ACTA ARITHMETICA
115.3 (2004)

A note on the Dirichlet characters of polynomials

by

WENPENG ZHANG and WEILI YAO (Xi'an)

1. Introduction. Let $q \geq 3$ be an integer and χ be a Dirichlet character modulo q. It is an important problem in analytic number theory to obtain a sharp upper bound estimate for character sums of polynomials,

$$\sum_{x=N+1}^{N+H} \chi(f(x)),$$

where $f(x)$ is a polynomial. If $q = p$ is an odd prime, it is a well known consequence of the theorem due to Weil (see [2]), stating that the Riemann Hypothesis is true for the zeta-function of an algebraic function field over a finite field, that if χ is a qth-order character to the prime modulus p, and if $f(x)$ is not a perfect qth power (mod p), then

$$\sum_{x=N+1}^{N+H} \chi(f(x)) \ll p^{1/2} \log p, \tag{1}$$

where $A \ll B$ denotes $|A| < kB$ for some constant k which in this case depends on the degree of f. The estimate (1) is the best possible. In fact, in [8] the following is proved:

Let $q \geq 3$ be an integer, χ be a primitive character modulo q, and m and n be two positive integers such that χ^m, χ^n and χ^{m+n} are primitive characters modulo q. Then for any integers r and s with $(r - s, q) = 1$, we have the identity

$$\Big|\sum_{a=1}^{q} \chi((a-r)^m (a-s)^n)\Big| = \sqrt{q}. \tag{2}$$

2000 *Mathematics Subject Classification*: 11L10, 11L40.

Key words and phrases: primitive Dirichlet characters, polynomials, Gauss sums, identity.

This work is supported by N.S.F. (10271093) and P.N.S.F. (2002A11) of P.R. China.

It is clear that the character sums in (2) can be viewed as Jacobi sums with Dirichlet characters. So the identity (2) provided new evaluations for the absolute value of some Jacobi sums.

This paper is a continuation of [8]. We will use the properties of Gauss sums and R. A. Smith's important work to give a new identity for the Dirichlet character of polynomials:

THEOREM. *Let $q \geq 3$ be a perfect square, χ be a primitive character modulo q, and m and n be two positive integers such that $(mn(m+n), q) = 1$. Then for any integers r and s with $(r-s, q) = 1$, we have the identity*

$$\sum_{a=1}^{q} \chi((a-r)^m (a-s)^n) = \sqrt{q}\, \chi((sm - rm)^m (rn - sn)^n) \overline{\chi}((m+n)^{m+n}).$$

In particular, for $r = 0$ and $s = 1$, from our Theorem we may immediately deduce a nice formula for calculating a Jacobi sum modulo an odd perfect square:

COROLLARY. *Let $q \geq 3$ be an odd perfect square and χ be any primitive character modulo q. Then for any positive integer m with $(m, q) = 1$, we have the identity*

$$\sum_{a=1}^{q} \chi(a^m (1-a)^m) = \sqrt{q}\, \overline{\chi}(4^m).$$

For general moduli q, whether there exists a similar formula is an open problem.

2. Some lemmas. To prove the Theorem, we need some lemmas. For convenience, we define the *Gauss sums* $G(m, \chi)$ as follows:

$$G(m, \chi) = \sum_{a=1}^{q} \chi(a) e\left(\frac{am}{q}\right),$$

where $e(y) = e^{2\pi i y}$. For $m = 1$, we write

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a) e\left(\frac{a}{q}\right).$$

Various properties and applications of $\tau(\chi)$ can be found in many analytic number theory books (see [1] and [4]). Here we first use the properties of Gauss sums to prove the following:

LEMMA 1. *Let p be a prime, α be a positive integer, and χ be a primitive character modulo p^α. Then for any two positive integers m and n with χ^m,*

χ^n and χ^{m+n} primitive characters modulo p^α, we have the identity

$$\tau(\chi^m)\tau(\chi^n) = \tau(\chi^{m+n}) \sum_{a=1}^{p^\alpha} \chi(a^m(1-a)^n).$$

Proof. This is a well known result on Jacobi sums (see [6] or Theorem 5.21 of [3]). ∎

LEMMA 2. *Let $q = q_1 q_2$ with $(q_1, q_2) = 1$. Then for any character χ modulo q, there exists a unique character χ_i modulo q_i $(i = 1, 2)$ such that $\chi = \chi_1\chi_2$ and*

$$\sum_{a=1}^{q} \chi(a^m(1-a)^n) = \prod_{i=1}^{2} \Big[\sum_{a_i=1}^{q_i} \chi_i(a_i^m(1-a_i)^n) \Big].$$

Proof. From the properties of the reduced residue system modulo q we have

$$\begin{aligned}\sum_{a=1}^{q} \chi(a^m(1-a)^n) &= \sum_{a_1=1}^{q_1} \sum_{a_2=1}^{q_2} \chi_1\chi_2((a_1q_2 + a_2q_1)^m(1 - a_1q_2 - a_2q_1)^n) \\ &= \sum_{a_1=1}^{q_1} \chi_1(a_1^m q_2^m (1 - a_1q_2)^n) \sum_{a_2=1}^{q_2} \chi_2(a_2^m q_1^m (1 - a_2q_1)^n) \\ &= \prod_{i=1}^{2} \Big[\sum_{a_i=1}^{q_i} \chi_i(a_i^m(1-a_i)^n) \Big]. \ \blacksquare\end{aligned}$$

LEMMA 3. *Let p be a prime, α be a positive integer, $q = p^{2\alpha}$, and χ be a primitive character modulo q. Then for any positive integer m with $(m, p) = 1$, we have the identity*

$$\tau^m(\chi) = q^{(m-1)/2}\tau(\chi^m)\overline{\chi}(m^m).$$

Proof. We introduce the *hyper-Kloosterman sum* by

$$K(q, m+1, z) = \sum_{\substack{x_1,\ldots,x_m \bmod q \\ (x_1,p)=\cdots=(x_m,p)=1}} e\Big(\frac{x_1 + \cdots + x_m + z\overline{x}_1\cdots\overline{x}_m}{q}\Big)$$

for $q = p^\alpha$, $m \geq 1$, and $p \nmid z$. Define an exponential sum by

$$I(q, m, z) = \sum_{\substack{x \bmod q \\ (x,p)=1}} e\Big(\frac{mx + z\overline{x}^m}{q}\Big).$$

From R. A. Smith [5] or Yangbo Ye [7] we know that

$$K(q, m+1, z) = q^{(m-1)/2} I(q, m, z) \quad \text{if } 2 \mid \alpha. \tag{3}$$

Now for any primitive character χ modulo $q = p^{2\alpha}$, from (3), the definition and properties of Gauss sums we have

$$\sum_{z=1}^{q} \chi(z) K(q, m+1, z) = q^{(m-1)/2} \sum_{z=1}^{q} \chi(z) I(q, m, z). \tag{4}$$

Note that $\tau(\chi) \neq 0$ and

$$\sum_{z=1}^{q} \chi(z) K(q, m+1, z) = \tau^{m+1}(\chi),$$
$$\sum_{z=1}^{q} \chi(z) I(q, m, z) = \tau(\chi)\tau(\chi^m)\overline{\chi}(m^m).$$

From (4) we immediately get the assertion. ■

3. Proof of the Theorem. From the three lemmas above, we can easily prove the Theorem. In fact, let $q = p^{2\alpha}$, and let χ be a primitive character modulo q. Then for any positive integers m and n with $(mn(m+n), q) = 1$, from Lemma 3 we have

$$\tau^m(\chi)\tau^n(\chi) = q^{(m+n-2)/2}\tau(\chi^m)\tau(\chi^n)\overline{\chi}(m^m n^n)$$

and

$$\tau^m(\chi)\tau^n(\chi) = \tau^{m+n}(\chi) = q^{(m+n-1)/2}\tau(\chi^{m+n})\overline{\chi}((m+n)^{m+n}).$$

Therefore, by Lemma 1 we get

$$\begin{aligned} q^{(m+n-1)/2}\tau(\chi^{m+n})\overline{\chi}((m+n)^{m+n}) \\ &= q^{(m+n-2)/2}\tau(\chi^m)\tau(\chi^n)\overline{\chi}(m^m n^n) \\ &= q^{(m+n-2)/2}\tau(\chi^{m+n})\overline{\chi}(m^m n^n)\sum_{a=1}^{q}\chi(a^m(1-a)^n). \end{aligned} \tag{5}$$

Since $\tau(\chi^{m+n}) \neq 0$, from (5) we obtain

$$\sum_{a=1}^{q} \chi(a^m(1-a)^n) = \sqrt{q}\,\overline{\chi}((m+n)^{m+n})\chi(m^m n^n). \tag{6}$$

This proves the Theorem with $q = p^{2\alpha}$, $r = 0$ and $s = 1$.

For a general perfect square q, let $q = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ be the factorization of q into prime powers. Then for any integers m and n with $(mn(m+n), q) = 1$, from the properties of primitive characters modulo q and Lemma 2 we know that (6) also holds.

Now for any integers r and s with $(r-s,q)=1$, from (6) we have

$$\begin{aligned}\sum_{a=1}^{q}\chi((a-r)^m(a-s)^n) &= \sum_{a=1}^{q}\chi(a^m(a+r-s)^n)\\ &= \sum_{a=1}^{q}\chi((a(s-r))^m(a(s-r)-(s-r))^n)\\ &= \chi((s-r)^{m+n})\sum_{a=1}^{q}\chi(a^m(a-1)^n)\\ &= \sqrt{q}\,\chi((sm-rm)^m(rn-sn)^n)\overline{\chi}((m+n)^{m+n}). \quad \blacksquare\end{aligned}$$

Acknowledgments. The authors express their gratitude to the referee for very helpful and detailed comments.

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, New York, 1976.

[2] D. A. Burgess, *On Dirichlet characters of polynomials*, Proc. London Math. Soc. 13 (1963), 537–548.

[3] R. Lidl and H. Niederreiter, *Finite Fields*, Addison-Wesley, Reading, MA, 1983.

[4] C. D. Pan and C. B. Pan, *Elements of Analytic Number Theory*, Science Press, Beijing, 1991.

[5] R. A. Smith, *On n-dimensional Kloosterman sums*, J. Number Theory 11 (1979), 324–343.

[6] J. Wang, *On the Jacobi sums modulo p^n*, ibid. 39 (1991), 50–64.

[7] Y. Ye, *Estimation of exponential sums of polynomials of higher degrees II*, Acta Arith. 93 (2000), 221–235.

[8] W. P. Zhang and Y. Yi, *On Dirichlet characters of polynomials*, Bull. London Math. Soc. 34 (2002), 469–473.

Research Center for Basic Science
Xi'an Jiaotong University
Xi'an, Shaanxi, P.R. China
E-mail: wpzhang@nwu.edu.cn
mimi5512867@sina.com

Received on 2.10.2003
and in revised form on 8.4.2004 (4639)