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Infinite 2-class field towers of some imaginary quadratic number fields

by

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1. Introduction. Let K be an imaginary quadratic number field with discriminant d, and C_K denote the ideal class group of K. We mean by the 2-*class field tower* of K the sequence of fields $K = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_i \subseteq \ldots$, where K_{i+1} is the Hilbert 2-class field (i.e. the maximal unramified abelian 2-extension) of K_i. If $K_{i+1} \neq K_i$ for all i, then we say that the 2-class field tower of K is *infinite*.

By the results of Golod–Shafarevich [3] and Vinberg–Gaschütz [12, 15], the 2-class field tower of K is infinite if 2-rank $C_K \geq 5$. On the other hand, Koch [6] and Hajir [4, 5] proved that the 2-class field tower of K is infinite if 4-rank $C_K \geq 3$. When 2-rank $C_K = 3$ and 4-rank $C_K = 0$, there are some examples of infinite families of K with infinite (resp. finite) 2-class field towers [4, 7]. However, when 2-rank $C_K = 4$, no example of K with finite 2-class field tower has ever been known. It has been conjectured [9] that the 2-class field tower of such a K is always infinite. In this direction, Benjamin [1, 2] proved that the 2-class field tower of K is infinite if 2-rank $C_K = 4$ and 4-rank $C_K = 2$, except some type of Rédei matrix of K.

In this paper, we study the case where 2-rank $C_K = 4$ and exactly one negative prime discriminant divides d, and prove that the 2-class field tower of such a K is infinite, except for one type of Rédei matrix of K. To prove our theorem, we use Martinet's inequalities [9] and their corollaries. We also use some properties of Rédei matrices [10, 11, 13, 14]. A similar problem for real quadratic number fields is treated by Maire [8], by a different method.

2. Martinet's inequalities and their corollaries. Let K be an imaginary quadratic number field.

MARTINET'S INEQUALITY (general case, [9]). *Let E/F be a quadratic extension of number fields. Denote by r_1 (resp. r_2) the number of real (resp.*

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imaginary) places of F. Also denote by t (resp. ϱ) the number of finite (resp. infinite) places of F which ramify in E. If

$$t \geq r_1 + r_2 - \varrho + 3 + 2\sqrt{2(r_1 + r_2) - \varrho + 1},$$

then the 2-class field tower of E is infinite.

MARTINET'S INEQUALITY I. *Let F be a totally real number field of degree n, and E a totally imaginary quadratic extension of F. Let t be the number of prime ideals of F which ramify in E. If*

$$t \geq 3 + 2\sqrt{n + 1},$$

then the 2-class field tower of E is infinite.

Proof. Since $r_1 = \varrho = n$ and $r_2 = 0$, the assertion follows from the general case of Martinet's inequality.

COROLLARY 1. *Let F be a real quadratic number field. Suppose that four rational primes split in F and ramify in K, or that a rational prime remains prime in F and three other rational primes split in F and these four rational primes ramify in K. Then the 2-class field tower of $E = FK$ is infinite.*

Proof. Since $n = 2$ and $t \geq 7 \geq 3 + 2\sqrt{2 + 1} = 6.464\ldots$ in these cases, the 2-class field tower of $E = FK$ is infinite by Martinet's inequality I.

COROLLARY 2. *Let F be a totally real number field of degree 4. Suppose that two rational primes split completely in F and ramify in K, or that a rational prime splits completely in F and two other rational primes are unramified and split into at least two primes in F and these three rational primes ramify in K. Then the 2-class field tower of $E = FK$ is infinite.*

Proof. Since $n = 4$ and $t \geq 8 \geq 3 + 2\sqrt{4 + 1} = 7.472\ldots$ in these cases, the 2-class field tower of $E = FK$ is infinite by Martinet's inequality I.

MARTINET'S INEQUALITY II. *Let F be a totally imaginary number field of degree n, and E a quadratic extension of F. Let t be the number of prime ideals of F which ramify in E. If*

$$t \geq n/2 + 3 + 2\sqrt{n + 1},$$

then the 2-class field tower of E is infinite.

Proof. Since $r_1 = \varrho = 0$ and $r_2 = n/2$, the assertion follows from the general case of Martinet's inequality.

COROLLARY 3. *Let F be an imaginary quadratic number field. Suppose that four rational primes split in F and ramify in K. Then the 2-class field tower of $E = FK$ is infinite.*

Proof. Since $n = 2$ and $t \geq 8 \geq 2/2 + 3 + 2\sqrt{2 + 1} = 7.464\ldots$, the 2-class field tower of $E = FK$ is infinite by Martinet's inequality II.

3. The case with one negative prime discriminant. Let K be an imaginary quadratic number field with discriminant d. First we recall some properties of Rédei matrices of quadratic number fields [10, 11, 13, 14].

A rational integer is called a *discriminant* if it is the discriminant of a quadratic number field or equal to 1. A discriminant which is divisible by only one prime is called a *prime discriminant*. Prime discriminants are denoted by $p^* = (-1)^{(p-1)/2}p$ (if p is an odd prime), or $p^* = -4$, 8 or -8 (if p is equal to 2). Let $d = p_1^* p_2^* \ldots p_t^*$ be the unique factorization of d into a product of prime discriminants. By genus theory, we have 2-rank $C_K = t-1$.

Using Kronecker symbols $\left(\frac{D}{p}\right)$, where D is a discriminant and p is a prime number satisfying $p \nmid D$, we define the Rédei matrix $R_K = (a_{ij}) \in \mathbf{M}_{t\times t}(\mathbb{Z}/2\mathbb{Z})$ of K by

$$(-1)^{a_{ij}} = \begin{cases} \left(\dfrac{p_i^*}{p_j}\right) & (i \neq j), \\ \left(\dfrac{d/p_i^*}{p_i}\right) & (i = j). \end{cases}$$

By the definition of the Kronecker symbol, we have $a_{ij} = 0$ $(i \neq j)$ if and only if the rational prime p_j splits in $\mathbb{Q}(\sqrt{p_i^*})$. Note that the sum of all row vectors of R_K is equal to the zero vector $\mathbf{0}$ in $(\mathbb{Z}/2\mathbb{Z})^t$ so that rank $R_K \leq t-1$ and the solution space X of the linear equations $\boldsymbol{x}R_K = \mathbf{0}$ $(\boldsymbol{x} \in (\mathbb{Z}/2\mathbb{Z})^t)$ contains the vector $\mathbf{1} = (1, 1, \ldots, 1)$ (t times). By the results of Rédei and Rédei–Reichardt, we have 4-rank $C_K = t - 1 - \operatorname{rank} R_K$.

In the case where $p_i^* \neq -4$ and $p_j^* \neq -4$, we have $a_{ij} = a_{ji}$ $(i \neq j)$ if and only if $p_i^* > 0$ or $p_j^* > 0$, by the quadratic reciprocity law. Therefore, if exactly one negative prime discriminant $(\neq -4)$ divides d, then R_K is a symmetric matrix.

THEOREM. *Let K be an imaginary quadratic number field with discriminant d. Suppose that* 2-rank $C_K = 4$ *and exactly one negative prime discriminant divides d. Let $d = p_1^* p_2^* p_3^* p_4^* p_5^*$ $(p_1^* < 0)$ be the unique factorization of d into a product of prime discriminants. Then the* 2*-class field tower of K is infinite*, *except the case where*

$$R_K = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix},$$

by changing the order of p_i's $(2 \leq i \leq 5)$. In the exceptional case, $p_1^* \neq -4$ and the 4*-rank of C_K is equal to* 0.

Proof. First, suppose that $p_1^* = -4$. Then we have $p_j \equiv 1 \pmod 4$ for any j $(2 \leq j \leq 5)$. Put $F = \mathbb{Q}(\sqrt{p_1^*}) = \mathbb{Q}(\sqrt{-1})$. Then the four rational primes p_j $(2 \leq j \leq 5)$ split in F and ramify in K. Hence, the 2-class field tower of $E = FK = K(\sqrt{-1})$ is infinite by Corollary 3. Since E/K is an unramified 2-extension, the 2-class field tower of K is also infinite.

In the following, we assume that $p_1^* \neq -4$. Therefore, R_K is a symmetric matrix. For each Rédei matrix R_K, if we could find a subfield $F = \mathbb{Q}(\sqrt{p_i}, \sqrt{p_j})$, $\mathbb{Q}(\sqrt{p_i}, \sqrt{p_j p_k})$ or $\mathbb{Q}(\sqrt{p_i p_j}, \sqrt{p_i p_k})$ $(i, j, k \in \{2, 3, 4, 5\})$ of the genus field $\mathbb{Q}(\sqrt{p_1^*}, \ldots, \sqrt{p_5^*})$ of K which satisfies the condition of Corollary 2, then the 2-class field tower of $E = FK$ would be infinite. Since E/K is an unramified 2-extension, we conclude that the 2-class field tower of K is also infinite, in those cases.

First, suppose that there exists a column vector $\boldsymbol{a}_j = (a_{ij})$ $(1 \leq j \leq 5)$ of R_K for which at least two of a_{ij}'s $(2 \leq i \leq 5,\ i \neq j)$ are 0. Then, assuming that $a_{ij} = a_{kj} = 0$ $(i \neq j, k \neq j)$, we put $F = \mathbb{Q}(\sqrt{p_i}, \sqrt{p_k})$. Since the rational prime p_j splits completely in F and ramifies in K, and two rational primes p_l, p_m $(\{i, j, k, l, m\} = \{1, 2, 3, 4, 5\})$ are unramified and split into at least two primes in F and ramify in K, the 2-class field tower of $E = FK$ is infinite by Corollary 2. Hence, the 2-class field tower of K is also infinite.

In the following, we assume that at most one of a_{ij}'s $(2 \leq i \leq 5, i \neq j)$ is 0 for each column vector $\boldsymbol{a}_j = (a_{ij})$ $(1 \leq j \leq 5)$ of R_K.

(i) *One of* a_{i1}*'s* $(2 \leq i \leq 5)$ *is* 0: In this case, we may assume that $a_{21} = 0$ and $a_{31} = a_{41} = a_{51} = 1$ without loss of generality. If $a_{32} = a_{42} = a_{52} = 1$, then we put $F = \mathbb{Q}(\sqrt{p_3 p_4}, \sqrt{p_3 p_5})$. Since $\left(\frac{p_i p_k}{p_j}\right) = (-1)(-1) = 1$ for each $j \in \{1, 2\}$ and $i, k \in \{3, 4, 5\}$, the two rational primes p_1 and p_2 split completely in F and ramify in K. Therefore, the 2-class field tower of $E = FK$ is infinite by Corollary 2, and the 2-class field tower of K is also infinite. On the other hand, if one of a_{i2}'s $(3 \leq i \leq 5)$ is 0, then we may assume that $a_{32} = 0$ and $a_{42} = a_{52} = 1$ without loss of generality. So, we have $a_{23} = 0$ and $a_{43} = a_{53} = 1$ by our assumption and

$$R_K = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & * & * \\ 1 & 1 & 1 & * & * \end{pmatrix},$$

where the asterisks "$*$" mean 0 or 1. We put $F = \mathbb{Q}(\sqrt{p_2}, \sqrt{p_4 p_5})$. Since $\left(\frac{p_2}{p_j}\right) = 1$ and $\left(\frac{p_4 p_5}{p_j}\right) = (-1)(-1) = 1$ for $j \in \{1, 3\}$, the two rational primes p_1 and p_3 split completely in F and ramify in K. Therefore, the 2-class field tower of $E = FK$ is infinite by Corollary 2, and the 2-class field tower of K is also infinite.

(ii) $a_{21} = a_{31} = a_{41} = a_{51} = 1$: If there exists a column vector $\boldsymbol{a}_j = (a_{ij})$ $(2 \le j \le 5)$ of R_K satisfying $a_{ij} = 1$ for all i $(2 \le i \le 5,\ i \ne j)$, then we put

$$F = \mathbb{Q}(\sqrt{p_k p_l}, \sqrt{p_k p_m}) \quad (\{j, k, l, m\} = \{2, 3, 4, 5\}).$$

In this case, as in the first half of the case (i), we see that the two rational primes p_1 and p_j split completely in F and ramify in K. Therefore, the 2-class field tower of $E = FK$ is infinite by Corollary 2, and the 2-class field tower of K is also infinite. However, if there exists no such column vector $\boldsymbol{a}_j$ $(2 \le j \le 5)$ of R_K, then we cannot find an appropriate field F which satisfies the condition of Martinet's inequality. In this case, we have $a_{23} = a_{32} = a_{45} = a_{54} = 0$, by changing the order of p_i's. So, R_K is as described in the assertion of our Theorem. This completes the proof of the Theorem.

Remark 1. In Theorem 1 of [1], Benjamin classified the case with only one negative prime discriminant ($\ne -4$) and 2-rank $C_K = 4$ into 32 types, by using "Kronecker symbol configurations". Among them, the infinitude of the 2-class field tower remained unsettled for 5 types. Actually, there are two more Kronecker symbol configurations

$$\left(\frac{p_1}{p_3}\right) = \left(\frac{p_1}{p_4}\right) = \left(\frac{p_2}{p_3}\right) = \left(\frac{p_2}{p_4}\right) = 1$$

with 4-rank $C_K = 0$, and

$$\left(\frac{p_1}{p_3}\right) = \left(\frac{p_1}{p_4}\right) = \left(\frac{p_2}{p_3}\right) = \left(\frac{p_2}{p_4}\right) = -1$$

with 4-rank $C_K = 2$. The numbers of Rédei matrices (= the numbers of Kronecker symbol configurations) with given 4-rank are as follows:

4-rank C_K	4	3	2	1	0	total
# of Rédei matrices	1	2	8	10	13	34

In our Theorem, we showed the infinitude of the 2-class field tower for 33 types, except the third case of Theorem 1(C) in [1] where p_1^* is negative ($\ne -4$) and

$$\left(\frac{p_2}{p_3}\right) = \left(\frac{p_4}{p_5}\right) = 1.$$

Examples. The following are some examples of imaginary quadratic number fields with only one negative prime discriminant ($\ne -4$), 2-rank C_K

$= 4$ and the Rédei matrix of exceptional type:

$$\mathbb{Q}(\sqrt{-3\cdot 5\cdot 29\cdot 8\cdot 17}),\qquad \mathbb{Q}(\sqrt{-7\cdot 5\cdot 41\cdot 13\cdot 17}),$$

$$\mathbb{Q}(\sqrt{-3\cdot 5\cdot 41\cdot 17\cdot 53}),\qquad \mathbb{Q}(\sqrt{-8\cdot 5\cdot 61\cdot 37\cdot 53}).$$

REMARK 2. In Theorems 1 and 2 of [2], Benjamin proved the infinitude of the 2-class field tower of an imaginary quadratic number field K with 2-rank $C_K = 4$ and 4-rank $C_K = 2$, in the case where R_K is not of the type

$$\begin{pmatrix} * & 1 & 1 & 0 & 0 \\ * & 1 & 1 & 1 & 1 \\ * & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & * & * \\ 1 & 1 & 1 & * & * \end{pmatrix}$$

with

$$p_1^* = -4,\quad p_2^* < 0,\quad p_3^* < 0,\quad p_4^* > 0,\quad p_5^* > 0.$$

With the methods above one can prove the theorems of Benjamin and Koch–Hajir as well.

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