified at v for all integers $m \geq 1$ relatively prime to $\mathrm{char}(k_v)$. It follows that if A has good reduction outside a finite set of places S, and if for each $v \notin S$, $\gcd(m, \mathrm{char}(k_v)) = 1$, then $K(A[m])/K$ is an unramified extension at all $v \notin S$.

Let $\chi : G_{\bar{K}/K} \to \mathrm{Aut}(A)$ be a character and A^χ the corresponding twist. We claim that A^χ has good reduction outside S if and only if the character χ is unramified outside S. This again will follow from the Néron–Ogg–Shafarevich criterion when one sees that the action of $\sigma \in I_v$ in the inertia subgroup on $A^\chi[m]$ is the same action by $\chi(\sigma)$ on $A[m]$. Showing that there are finitely many twists of A over K with good reduction outside S now reduces to showing that there are only finitely many characters unramified outside S. Proposition C.4.2 of [5] shows this always to be the case when $\mathrm{Aut}(A)$ is finite, in particular for elliptic curves and polarized abelian varieties.

The critical step in this proof is the equivalence of the following three statements:

(1) A has good reduction at v.
(2) $A[m]$ is unramified at v for all m such that m and $\mathrm{char}(k_v)$ are relatively prime.
(3) $K(A[m])/K$ is unramified at v for all m such that m and $\mathrm{char}(k_v)$ are relatively prime.

Such a theorem does not exist in the dynamical case. A theorem does exist going one way which states that for *almost all places* the extension generated by points of exact period m is unramified when ϕ has good reduction and m is relatively prime to $\mathrm{char}(k_v)$ (see [10, Proposition 3.63]). It is not known to the author whether a converse exists.

2. Preliminaries

2.1. Review of rational morphisms on projective space. We will fix the notation and definitions regarding the dynamics of rational morphisms on projective space; for more details see [10].

Fix coordinates $(X_0 : \ldots : X_n)$ of $\mathbb{P}^n(\bar{K})$. An arbitrary rational morphism $\phi : \mathbb{P}^n \to \mathbb{P}^n$ defined over $\bar{K}$ is given by an $(n+1)$-tuple

$$\phi(X_0 : \ldots : X_n) = (F_0(X_0 : \ldots : X_n) : \ldots : F_n(X_0 : \ldots : X_n))$$

where $F_i(X_0, \ldots, X_n)$ is a homogeneous polynomial of degree d for $i = 0, \ldots, n$ and $F_0, \ldots, F_n$ have no non-trivial common solutions.

Let $j = (j_0, \ldots, j_n)$ be a multi-index where $0 \leq j_k \leq d$ for each k and $j_0 + \cdots + j_n = d$. We will let X^j denote the monomial $X_0^{j_0} \cdots X_n^{j_n}$ and then write

$$F_i = \sum_j a_{ij} X^j$$

with coefficients $a_{ij} \in \bar{K}$. There are $\binom{n+d}{d}$ monomials of degree d in $n+1$ variables and so ϕ can be identified with a point $(a_{0j} : \ldots : a_{nj}) \in \mathbb{P}^N$ where $N = N(n,d) = (n+1)\binom{n+d}{d} - 1$. Conversely, any point of $\mathbb{P}^N$ determines a rational map $\phi : \mathbb{P}^n \to \mathbb{P}^n$, although this map may not be a morphism. The requirement that ϕ be a morphism is equivalent to the non-vanishing of the Macaulay resultant polynomial, i.e. $\mathrm{Res}(F_0, \ldots, F_n) \neq 0$, where the resultant polynomial is a multi-homogeneous polynomial over $\mathbb{Z}$ in the coefficients a_{ij}. For further details about this polynomial and proof of the above claim with regard to when a rational map is a morphism, see [1, Chapter 3].

There is a natural $\mathrm{PGL}_{n+1}(\bar{K})$ action on rational morphisms which sends $f \in \mathrm{PGL}_{n+1}(\bar{K})$ and ϕ to $\phi^f = f \circ \phi \circ f^{-1}$. When ϕ is a rational morphism it may be iterated and we write $\phi^n = \phi \circ \cdots \circ \phi$ to denote the nth iterate of ϕ. The action of conjugation is compatible with iteration in the sense that $(\phi^f)^n = (\phi^n)^f$.

There are natural sets of points in $\mathbb{P}^n$ which can be associated to a rational morphism $\phi : \mathbb{P}^n \to \mathbb{P}^n$. A point $P \in \mathbb{P}^n(\bar{K})$ is *periodic* if $\phi^m(P) = P$ for some positive integer $m \geq 1$, and *preperiodic* if some iterate $\phi^n(P)$ is periodic. Equivalently, P is preperiodic if and only if the forward orbit of P is finite.

We use $\mathrm{Per}(\phi)$ and $\mathrm{PrePer}(\phi)$ to denote the set of all periodic points and all preperiodic points, respectively, for a fixed rational morphism ϕ. Let

$\mathcal{O}_\phi(P)$ denote the forward orbit of P under ϕ. We write

$$\mathrm{PrePer}(\phi, M) = \{P \in \mathbb{P}^n \mid |\mathcal{O}_\phi(P)| \le M\} \tag{1}$$

for the set of preperiodic points with forward orbit of length at most M. We use $\mathrm{Fix}(\phi) = \{P \in \mathbb{P}^n \mid \phi(P) = P\}$ to denote the set of fixed points of ϕ.

2.2. Review of twists of rational morphisms. Suppose that $\phi, \psi : \mathbb{P}^n \to \mathbb{P}^n$ are rational morphisms of degree d defined over K. We have already stated the definition of twist, K-isomorphism, and $\bar{K}$-isomorphism in Section 1. It is further possible to give a description of twists using Galois cohomology of $\mathrm{PGL}_n(\bar{K})$ and the automorphism group $\mathrm{Aut}(\phi)$. For more details, see [10, Chapter 4, Section 9]. We will not make use of this description in this paper.

Two rational morphisms ϕ, ψ defined over K which are twists have identical geometric properties as morphisms on $\mathbb{P}^n(\bar{K})$ but may have significantly different arithmetic properties as morphisms on $\mathbb{P}^n(K)$. Let Hom_d^n denote the parameter space of rational morphisms of degree d on $\mathbb{P}^n$. Then ϕ, ψ are twists if and only if they descend to the same point in the moduli space $\mathcal{M}_d^n$ under the quotient map

$$\mathrm{Hom}_d^n \to \mathrm{Hom}_d^n/\mathrm{PGL}_{n+1} = \mathcal{M}_d^n.$$

The space $\mathcal{M}_d^n$ is known to be a coarse solution to the moduli problem for rational maps. Moreover, it is a good geometric quotient in the sense of geometric invariant theory. For more details regarding $\mathcal{M}_d^n$ and its structure as a variety, see [12, Chapter 2].

When $n = 1$, it is essential to note that the theorem proved in this paper is fundamentally different from the other dynamical-Shafarevich results of [6], [8], and [13] in the sense that the finiteness theorem of this paper holds only within a single $\bar{K}$-isomorphism class of morphisms defined over K.

2.3. Review of number-theoretic preliminaries. Let M_K denote the set of places of the number field K. For any place $v \in M_K$ let $|\cdot|_v$ denote any absolute value on K associated to v. If v is non-Archimedean, let K_v denote the completion of K with respect to v, and

$$\mathcal{O}_v = \{x \in K_v \mid |x|_v \le 1\}, \quad \mathcal{O}_v^\times = \{x \in K_v \mid |x|_v = 1\}$$

denote the subring of v-integral elements and the group of v-adic units in $\mathcal{O}_v$, respectively. Then $\mathcal{O}_v$ is a discrete local ring with maximal ideal $\mathfrak{m}_v = \{x \in K \mid |x|_v < 1\}$. Let $\mathcal{O}_v \to k_v = \mathcal{O}_v/\mathfrak{m}_v$ be the reduction map onto the residue field k_v. For $x \in \mathcal{O}_v$ we denote the image of this map by $\tilde{x}_v$ or just $\tilde{x}$ if v is understood.

For a rational morphism $\phi : \mathbb{P}^n \to \mathbb{P}^n$ defined over K and $v \in M_K$ we can define the *reduction* of ϕ at the place v in the following manner. Having the natural embedding $K \to K_v$ one can consider ϕ to be a rational morphism over K_v. As $K_v = \mathrm{Frac}(\mathcal{O}_v)$ and $\mathcal{O}_v$ is a PID, one can choose homogeneous coefficients for $\phi = (a_{0j} : \ldots : a_{nj})$ as a point in $\mathbb{P}^N$ such that $|a_{ij}|_v \le 1$ and $\max_{i,j}(|a_{ij}|_v) = 1$.

Definition. The *reduction of ϕ at v* is the rational map

$$\tilde{\phi}_v = (\tilde{a}_{0j} : \ldots : \tilde{a}_{nj}) \in \mathbb{P}^N(k_v)$$

This reduction is independent of the choice of homogeneous coordinates. The reduction of a morphism may or may not be a morphism over the residue field.

Definition. For a rational morphism $\phi : \mathbb{P}^n \to \mathbb{P}^n$ of degree d we say that ϕ has *good reduction* at v if there exists some $f \in \mathrm{PGL}_{n+1}(K)$ such that $\widetilde{\phi_v^f} : \mathbb{P}^n \to \mathbb{P}^n$ is a morphism defined over k_v and $\deg(\widetilde{\phi_v^f}) = d$. We say ϕ has *bad reduction* otherwise.

By definition, the notion of good reduction is seen to be a $\mathrm{PGL}_2(K)$-invariant concept and is therefore well defined for a K-equivalence class $[\phi]_K$.

In this paper S denotes a finite subset of M_K which includes all of the Archimedean places, $\mathcal{O}_S$ the S-integers of K, $\mathcal{O}_S^\times$ the S-unit group of K. More specifically,

$$\mathcal{O}_S = \{x \in K \mid |x|_v \le 1 \text{ for all } v \notin S\},$$
$$\mathcal{O}_S^\times = \{x \in K \mid |x|_v = 1 \text{ for all } v \notin S\}.$$

Definition. The *absolute S-integers of $\bar{K}$* will consist of all elements of $\bar{K}$ which are w-integral for every place w of $\bar{K}$ whose restriction to K is not in S. The absolute S-integers of $\bar{K}$ are denoted by $\overline{\mathcal{O}}_S$.

For a point $P = (p_0 : \ldots : p_n) \in \mathbb{P}^n(K)$ and a place v we say that the coordinates are *normalized* with respect to v, or *v-normalized*, if $|p_i|_v \le 1$ for $0 \le i \le n$ and $|p_i| = 1$ for some i. The following lemma is well known, so we omit the proof.

Lemma 2. *Let $P \in \mathbb{P}^n(K)$ and S be sufficiently large such that $\mathcal{O}_S$ is a principal ideal domain. Then there exist coordinates $(p_0 : \ldots : p_n)$ for P which are v-normalized for all $v \notin S$.*

We will also call such coordinates normalized, and context will make it clear whether we refer to a single place v or to all places $v \notin S$.

3. A Diophantine result. Fix a number field K, an algebraic closure $\bar{K}$, and a morphism $\phi : \mathbb{P}^n \to \mathbb{P}^n$ of degree at least $d \ge 2$ defined over K. Fix projective coordinates $(X_0 : \ldots : X_n)$ on $\mathbb{P}^n$ and let v denote a non-Archimedean place of K.

Definition. A form $F \in K[X_0, \dots, X_n]$ is called *decomposable* defined over K if it can be factored over $\bar{K}$ as $F = \ell_1^{k_1} \cdots \ell_t^{k_t}$ where $\ell_1, \dots, \ell_t$ are pairwise non-proportional homogeneous linear polynomials over $\bar{K}$ and $k_1, \dots, k_t$ are positive integers such that $k_1 + \dots + k_t = \deg(F)$.

This type of form is studied by Evertse and Győry in [2]. Each decomposable form has an associated discriminant which is a fractional ideal of $\mathcal{O}_S$. From this point on, by "ideal" we mean fractional ideal.

Let $\mathcal{V} \subset \mathbb{P}^n(\bar{K})$ be a finite subset with $|\mathcal{V}| > n$. Let $P_0, \dots, P_n \in \mathcal{V}$ be a collection of $n+1$ points. We define $\det(P_0, \dots, P_n)$ to be the determinant of the $n+1$ by $n+1$ matrix of the coordinates of the P_i. This determinant depends on the choice of representation used to compute it. If the coordinates of P_i are replaced by $(rp_{i0} : \ldots : rp_{in})$ with $r \in K$, then the determinant changes by a factor of r as well. Any linear homogeneous form $\ell \in K[X_0, \dots, X_n]$ in $n+1$ variables can be identified as a point in projective space by

$$\ell = \sum_i p_i X_i \mapsto P = (p_0 : \ldots : p_n). \tag{2}$$

We can therefore define $\det(\ell_0, \dots, \ell_n)$ for $n+1$ linear forms of $K[X_0, \dots, X_n]$ in the analogous way.

Definition. Suppose $F(X_0, \dots, X_n)$ is a decomposable form and $F = \ell_1^{k_1} \cdots \ell_t^{k_t}$ as a polynomial over $\bar{K}$. Let $\mathcal{I}(F)$ be the collection of $\bar{K}$-linearly independent subsets $\{\ell_{i_0}, \dots, \ell_{i_n}\}$ of $\{\ell_0, \dots, \ell_t\}$, and let (ℓ_i) denote the ideal generated by the coefficients of ℓ_i. Assume $\mathcal{I}(F) \neq \emptyset$. Then the *discriminant* of F, denoted D_F, is an ideal of $\mathcal{O}_S$ defined as follows:

$$D_F = \prod_{\mathcal{I}(F)} \left(\frac{\det(\ell_{i_0}, \dots, \ell_{i_n})}{(\ell_{i_0}) \cdots (\ell_{i_n})} \right)^2. \tag{3}$$

Remark. For each linear form ℓ_i, the ideal (ℓ_i) is actually an ideal of the integral closure of $\mathcal{O}_S$ in some extension of K, but D_F is an ideal of $\mathcal{O}_S$. For more details see the introduction of [2].

Definition. Let F and G be two decomposable forms in $n+1$ variables of degree d. The forms F and G are *weakly $\mathcal{O}_S$-equivalent* if

$$F(X_0, \dots, X_n) = \lambda G(A(X_0, \dots, X_n))$$

for some $\lambda \in K^{\times}$ and some $A \in \mathrm{GL}_{n+1}(\mathcal{O}_S)$.

To any finite subset $\mathcal{V} \subset \mathbb{P}^n(\bar{K})$ which is $\mathrm{Gal}(\bar{K}/K)$-stable we can associate a decomposable homogeneous form $F_{\mathcal{V}} \in K[X_0, \dots, X_n]$ in the following manner: Let L denote the field of definition of the set $\mathcal{V}$, i.e. L is the smallest extension of K such that for each $P \in \mathcal{V}$, $P \in \mathbb{P}^n(L)$. For $P = (p_0 : \ldots : p_n) \in \mathcal{V}$ the homogeneous form of degree 1 is $\ell_P = \sum_i p_i X_i$, and this form is defined over L. The form ℓ_P is well defined up to multiplication by a non-zero scalar of L. Set

$$F_{\mathcal{V}} = \prod_{P \in \mathcal{V}} \ell_P. \tag{4}$$

As $\mathcal{V}$ is $\mathrm{Gal}(\bar{K}/K)$-stable, it follows that $F_{\mathcal{V}}$ is a decomposable form of $K[X_0, \ldots, X_n]$ of degree $|\mathcal{V}|$ and is well defined up to multiplication by a non-zero scalar of K. Furthermore, it follows that $D_{F_{\mathcal{V}}}$ is a well defined fractional ideal of $\mathcal{O}_S$. The discriminant ideal is unchanged if we multiply $F_{\mathcal{V}}$ by a scalar of $\gamma \in K^{\times}$, so we may choose some γ and, after multiplying $F_{\mathcal{V}}$ through, assume that $F_{\mathcal{V}} \in \mathcal{O}_S[X_0, \ldots, X_n]$.

Definition. Let $N \geq n+1$ be an integer and consider the set $\mathcal{P}(S, N)$ of all subsets $\mathcal{V} \subset \mathbb{P}^n(\bar{K})$ such that:

(1) $|\mathcal{V}| = N$.
(2) $\mathcal{V}$ is $\mathrm{Gal}(\bar{K}/K)$-stable.
(3) $\mathcal{V}$ contains at least one linearly independent $(n+1)$-point subset.
(4) $D_{F_{\mathcal{V}}} = \mathcal{O}_S$.

Condition (4) generalizes the familiar notion of pairwise S-integrality for points of projective space. Recall that pairwise S-integrality in $\mathbb{P}^1$ merely requires that two distinct points $P, Q \in \mathbb{P}^1(K)$ reduce to distinct points in $\mathbb{P}^1(k_v)$ for every $v \notin S$. The above requirement on the discriminant ideal is strictly stronger than pairwise S-integrality as it requires that linearly independent points of $\mathbb{P}^n(\bar{K})$ not only remain distinct after reduction at each place v, but also that they remain linearly independent. In general, for any place v of K with $v(D_{F_{\mathcal{V}}}) > 0$ (of which there can only be finitely many), some set of $n+1$ linearly independent points $P_0, \ldots, P_n \in \mathcal{V}$ descend to linearly dependent points in $\tilde{P}_0, \ldots, \tilde{P}_n \in \mathbb{P}^n(k_v)$.

Lemma 3. *Let $\mathcal{V}, \mathcal{W} \subset \mathbb{P}^n(\bar{K})$ be two $\mathrm{Gal}(\bar{K}/K)$-invariant subsets such that $|\mathcal{V}| = |\mathcal{W}| = N$, both contain at least one linearly independent $(n+1)$-point subset, and $f(\mathcal{V}) = \mathcal{W}$ for some $f \in \mathrm{PGL}_{n+1}(\overline{\mathcal{O}}_S)$. Then $D_{F_{\mathcal{V}}} = D_{F_{\mathcal{W}}}$. In particular, $D_{F_{\mathcal{V}}} = \mathcal{O}_S$ if and only if $D_{F_{f(\mathcal{V})}} = \mathcal{O}_S$.*

Proof. Let L/K be a finite extension such that $\mathcal{V}, \mathcal{W}$ are subsets of $\mathbb{P}^n(L)$ and such that $f \in \mathrm{PGL}_n(\mathcal{O}_T)$, where $\mathcal{O}_T$ refers to the integral closure of $\mathcal{O}_S$ in L. Let $A \in \mathrm{GL}_{n+1}(\mathcal{O}_T)$ be a lift of f. Let A^t denote the transpose of A. It follows that A^t defines a weak $\mathcal{O}_S$-equivalence between $F_{\mathcal{V}}$ and $F_{\mathcal{W}}$. By Section 1 of [2] the discriminant is invariant under weak $\mathcal{O}_S$-equivalence. Consequently, $D_{F_{\mathcal{V}}} = D_{F_{\mathcal{W}}}$. ∎

Lemma 4. *There exists a group action*

$$\mathrm{PGL}_{n+1}(\mathcal{O}_S) \times \mathcal{P}(S, N) \to \mathcal{P}(S, N)$$

defined by $(f, \mathcal{V}) \mapsto f(\mathcal{V}) = \{f(P) \mid P \in \mathcal{V}\}$.

Proof. Let $f, g, h \in \mathrm{PGL}_{n+1}(\mathcal{O}_S)$ and $\mathcal{V} \in \mathcal{P}(S,N)$. We must show that conditions (1) to (4) in the definition of $\mathcal{P}(S,N)$ hold for $f(\mathcal{V})$. Requirements (1) and (4) are obviously satisfied since f is an automorphism of $\mathbb{P}^n$. Let $\sigma \in \mathrm{Gal}(\bar{K}/K)$ and $f(P) \in f(\mathcal{V})$. Then, since f is defined over $K = \mathrm{Frac}(\mathcal{O}_S)$ and $\mathcal{V}$ is $\mathrm{Gal}(\bar{K}/K)$-stable we have $\sigma \cdot f(P) = f(\sigma \cdot P) = f(Q) \in f(\mathcal{V})$ for some $Q \in \mathcal{V}$, proving condition (2). Lastly, (3) follows from Lemma 3. ∎

THEOREM 5. *The group action of* $\mathrm{PGL}_{n+1}(\mathcal{O}_S)$ *on* $\mathcal{P}(S,N)$ *has only finitely many orbits.*

Proof. This theorem is a reformulation of Corollary 2 of Evertse and Győry [2]. Their corollary states that there are only finitely many weak $\mathcal{O}_S$-equivalence classes of decomposable forms in $K[X_0,\ldots,X_n]$ of fixed degree N and given discriminant ideal D.

To conclude the proof of Theorem 5 it suffices to show that if

$$\mathcal{V} = \{P_1,\ldots,P_N\}, \mathcal{W} = \{Q_1,\ldots,Q_N\} \in \mathcal{P}(S,N)$$

and if the forms $F_{\mathcal{V}}$ and $F_{\mathcal{W}}$ are weakly $\mathcal{O}_S$-equivalent, then the sets $\mathcal{V}$ and $\mathcal{W}$ are in the same $\mathrm{PGL}_{n+1}(\mathcal{O}_S)$-orbit.

Let F and G denote $F_{\mathcal{V}}$ and $F_{\mathcal{W}}$, respectively. If F and G are weakly $\mathcal{O}_S$-equivalent then there exist $\lambda \in K^{\times}$ and $A \in \mathrm{GL}_{n+1}(\mathcal{O}_S)$ such that

$$F(X_0,\ldots,X_n) = \lambda G(A(X_0,\ldots,X_n)).$$

It follows that

$$\prod_{1\le i\le N} \ell_{P_i}(X_0,\ldots,X_n) = \lambda \prod_{1\le i\le N} \ell_{Q_i}(A(X_0,\ldots,X_n)).$$

Let $A^t \in \mathrm{GL}_{n+1}(\mathcal{O}_S)$ be the transpose of A. After reordering we have

$$\ell_{P_i}(X_0,\ldots,X_n) = \lambda_i \ell_{Q_i}(A(X_0,\ldots,X_n))$$

for $\lambda_i \in K^{\times}$. Equating coefficients gives

$$(p_{0i},\ldots,p_{ni}) = \lambda_i A^t(q_{0i},\ldots,q_{ni}).$$

Let $a \in \mathrm{PGL}_{n+1}(\mathcal{O}_S)$ be the projective linear transformation corresponding to A^t. Note that when we pass to projective space the scalars λ_i become irrelevant. Then $P_i = a(Q_i)$ and therefore $\mathcal{V} = a(\mathcal{W})$. ∎

4. Main theorem. Let $M \ge 1$. Recall that $\mathrm{PrePer}(\phi, M)$ denotes the set of all points in $\mathbb{P}^n(\bar{K})$ which are ϕ-preperiodic and whose forward orbit has size at most M. We remark that the set $\mathrm{PrePer}(\phi, M)$ is finite. It follows from Northcott's theorem that the set of preperiodic points for a morphism of projective space, defined over a number field, and degree greater than one, is a set of bounded height. Further requiring that the forward orbit of these preperiodic points be less than M bounds their degree of definition. Sets of bounded height and degree are finite. See [10, Chapter 3, Section 2] for proofs of these two facts. Let $N = |\mathrm{PrePer}(\phi, M)|$. Every rational morphism of degree at least 2 has infinitely many preperiodic points, so by increasing M we may assume that $N \geq n + 2$.

Definition. A subset $\mathcal{V} \subset \mathbb{P}^n(\bar{K})$ with $|\mathcal{V}| \geq n+1$ is in *general position* if no $(n+1)$-point subset of $\mathcal{V}$ lies in a hyperplane.

From Fakhruddin's result on the Zariski density of preperiodic points (see [3, Theorem 5.1]), by increasing M we may assume that there is a subset of $\mathrm{PrePer}(\phi, M)$ consisting of $n + 2$ points which lie in general position.

Lemma 6. *Let $\mathcal{V}, \mathcal{W} \subset \mathbb{P}^n(\bar{K})$ be finite, and assume that $\mathcal{V}$ has a subset $\mathcal{V}_0$ in general position with $|\mathcal{V}_0| = n+2$. Then there exist only finitely many automorphisms $f \in \mathrm{PGL}_{n+1}(\bar{K})$ such that $f(\mathcal{V}) = \mathcal{W}$.*

Proof. Suppose that $f_1, f_2, \ldots \in \mathrm{PGL}_{n+1}(\bar{K})$ is an infinite sequence of distinct automorphisms such that $f_i(\mathcal{V}) = \mathcal{W}$. Since $\mathcal{W}$ is finite, it has only finitely many subsets, and we may assume, after perhaps passing to an infinite subsequence, that there exists a subset $\mathcal{W}_0 \subset \mathcal{W}$ in general position with $|\mathcal{W}_0| = n + 2$ and $f_i(\mathcal{V}_0) = \mathcal{W}_0$ for all i. Choose $g \in \mathrm{PGL}_{n+1}(\bar{K})$ such that $g(\mathcal{W}_0) = \mathcal{V}_0$. Then the compositions $g \circ f_i$ form an infinite sequence of distinct automorphisms in $\mathrm{PGL}_{n+1}(\bar{K})$ such that $g \circ f_i(\mathcal{V}_0) = \mathcal{V}_0$. This gives a contradiction as there are only $(n + 2)!$ such automorphisms. ∎

Definition. A *homogeneous lift* of ϕ is a map $\Phi : \mathring{\mathbb{A}}^{n+1} \to \mathring{\mathbb{A}}^{n+1}$ given by an $(n + 1)$-tuple $(F_0, \ldots, F_n)$ of forms F_i such that

$$\phi(p_0 : \ldots : p_n) = (F_0(p_0 : \ldots : p_n) : \ldots : F_n(p_0 : \ldots : p_n))$$

for all points $P = (p_0 : \ldots : p_n) \in \mathbb{P}^n$.

Proposition 7. *Assume that $\mathcal{O}_S$ is a PID. Let $\phi, \psi : \mathbb{P}^n \to \mathbb{P}^n$ be rational morphisms defined over K of degree d, both having good reduction at all places v of K outside S. Assume that $[\psi]_K \in \mathrm{Twist}(\phi/K)$.*

(a) *There exist rational morphisms $\phi_0 \in [\phi]_K$ and $\psi_0 \in [\psi]_K$, and homogeneous lifts $\Phi, \Psi : \mathring{\mathbb{A}}^{n+1} \to \mathring{\mathbb{A}}^{n+1}$ of ϕ_0 and ψ_0, respectively, such that Φ, Ψ have coefficients in $\mathcal{O}_S$ and resultants $\mathrm{Res}(\Phi), \mathrm{Res}(\Psi) \in \mathcal{O}_S^\times$.*
(b) *There exists $A \in \mathrm{GL}_{n+1}(\overline{\mathcal{O}}_S)$ such that $\Psi = \Phi^A$.*
(c) *For each integer $M \geq 1$, we have*

$$\mathrm{PrePer}(\psi_0, M) = f(\mathrm{PrePer}(\phi_0, M)),$$

where $f : \mathbb{P}^n \to \mathbb{P}^n$ is the automorphism associated to A. Moreover, $\mathrm{PrePer}(\phi_0, M) \in \mathcal{P}(S, N)$ if and only if $\mathrm{PrePer}(\psi_0, M) \in \mathcal{P}(S, N)$.

Proof. Part (a) follows from the main theorem of [7] regarding the existence of global minimal models over principal ideal domains. The existence of $A \in \mathrm{GL}_{n+1}(\bar{K})$ follows from Φ, Ψ being lifts of twists. That $A \in \mathrm{GL}_{n+1}(\overline{\mathcal{O}}_S)$ follows from [9, Lemma 6]. For (c), that $\mathrm{PrePer}(\phi_0, M) = f(\mathrm{PrePer}(\psi_0, M))$

is a consequence of the definition of these sets. As $A \in \mathrm{GL}_{n+1}(\overline{\mathcal{O}}_S)$ it follows that $f \in \mathrm{PGL}_{n+1}(\overline{\mathcal{O}}_S)$, and it is immediate from Lemma 3 that $\mathrm{PrePer}(\phi_0, M) \in \mathcal{P}(S,N)$ if and only if $\mathrm{PrePer}(\psi_0, M) \in \mathcal{P}(S,N)$. ∎

We are now ready to prove the main theorem.

Theorem 8. *Let $\phi : \mathbb{P}^n \to \mathbb{P}^n$ be a rational morphism of degree $d > 1$ defined over K and $\mathrm{Twist}(\phi/K)$ the set of K-twists. Then there are only finitely many twists $[\psi]_K \in \mathrm{Twist}(\phi/K)$ which have good reduction at all places $v \notin S$.*

Proof. If no twists of ϕ have good reduction outside S then there is nothing to prove. Therefore, assume that at least one such twist exists, and since being twists is an equivalence relation, without loss of generality assume that ϕ has good reduction outside S. Let $[\psi]_K$ be a twist of ϕ with good reduction outside S and let $f \in \mathrm{PGL}_{n+1}(\bar{K})$ be the linear fractional transformation such that $\psi = \phi^f$.

We may assume by Fakhruddin's result on Zariski density of preperiodic points (see [3]) that we can increase M so that a set of $n+2$ points of $\mathbb{P}^n$ in general position is contained in $\mathrm{PrePer}(\phi, M)$. Because ϕ is defined over K the set $\mathrm{PrePer}(\phi, M)$ is $\mathrm{Gal}(\bar{K}/K)$-stable. There are only finitely many combinations of $n+1$ points of $\mathrm{PrePer}(\phi, M)$ which are linearly independent; let $D_0, \ldots, D_t$ be their determinants. There are only finitely many places v of K such that $v(D_i) > 0$ for at least one $i = 0, \ldots, t$. Enlarge S by these places. It follows that $D = \mathcal{O}_S$ where D is the discriminant associated to the set $\mathrm{PrePer}(\phi, M)$, and further enlarging S does not change this condition. Hence,

$$\mathrm{PrePer}(\phi, M) \in \mathcal{P}(S,N)$$

where $N = |\mathrm{PrePer}(\phi, M)|$. Finally, we may further enlarge S until $\mathcal{O}_S$ is a PID. Consequently, by Proposition 7 we have $\mathrm{PrePer}(\psi, M) \in \mathcal{P}(S,N)$, perhaps after replacing ψ with some K-isomorphic morphism within $[\psi]_K$.

By Theorem 5, there are only finitely many $\mathrm{PGL}_{n+1}(\mathcal{O}_S)$-equivalence classes in $\mathcal{P}(S,N)$. It will suffice to prove the theorem under the assumption that $\mathrm{PrePer}(\phi, M)$ and $\mathrm{PrePer}(\psi, M)$ lie in the same equivalence class. It follows that there exists a linear transformation $g \in \mathrm{PGL}_{n+1}(\mathcal{O}_S)$ such that

$$\mathrm{PrePer}(\psi, M) = g(\mathrm{PrePer}(\phi, M))$$

and therefore

$$\mathrm{PrePer}(\psi^g, M) = g^{-1}(\mathrm{PrePer}(\psi, M)) = \mathrm{PrePer}(\phi, M).$$

As ψ^g also has good reduction at all $v \notin S$, it suffices to replace ψ with ψ^g and assume that $\mathrm{PrePer}(\psi, M) = \mathrm{PrePer}(\phi, M)$.

Consequently, f gives a bijection

$$f : \mathrm{PrePer}(\phi, M) \to \mathrm{PrePer}(\phi, M). \tag{5}$$

As $\mathrm{PrePer}(\phi, M)$ is finite and contains a subset of $n+2$ points in general position, it follows from Lemma 6 that only finitely many f can exist. Therefore, there can only be finitely many twists of ϕ with good reduction at all $v \notin S$. ∎

Acknowledgements. The author would like to thank Clayton Petsche for his guidance over the last several years and for his generous friendship. Furthermore, he would like to thank the referee for his constructive comments and insights. He would also like to thank Lucien Szpiro, Victor Kolyvagin, and Ken Kramer for their support under NSF grant DMS-0739346.

References

[1] D. A. Cox, J. Little and D. O'Shea, *Using Algebraic Geometry*, 2nd ed., Grad. Texts in Math. 185, Springer, New York, 2005.

[2] J.-H. Evertse and K. Győry, *Effective finiteness theorems for decomposable forms of given discriminant*, Acta Arith. 60 (1992), 233–277.

[3] N. Fakhruddin, *Questions on self maps of algebraic varieties*, J. Ramanujan Math. Soc. 18 (2003), 109–122.

[4] G. Faltings, *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Invent. Math. 73 (1983), 349–366; Erratum, ibid. 75 (1984), 381.

[5] M. Hindry and J. H. Silverman, *Diophantine Geometry: An Introduction*, Grad. Texts in Math. 201, Springer, New York, 2000.

[6] C. Petsche, *Critically separable rational maps in families*, Compos. Math. 148 (2012), 1880–1896.

[7] C. Petsche and B. Stout, *Global minimal models for endomorphisms of projective space*, J. Théor. Nombres Bordeaux, to appear.

[8] C. Petsche and B. Stout, *On quadratic rational maps with prescribed good reduction*, Proc. Amer. Math. Soc., to appear.

[9] C. Petsche, L. Szpiro and M. Tepper, *Isotriviality is equivalent to potential good reduction for endomorphism of $\mathbb{P}^N$ over function fields*, J. Algebra 322 (2009), 3345–3365.

[10] J. H. Silverman, *The Arithmetic of Dynamical Systems*, Grad. Texts in Math. 241, Springer, New York, 2007.

[11] J. H. Silverman, *The Arithmetic of Elliptic Curves*, Grad. Texts in Math. 106, Springer, New York, 1992.

[12] J. H. Silverman, *Moduli Spaces and Arithmetic Dynamics*, CRM Monograph Ser. 30, Amer. Math. Soc., Providence, RI, 2012.

[13] L. Szpiro and T. J. Tucker, *A Shafarevich–Faltings theorem for rational functions*, Pure Appl. Math. Quart. 4 (2008), 715–728.

Brian Justin Stout
Department of Mathemaics
U.S. Naval Academy
Chauvenet Hall
Annapolis, MD, 21401-1363, U.S.A.
E-mail: stout@usna.edu

Received on 17.1.2014
and in revised form on 23.7.2014 (7706)