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Lattice points in a circle: An improved mean-square asymptotics

by

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1. Introduction. Let as usual

$$r(n) := \#\{(u,v) \in \mathbb{Z}^2 : u^2 + v^2 = n\} \tag{1.1}$$

denote the number of ways to write the integer n as a sum of two squares. Then the classic circle problem, which goes back to C. F. Gauß, is concerned with the asymptotic behaviour of the quantity

$$P(x) := \sum_{0 \le n \le x} r(n) - \pi x. \tag{1.2}$$

In other words, this is the *lattice point discrepancy* of a compact, origin-centered circular disc with radius $\sqrt{x}$. For detailed and enlightening expositions of the rich history of this topic, the reader should consult the monographs of E. Krätzel [10], [11].

The sharpest upper bound has been established quite recently by M. Huxley [7] (as a slight improvement on Huxley [6]) and reads

$$P(x) \ll x^{131/416} (\log x)^{18637/8320}, \tag{1.3}$$

where $131/416 = 0.3149\ldots$ Concerning lower bounds, significant progress has just been achieved by K. Soundararajan [20] who proved that

$$P(x) \neq o(x^{1/4} (\log x)^{1/4} (\log\log x)^{3(2^{1/3}-1)/4} (\log\log\log x)^{-5/8}). \tag{1.4}$$

For a longer time, it has been known that

$$\liminf_{x\to\infty} \left(\frac{P(x)}{x^{1/4}(\log x)^{1/4}\omega_1(x)} \right) < 0, \qquad \limsup_{x\to\infty} \left(\frac{P(x)}{x^{1/4}\omega_2(x)} \right) > 0, \tag{1.5}$$

with

$$\omega_1(x) := (\log\log x)^{(\log 2)/4} \exp(-c_1 (\log\log\log x)^{1/2}),$$
$$\omega_2(x) := \exp(c_2 (\log\log x)^{1/4} (\log\log\log x)^{-3/4}),$$

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c_1, c_2 appropriate positive constants. These results are due to J. L. Hafner [4], resp. K. Corrádi and I. Kátai [2].

It is usually conjectured that $x^{1/4}$, as it appears in (1.4), essentially meets the "true" order of $P(x)$, i.e., that

$$\inf\{\lambda : P(x) \ll_\lambda x^\lambda\} = 1/4.$$

In favour of this hypothesis, there are quite precise mean-square asymptotics of the shape

$$\int_0^X (P(x))^2\,dx = CX^{3/2} + Q(X), \tag{1.6}$$

with

$$C = \frac{1}{3\pi^2}\sum_{n=1}^{\infty} r^2(n) n^{-3/2} = \frac{16}{3\pi^2}\,\frac{\zeta^2(3/2)L^2(3/2)}{\zeta(3)}\,(1+2^{-3/2})^{-1} \approx 1.69396. \tag{1.7}$$

Here and throughout, $L(s)$ denotes the L-series corresponding to the non-principal character modulo 4.

The estimation of the remainder $Q(X)$ has been subject of intensive research, by increasingly sophisticated methods. We mention the results of H. Cramér [3]: $Q(X) \ll X^{5/4+\varepsilon}$, E. Landau [12]: $Q(X) \ll X^{1+\varepsilon}$, and A. Walfisz [21]: $Q(X) \ll X(\log X)^3$. The sharpest bound to date is due to I. Kátai [9] and reads

$$Q(X) \ll X(\log X)^2. \tag{1.8}$$

More recently, E. Preissmann [17] found a short and elegant proof for this result, using a deep inequality of Montgomery and Vaughan. (See Lemma B below.)

The objective of the present note is to obtain a further reduction of the log-exponent in the estimate (1.8).

THEOREM. *The error term $Q(X)$ defined by* (1.6), (1.7) *satisfies, as* $X \to \infty$,

$$Q(X) \ll X(\log X)^{3/2}\log\log X\,.$$

REMARKS. 1. Compared to the previous authors cited, we simplify the analysis by an approach suggested by T. Meurman for the divisor problem [14]. The basic idea of our refinement is that most integers are *not* representable as a sum of two squares, and that the so-called $\mathfrak{B}$-numbers (i.e., those with $r(n) > 0$) are usually well-spaced—i.e., "$\mathfrak{B}$-twins" are still less frequent. This fact permits a small extra saving when applying the Montgomery–Vaughan bound with care.

2. It may be instructive to report the present state-of-the-art with the analogous error term $Q_\Delta(X)$ corresponding to the Dirichlet divisor problem. According to E. Preissmann [17], it is known that $Q_\Delta(X) \ll X(\log X)^4$, while Y.-K. Lau and K.-M. Tsang [13] recently proved that $Q_\Delta(X)$ is not a $o(X(\log X)^2)$. This might suggest that our result is not far away from the best possible bound.

2. Some auxiliary results

NOTATION. Variables of summation automatically range over all integers satisfying the conditions indicated. p denotes primes throughout, and $\mathbb{P}$ is the set of *all* (rational) primes. For any subset $\mathfrak{P} \subseteq \mathbb{P}$, we write $\mathfrak{D}(\mathfrak{P})$ for the set of all positive integers m whose prime divisors are all in $\mathfrak{P}$. All constants implied in the symbols $O(\cdot)$, $\ll$, $\gg$, etc., are absolute, and ε denotes a sufficiently small positive constant.

LEMMA A. *Suppose that $x \geq 1$, $x \notin \mathbb{Z}$, and $x \leq M \leq x^A$, where $A > 1$ is some fixed constant. Denote by $\|\cdot\|$ the distance from the nearest integer. Then, for every $\varepsilon > 0$,*

$$P(x) = \sqrt{x} \sum_{1 \leq n < M} r(n) n^{-1/2} J_1(2\pi\sqrt{nx}) + O(\min(x^{5/4}M^{-1/2} + x^{1/2+\varepsilon}M^{-1/2}\|x\|^{-1} + x^{1/4}M^{-1/4}, x^{\varepsilon}))$$

where J_1 is a Bessel function.

Proof. This is Lemma 1 in A. Ivić [8], combined with his eq. (1.7). ∎

LEMMA B. *For an arbitrary finite index set $\mathcal{J}$, let $(a_j)_{j\in\mathcal{J}}$ be a complex sequence and let $(\lambda_j)_{j\in\mathcal{J}}$ be a sequence of pairwise distinct reals. Write*

$$\delta_j := \min_{k\in\mathcal{J},\, k\neq j} |\lambda_k - \lambda_j|.$$

Then, for arbitrary real T_0 and $T > 0$,

$$\int_{T_0}^{T_0+T} \Big|\sum_{j\in\mathcal{J}} a_j \exp(i\lambda_j t)\Big|^2 dt = T \sum_{j\in\mathcal{J}} |a_j|^2 + O\bigg(\sum_{j\in\mathcal{J}} \frac{|a_j|^2}{\delta_j}\bigg),$$

where the O-constant is absolute.

Proof. This is an obvious variant of Corollary 2 in H. L. Montgomery and R. C. Vaughan [15]. ∎

LEMMA C. *For each prime power p^α, $\alpha \geq 1$, let $\overline{\Omega}(p^\alpha)$ be a set of distinct residue classes $\bar{\mathfrak{c}}$ modulo p^α. Define further*

$$\Omega(p^\alpha) = \Big\{n \in \mathbb{Z}_+ : n \in \bigcup_{\bar{\mathfrak{c}}\in\overline{\Omega}(p^\alpha)} \bar{\mathfrak{c}}\Big\}, \quad \theta(p^\alpha) = 1 - \sum_{j=1}^{\alpha} \frac{\#\overline{\Omega}(p^j)}{p^j}, \quad \theta(1) = 1.$$

Suppose that $\Omega(p^\alpha)\cap\Omega(p^\beta)=\emptyset$ *for all primes* p *and positive integers* $\alpha\neq\beta$. *For real* $x>0$, *let finally*

$$A(x)=\Big\{n\in\mathbb{Z}_+ : n\le x \ \textit{and}\ n\notin \bigcup_{p\in\mathbb{P},\,\alpha\in\mathbb{Z}_+}\Omega(p^\alpha)\Big\}.$$

Then, *for arbitrary real* $D>1$,

$$\#A(x)\le\frac{x+D^2}{V_D},\quad \textit{where}\quad V_D:=\sum_{0<d<D}\prod_{p^\alpha\|d}\left(\frac{1}{\theta(p^\alpha)}-\frac{1}{\theta(p^{\alpha-1})}\right).$$

Proof. This is a deep sieve theorem due to A. Selberg [19]. It can be found in Y. Motohashi [16, p. 11, Theorem 2], and also in T. Cochrane and R. E. Dressler [1]. ∎

LEMMA D. *Let* $(a_n)_{n\in\mathbb{Z}_+}$ *be a sequence of nonnegative reals*, *and suppose that the Dirichlet series*

$$f(s)=\sum_{n=1}^{\infty}a_n\,n^{-s}$$

converges for $\Re(s)>1$. *Assume further that*, *for some real constants* A *and* $\gamma>0$,

$$f(s)=(A+o(1))(s-1)^{-\gamma}$$

as $s\to 1+$. *Then*, *as* $x\to\infty$,

$$\sum_{1\le n\le x}\frac{a_n}{n}=\left(\frac{A}{\Gamma(1+\gamma)}+o(1)\right)(\log x)^{\gamma}.$$

Proof. This is a standard Tauberian theorem. For the present formulation, cf. T. Cochrane and R. E. Dressler [1, Lemma B]. ∎

LEMMA E. *Let as usual*

$$\mathfrak{B}=\{n\in\mathbb{Z}: r(n)>0\},$$

and let $\mathbf{b}:\mathbb{Z}\to\{0,1\}$ *denote the indicator function of* $\mathfrak{B}$. *Then*, *for each integer* $n>0$,

$$r^2(n)\le\sum_{\substack{km=n\\k,m>0}}r(k)r(m),\tag{2.1}$$

and

$$r(n)\le 4\sum_{\substack{km=n\\k,m>0}}\mathbf{b}(k)\mathbf{b}(m).\tag{2.2}$$

Proof. Recall the explicit formula for $r(n)$ (cf., e.g., [5, p. 60]): $\frac{1}{4}r(n)$ (and hence also $\mathbf{b}(n)$) is multiplicative; for p prime and any integer $k \geq 0$,

$$\frac{1}{4}\, r(p^k) = \begin{cases} k+1 & \text{if } p \equiv 1 \bmod 4, \\ 0 & \text{if } p \equiv 3 \bmod 4 \text{ and } k \text{ is odd}, \\ 1 & \text{if } p \equiv 3 \bmod 4 \text{ and } k \text{ is even}, \\ 1 & \text{if } p = 2. \end{cases} \tag{2.3}$$

By multiplicativity, it suffices to verify (2.1) and (2.2) for prime powers. The only case not completely obvious is that of (2.1) for $p \equiv 1 \bmod 4$, where we have to show that

$$(k+1)^2 \leq \sum_{j=0}^{k} (j+1)(k-j+1).$$

Since the right hand side equals

$$\frac{1}{6}\,(k+1)(k+2)(k+3) = (k+1)^2 + \frac{1}{6}\, k(k+1)(k-1),$$

this is clear as well. ∎

Lemma F. *Let ϕ denote the Euler totient function. Then, for $y \geq 2$,*

$$\sum_{0<n\leq y} \left(\frac{n}{\phi(n)}\right)^2 \frac{\mathbf{b}(n)}{n} \ll (\log y)^{1/2}, \tag{2.4}$$

$$\sum_{0<n\leq y} \left(\frac{n}{\phi(n)}\right)^2 \frac{r(n)}{n} \ll \log y. \tag{2.5}$$

Proof. Recall that, for $\Re(s) > 1$, the Dedekind zeta-function of the Gaussian field satisfies

$$\zeta_{\mathbb{Q}(i)}(s) = \sum_{n=1}^{\infty} \frac{1}{4}\, r(n) n^{-s} = g_1(s) \prod_{p\equiv 1 \bmod 4} (1 - p^{-s})^{-2};$$

here and throughout, $g_1(s), g_2(s), \ldots$ denote functions which are holomorphic (at least) in the closed half-plane $\Re(s) \geq 1$. Hence, for $\Re(s) > 1$,

$$\begin{aligned} f_1(s) &:= \sum_{n=1}^{\infty} \left(\frac{n}{\phi(n)}\right)^2 \mathbf{b}(n) n^{-s} \\ &= g_2(s) \prod_{p\equiv 1 \bmod 4} \left(1 + \left(1 - \frac{1}{p}\right)^{-2} \sum_{k=1}^{\infty} p^{-ks}\right) \\ &= g_3(s)(\zeta_{\mathbb{Q}(i)}(s))^{1/2} \\ &\quad \times \prod_{p\equiv 1 \bmod 4} \left\{\left(1 + \left(1 - \frac{1}{p}\right)^{-2} \frac{p^{-s}}{1-p^{-s}}\right)(1 - p^{-s})\right\} \\ &= g_3(s)(\zeta_{\mathbb{Q}(i)}(s))^{1/2} \prod_{p\equiv 1 \bmod 4} \left(1 + \frac{2p-1}{(p-1)^2}\, p^{-s}\right). \end{aligned}$$

Since the last product converges absolutely for $\Re(s) > 0$, it follows that $f_1(s) = g_4(s)(\zeta_{\mathbb{Q}(i)}(s))^{1/2}$ in $\Re(s) > 1$, hence

$$f_1(s) = \left(g_4(1)\,\frac{\sqrt{\pi}}{2} + o(1)\right)(s-1)^{-1/2} \qquad \text{as } s \to 1+.$$

Thus Lemma D immediately implies (2.4).

To establish (2.5), we similarly consider, for $\Re(s) > 1$,

$$\begin{aligned} f_2(s) &:= \sum_{n=1}^{\infty} \left(\frac{n}{\phi(n)}\right)^2 \frac{1}{4}\, r(n) n^{-s} \\ &= g_5(s) \prod_{p \equiv 1 \bmod 4} \left(1 + \left(1 - \frac{1}{p}\right)^{-2} \sum_{k=1}^{\infty} (k+1) p^{-ks}\right) \\ &= g_6(s)\zeta_{\mathbb{Q}(i)}(s) \\ &\quad \times \prod_{p \equiv 1 \bmod 4} \left\{\left(1 + \left(1 - \frac{1}{p}\right)^{-2} \frac{p^{-s}(2-p^{-s})}{(1-p^{-s})^2}\right)(1-p^{-s})^2\right\}. \end{aligned}$$

The last product (call it $H(s)$) converges absolutely for $s = 1$, namely

$$H(1) = \prod_{p \equiv 1 \bmod 4} \left(1 + \frac{(2p-1)^2}{p^2(p-1)^2}\right).$$

Thus

$$f_2(s) = \left(g_6(1)H(1)\,\frac{\pi}{4} + o(1)\right)(s-1)^{-1} \qquad \text{as } s \to 1+,$$

and one more appeal to Lemma D completes the proof of Lemma F. ∎

3. Sums over $\mathfrak{B}$-twins

Proposition 1. *For integers $k > 0$ and $h \neq 0$, and large real x,*

$$S_0(k,h;x) := \sum_{0<n\le x} \mathbf{b}(n)\mathbf{b}(kn+h) \ll \left(\frac{k|h|}{\phi(k|h|)}\right)^2 \frac{x}{\log x}, \tag{3.1}$$

where ϕ is the Euler totient function.

Proof. For $k = 1$ this is a classic and celebrated result of G. J. Rieger [18]. Instead of working out his argument for the general case, we prefer to follow the approach of T. Cochrane and R. E. Dressler [1] who used a deeper theorem of Selberg's (our Lemma C) to deal with the problem of $\mathfrak{B}$-triples. Let

$$\mathfrak{P} = \mathfrak{P}_{k,h} = \{p \in \mathbb{P} : p \equiv 3 \bmod 4 \text{ and } p \nmid kh\}.$$

Further, for $p \in \mathfrak{P}$, $\alpha \in \mathbb{Z}_+$, denote by $\overline{k^*_{p^\alpha}}$ the residue class modulo p^α for which $\overline{k}\,\overline{k^*_{p^\alpha}} = \overline{1}$. In the notation of Lemma C, we choose

$$\overline{\Omega}(p^\alpha) := \{\overline{jp^{\alpha-1}} : j = 1, \ldots, p-1\} \cup \{\overline{k^*_{p^\alpha}(jp^{\alpha-1} - h)} : j = 1, \ldots, p-1\}$$

if and only if

$$(*) \qquad p \in \mathfrak{P} \text{ and } \alpha \text{ is an even positive integer,}$$

and $\overline{\Omega}(p^\alpha) := \emptyset$ in all other cases. Thus it is easy to see that, in the case $(*)$, $\#\overline{\Omega}(p^\alpha) = 2(p-1)$, and

$$n \in \Omega(p^\alpha) \ \Rightarrow\ (p^{\alpha-1} \,\|\, n \text{ or } p^{\alpha-1} \,\|\, (kn+h)) \ \Rightarrow\ \mathbf{b}(n)\mathbf{b}(kn+h) = 0.$$

Therefore, in the terminology of Lemma C,

$$S_0(k, h; x) \le \# A(x) \le \frac{x + D^2}{V_D} \le \frac{2x}{V_D}, \tag{3.2}$$

if we choose $D := \sqrt{x}$. To find a lower bound for V_D, we note that

$$\theta(p^\alpha) = 1 - 2(p-1) \sum_{0 < 2\beta \le \alpha} p^{-2\beta} \qquad \text{for } p \in \mathfrak{P},$$

hence

$$\theta(p^\alpha) = \begin{cases} 1 - \dfrac{2}{p+1} + \dfrac{2}{p^\alpha(p+1)} & \text{for } p \in \mathfrak{P} \text{ and } \alpha > 0 \text{ even,} \\ \theta(p^{\alpha-1}) & \text{for } p \in \mathfrak{P} \text{ and } \alpha \text{ odd,} \\ 1 & \text{if } p \notin \mathfrak{P}. \end{cases} \tag{3.3}$$

Therefore,

$$\frac{1}{\theta(p^\alpha)} - \frac{1}{\theta(p^{\alpha-1})} = \begin{cases} \dfrac{2p^\alpha(p+1)(p^2-1)}{(p^\alpha(p-1)+2)(p^\alpha(p-1)+2p^2)} \\ \qquad \text{if } p \in \mathfrak{P} \text{ and } \alpha > 0 \text{ even,} \\ 0 \quad \text{in all other cases.} \end{cases}$$

Consequently, in the sum defining V_D, we can restrict d to perfect squares, and moreover to the set $\mathfrak{D}(\mathfrak{P})$. Thus

$$\begin{aligned} V_D &= \sum_{\substack{0 < d_1 < \sqrt{D} \\ d_1 \in \mathfrak{D}(\mathfrak{P})}} \prod_{p^\beta \| d_1} \left(\frac{1}{\theta(p^{2\beta})} - \frac{1}{\theta(p^{2\beta-2})} \right) \\ &\ge \sum_{\substack{0 < d_1 < \sqrt{D} \\ d_1 \in \mathfrak{D}(\mathfrak{P})}} \mu^2(d_1) \prod_{p \mid d_1} \left(\frac{1}{\theta(p^2)} - 1 \right), \end{aligned}$$

where $\mu(\cdot)$ is the Möbius function. By (3.3),

$$\frac{1}{\theta(p^2)} - 1 = \frac{2(p-1)}{p^2 - 2p + 2} \ge \frac{2}{p},$$

hence

$$V_D \geq \sum_{\substack{0<d_1<\sqrt{D}\\ d_1\in\mathfrak{D}(\mathfrak{P})}} \mu^2(d_1) \prod_{p\mid d_1} \frac{2}{p} = \sum_{\substack{0<d_1<\sqrt{D}\\ d_1\in\mathfrak{D}(\mathfrak{P})}} \mu^2(d_1)\, \frac{2^{\omega(d_1)}}{d_1}, \tag{3.4}$$

where $\omega(m)$ denotes the number of (distinct) prime divisors of $m \in \mathbb{Z}_+$. Let

$$\mathfrak{P}' := \{p \in \mathbb{P} :\ p \equiv 3 \bmod 4 \text{ and } p \mid kh\}, \qquad \gamma(k,h) := \prod_{p\in\mathfrak{P}'} \left(1 + \frac{2}{p}\right).$$

Obviously,

$$\gamma(k,h) \leq \left(\frac{k|h|}{\phi(k|h|)}\right)^2. \tag{3.5}$$

Furthermore, by (3.4),

$$\begin{aligned} \gamma(k,h)V_D &\geq \sum_{m_1\in\mathfrak{D}(\mathfrak{P}')} \mu^2(m_1)\, \frac{2^{\omega(m_1)}}{m_1} \sum_{\substack{m_2<\sqrt{D}\\ m_2\in\mathfrak{D}(\mathfrak{P})}} \mu^2(m_2)\, \frac{2^{\omega(m_2)}}{m_2} \\ &\geq \sum_{\substack{m<\sqrt{D}\\ m\in\mathfrak{D}(\mathbb{P}_3)}} \mu^2(m)\, \frac{2^{\omega(m)}}{m}, \end{aligned} \tag{3.6}$$

where

$$\mathbb{P}_3 := \{\, p \in \mathbb{P} \ :\ p \equiv 3 \bmod 4\}.$$

To estimate the last expression in (3.6), we consider the generating function, for $\Re(s) > 1$,

$$f(s) = \prod_{p\in\mathbb{P}_3} \left(1 + \frac{2}{p^s}\right) = \sum_{m\in\mathfrak{D}(\mathbb{P}_3)} \mu^2(m)\, 2^{\omega(m)}\, m^{-s}.$$

Evidently,

$$f(s) = g_7(s)\, \frac{\zeta(s)^2}{\zeta_{\mathbb{Q}(i)}(s)} = g_7(s)\, \frac{\zeta(s)}{L(s)},$$

where $g_7(s)$ is holomorphic in $\Re(s) \geq 1$, and $g_7(1) \neq 0$. Hence, for $s \to 1+$,

$$f(s) \sim g_7(1)\, \frac{4}{\pi}\, (s-1)^{-1}.$$

Thus Lemma D implies that

$$\sum_{\substack{m<\sqrt{D}\\ m\in\mathfrak{D}(\mathbb{P}_3)}} \mu^2(m)\, \frac{2^{\omega(m)}}{m} \gg \log D.$$

Together with (3.6) and (3.5), this implies that

$$V_D \gg \left(\frac{k|h|}{\phi(k|h|)}\right)^{-2} \log D.$$

Recalling (3.2) and our choice $D = \sqrt{x}$, we complete the proof of Proposition 1. ∎

PROPOSITION 2. *For integers $k > 0$ and $h \neq 0$, and large real x,*

$$\begin{aligned} S_1(k, h; x) &:= \sum_{0<n<x} r(n)\mathbf{b}(kn + h) \\ &\ll \left(\frac{k|h|}{\phi(k|h|)}\right)^2 x(\log x)^{-1/2}. \end{aligned} \tag{3.7}$$

Furthermore, for each integer $h \neq 0$,

$$S_2(h, x) := \sum_{0<n<x} r^2(n)\, \mathbf{b}(n + h) \ll \left(\frac{|h|}{\phi(|h|)}\right)^2 x(\log x)^{1/2}. \tag{3.8}$$

Proof. By (2.2) and a crude form of the *hyperbola method*,

$$\begin{aligned} S_1(k, h; x) &\le 4 \sum_{\substack{0<n_1n_2<x \\ n_1,n_2>0}} \mathbf{b}(n_1)\mathbf{b}(n_2)\mathbf{b}(kn_1n_2 + h) \\ &\le 8 \sum_{0<n_1<\sqrt{x}} \mathbf{b}(n_1) \sum_{0<n_2\le x/n_1} \mathbf{b}(n_2)\mathbf{b}(kn_1n_2 + h) \\ &= 8 \sum_{0<n_1<\sqrt{x}} \mathbf{b}(n_1) S_0\left(kn_1, h; \frac{x}{n_1}\right). \end{aligned}$$

Hence, by Proposition 1 and (2.4) of Lemma F,

$$\begin{aligned} S_1(k, h; x) &\ll \sum_{0<n_1<\sqrt{x}} \mathbf{b}(n_1)\left(\frac{kn_1|h|}{\phi(kn_1|h|)}\right)^2 \frac{x}{n_1} \left(\log\left(\frac{x}{n_1}\right)\right)^{-1} \\ &\ll \left(\frac{k|h|}{\phi(k|h|)}\right)^2 \frac{x}{\log x} \sum_{0<n_1<\sqrt{x}} \left(\frac{n_1}{\phi(n_1)}\right)^2 \frac{\mathbf{b}(n_1)}{n_1} \\ &\ll \left(\frac{k|h|}{\phi(k|h|)}\right)^2 x(\log x)^{-1/2}, \end{aligned}$$

which establishes (3.7).

Similarly, in order to show (3.8), we conclude by (2.1) that

$$\begin{aligned} S_2(h,x) &\le \sum_{\substack{0<mk<x\\ m,k>0}} r(m)\,r(k)\mathbf{b}(km+h) \\ &\le 2\sum_{0<k<\sqrt{x}} r(k) \sum_{0<m\le x/k} r(m)\mathbf{b}(km+h) \\ &= 2\sum_{0<k<\sqrt{x}} r(k) S_1\Big(k,h;\frac{x}{k}\Big). \end{aligned}$$

Therefore, by (3.7),

$$\begin{aligned} S_2(h,x) &\ll \sum_{0<k<\sqrt{x}} r(k)\Big(\frac{k|h|}{\phi(k|h|)}\Big)^2 \frac{x}{k}\left(\log\left(\frac{x}{k}\right)\right)^{-1/2} \\ &\ll \left(\frac{|h|}{\phi(|h|)}\right)^2 x(\log x)^{-1/2} \sum_{0<k<\sqrt{x}} \left(\frac{k}{\phi(k)}\right)^2 \frac{r(k)}{k} \\ &\ll \left(\frac{|h|}{\phi(|h|)}\right)^2 x(\log x)^{1/2}, \end{aligned}$$

in view of (2.5). This completes the proof of Proposition 2. ∎

4. Proof of the Theorem

PROPOSITION 3. *For $X \ge 2$ and $\frac{1}{2}X \le x \le X$, define*

$$H(X,x) := \frac{1}{\pi}\, x^{1/4} \sum_{1\le n<X^5} \frac{r(n)}{n^{3/4}} \cos\left(2\pi\sqrt{nx} - \frac{3}{4}\,\pi\right), \tag{4.1}$$

and

$$R(X,x) := P(x) - H(X,x).$$

Then

$$\int_{X/2}^{X} R(X,x)^2\,dx \ll X^{1/2}. \tag{4.2}$$

Proof. By the usual asymptotics for Bessel functions,

$$J_1(2\pi\sqrt{nx}) = \frac{1}{\pi}\,(nx)^{-1/4}\cos\left(2\pi\sqrt{nx} - \frac{3}{4}\,\pi\right) + O((nx)^{-3/4}). \tag{4.3}$$

Using this in Lemma A, with $M = X^5$, the main terms of (4.3) obviously add up to $H(X,x)$. The total contribution of the O-terms from (4.3) to the left hand side of (4.2) is

$$\ll \int_{X/2}^{X} \Big(x^{-1/4} \sum_{1\le n<X^5} \frac{r(n)}{n^{5/4}} \Big)^2 \, dx \ll X^{1/2}. \tag{4.4}$$

Squaring and integrating the O-term of Lemma A, with $M = X^5$, we get

$$\int_{X/2}^{X} \min(X^{-4+2\varepsilon}\|x\|^{-2} + X^{-2}, X^{2\varepsilon})\, dx = \int\limits_{\substack{X/2\le x\le X\\ \|x\|\le\omega}} + \int\limits_{\substack{X/2\le x\le X\\ \|x\|>\omega}},$$

say, where $\omega = X^{-4/3}$. Obviously,

$$\int\limits_{\substack{X/2\le x\le X\\ \|x\|\le\omega}} \ll \int\limits_{\substack{X/2\le x\le X\\ \|x\|\le\omega}} X^{2\varepsilon}\, dx \ll \omega X^{1+2\varepsilon} \ll X^{-1/3+2\varepsilon}$$

and

$$\int\limits_{\substack{X/2\le x\le X\\ \|x\|>\omega}} \ll (X^{-4+2\varepsilon}\omega^{-2} + X^{-2})X \ll X^{-1/3+2\varepsilon}.$$

Together with (4.4) this verifies Proposition 3. ∎

Proposition 4. *For positive integers n, let*

$$\Delta_{\mathfrak{B}}(n) = \min_{\substack{k\in\mathfrak{B}\\ k\ne n}} |k-n|,$$

and let $H(X,x)$ be defined by (4.1). *Then, for $X \ge 2$,*

$$\int_{X/2}^{X} H(X,x)^2\, dx = C\Big(X^{3/2} - \Big(\frac{1}{2}X\Big)^{3/2}\Big) + O\Big(X \sum_{1\le n<X^5} \frac{r^2(n)}{n\Delta_{\mathfrak{B}}(n)}\Big),$$

where C is the constant given in (1.7).

Proof. For $u \ge \sqrt{X/2}$, we put

$$G(X,u) := \int_{\sqrt{X/2}}^{u} \Big(\sum_{1\le n<X^5} \frac{r(n)}{n^{3/4}} \cos\Big(2\pi\sqrt{n}\, t - \frac{3}{4}\pi\Big)\Big)^2 \, dt. \tag{4.5}$$

Then, in view of (4.1),

$$\begin{aligned} \int_{X/2}^{X} H(X,x)^2\, dx &= \frac{2}{\pi^2} \int_{\sqrt{X/2}}^{\sqrt{X}} u^2 \frac{\partial G}{\partial u}(X,u)\, du \\ &= \frac{2}{\pi^2} XG(X,\sqrt{X}) - \frac{4}{\pi^2} \int_{\sqrt{X/2}}^{\sqrt{X}} uG(X,u)\, du, \end{aligned} \tag{4.6}$$

after a change of variable and integration by parts. We apply Lemma B to evaluate $G(X,u)$, using the identity $\cos(\alpha) = \frac{1}{2}(\exp(i\alpha) + \exp(-i\alpha))$. We choose

$$\mathcal{J} = \{j \in \mathbb{Z} : 0 < |j| < X^5,\ |j| \in \mathfrak{B}\},$$

and, for all $j \in \mathcal{J}$,

$$a_j = \frac{1}{2}\,\frac{r(|j|)}{|j|^{3/4}}\,\exp\Big(-\operatorname{sgn}(j)\,\frac{3}{4}\,\pi i\Big), \qquad \lambda_j = 2\pi\ \operatorname{sgn}(j)\sqrt{|j|}.$$

For integers $n \ge 1$, write

$$m_{\mathfrak{B}}^{>}(n) := \min\{m \in \mathfrak{B} : m > n\}, \qquad m_{\mathfrak{B}}^{<}(n) := \max\{m \in \mathfrak{B} : m < n\}.$$

Then $m_{\mathfrak{B}}^{>}(n) \asymp n \asymp m_{\mathfrak{B}}^{<}(n)$, e.g., as an immediate consequence of the classic asymptotics for $\sum \mathbf{b}(n)$. Hence, in the notation of Lemma B,

$$\begin{aligned}
\delta_j &= \min_{\substack{k\in\mathcal{J}\\ k\neq j}} |\lambda_k - \lambda_j| \\
&\gg \min\left(\frac{\big|m_{\mathfrak{B}}^{>}(|j|) - |j|\big|}{\sqrt{m_{\mathfrak{B}}^{>}(|j|)} + \sqrt{|j|}},\ \frac{\big|m_{\mathfrak{B}}^{<}(|j|) - |j|\big|}{\sqrt{m_{\mathfrak{B}}^{<}(|j|)} + \sqrt{|j|}}\right) \\
&\gg \frac{\Delta_{\mathfrak{B}}(|j|)}{\sqrt{|j|}}.
\end{aligned}$$

Therefore, Lemma B yields

$$\begin{aligned}
G(X,u) &= \Big(u - \sqrt{\tfrac{1}{2}X}\Big)\,\frac{1}{2}\sum_{1\le n<X^5} \frac{r^2(n)}{n^{3/2}} + O\Big(\sum_{1\le n<X^5} \frac{r^2(n)}{n\Delta_{\mathfrak{B}}(n)}\Big) \\
&= \frac{1}{2}\Big(u - \sqrt{\tfrac{1}{2}X}\Big)\sum_{n=1}^{\infty} \frac{r^2(n)}{n^{3/2}} \\
&\qquad + O(uX^{-5/2+\varepsilon}) + O\Big(\sum_{1\le n<X^5} \frac{r^2(n)}{n\Delta_{\mathfrak{B}}(n)}\Big).
\end{aligned}$$

Using this in (4.6), we immediately establish Proposition 4. ■

We are now ready to complete the proof of our Theorem. Combining Propositions 3 and 4, with an appeal to Cauchy's inequality, and summing over all intervals $\big[\frac{1}{2}X, X\big]$, $\big[\frac{1}{4}X, \frac{1}{2}X\big], \ldots$, we obtain

$$Q(X) \ll X \sum_{1\le n<X^5} \frac{r^2(n)}{n\Delta_{\mathfrak{B}}(n)}. \tag{4.7}$$

Now, for real numbers $Z \geq 2$ and $W \geq 2$,

$$\sum_{\substack{\frac{1}{2}Z \leq n < Z \\ \frac{1}{2}W \leq \Delta_{\mathfrak{B}}(n) < W}} \frac{r^2(n)}{n\Delta_{\mathfrak{B}}(n)} \ll \frac{1}{WZ} \sum_{\substack{\frac{1}{2}Z \leq n < Z \\ \Delta_{\mathfrak{B}}(n) < W}} r^2(n)$$

$$\leq \frac{1}{WZ} \sum_{0<|h|<W} \Big(\sum_{\frac{1}{2}Z \leq n < Z} r^2(n)\mathbf{b}(n+h) \Big)$$

$$\ll \frac{1}{W} (\log Z)^{1/2} \sum_{0<|h|<W} \left(\frac{|h|}{\phi(|h|)} \right)^2 \ll (\log Z)^{1/2},$$

by an appeal to (3.8). We use this estimate for $W = 2, 4, 8, \ldots, 2^w \asymp \log X$, thus $w \ll \log\log X$, and

$$\sum_{\substack{\frac{1}{2}Z \leq n < Z \\ \Delta_{\mathfrak{B}}(n) \geq 2^w}} \frac{r^2(n)}{n\Delta_{\mathfrak{B}}(n)} \ll \frac{1}{Z\log X} \sum_{\frac{1}{2}Z \leq n < Z} r^2(n) \ll \frac{\log Z}{\log X} \ll 1,$$

as long as $\log Z \ll \log X$. Therefore, altogether,

$$\sum_{\frac{1}{2}Z \leq n < Z} \frac{r^2(n)}{n\Delta_{\mathfrak{B}}(n)} \ll (\log X)^{1/2} \log\log X,$$

if $\log Z \ll \log X$. Summing finally over $Z = X^5, \frac{1}{2}X^5, \frac{1}{4}X^5, \ldots$ (up to $O(\log X)$ terms) and recalling (4.7), we complete the proof of our Theorem. ■

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