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Car–Pólya and Gel'fond's theorems for $\mathbb{F}_q[T]$

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1. Introduction. In 1915, Pólya proved the following result:

Theorem 1 ([12]). *Let f be an entire function on $\mathbb{C}$ such that*

$$f(\mathbb{Z}) \subset \mathbb{Z}, \qquad \overline{\lim_{r\to+\infty}} \frac{\ln|f|_r}{r} < \ln\left(\frac{3+\sqrt{5}}{2}\right),$$

resp.

$$f(\mathbb{N}) \subset \mathbb{Z}, \qquad \overline{\lim_{r\to+\infty}} \frac{\ln|f|_r}{r} < \ln 2,$$

where $|f|_r = \sup_{|z|\le r}|f(z)|$. Then f is a polynomial of $\mathbb{Q}[X]$. Moreover, $\ln\left(\frac{3+\sqrt{5}}{2}\right)$ (resp. $\ln 2$) is optimal.

The constant is optimal, since the function

$$f(z) = \frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^z - \left(\frac{3-\sqrt{5}}{2}\right)^z\right]$$

(resp. $f(z) = 2^z$) is a non-polynomial entire function such that $f(\mathbb{Z}) \subset \mathbb{Z}$ (resp. $f(\mathbb{N}) \subset \mathbb{Z}$) and its exponential type is $\ln\left(\frac{3+\sqrt{5}}{2}\right)$ (resp. $\ln 2$).

In 1933, Gel'fond proved:

Theorem 2 ([9]). *Let f be an entire function on $\mathbb{C}$ and $q \ge 2$ be an integer. If, for all $n \in \mathbb{N}$, $f(q^n) \in \mathbb{Z}$ and if*

$$\ln|f|_r < \frac{1}{4\ln q}(\ln r)^2 - \frac{1}{2}\ln r - \omega(r)$$

where $\omega(r) \to +\infty$ as $r \to +\infty$, then f is a polynomial in $\mathbb{Q}[X]$.

The coefficients $1/4\ln q$ and $-1/2$ are optimal, since the entire function

$$\varphi(z) = \sum_{n\ge 0}\prod_{k=0}^{n-1}\frac{z-q^k}{q^n-q^k}$$

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satisfies

$$\ln|\varphi(z)| < \frac{1}{4\ln q}(\ln r)^2 - \frac{1}{2}\ln r + O(1) \quad \text{for } |z| = r \text{ as } r \to +\infty,$$

$\varphi(q^n) \in \mathbb{Z}$ for all $n \in \mathbb{N}$, and φ is not a polynomial.

Here, we are interested in analogous results in function fields.

Let q be a power of a prime number, $\mathbb{F}_q$ be a field with q elements, $\mathbb{F}_q[T]$ be the ring of polynomials in T over $\mathbb{F}_q$, $\mathbb{F}_q(T)$ be its quotient field. Let $\mathbb{F}_q(T)_\infty$ be the completion of $\mathbb{F}_q(T)$ for the infinite $1/T$-adic valuation v normalized by $v(1/T) = 1$ and let Ω be the completion of an algebraic closure of $\mathbb{F}_q(T)_\infty$. The valuation v extends to a (non-discrete) valuation on Ω that we still denote by v. For all $z \in \Omega$, we put

$$\deg(z) = -v(z).$$

Let $f(X) = \sum_{n\ge 0} c_n X^n$ be an entire function on Ω and r be a real number. We put

$$M(f,r) = \sup_{\deg(z)\le r}\{\deg(f(z))\}.$$

Schnirelmann showed (see [13] or [8, Appendice]) that, for all $r \in \mathbb{Q}^+$,

$$M(f,r) = \sup_{n\in\mathbb{N}}\{\deg(c_n) + nr\}.$$

By continuity, for all $r \in \mathbb{R}^+$ we have

$$M(f,r) = \sup_{n\in\mathbb{N}}\{\deg(c_n) + nr\}. \tag{1}$$

Mireille Car proved the following analog of Pólya's theorem for $\mathbb{F}_q[T]$.

Theorem 3 ([4, Theorem 2]). *Let f be an entire function on Ω such that*

$$f(\mathbb{F}_q[T]) \subset \mathbb{F}_q[T], \qquad \overline{\lim_{r\to+\infty}} \frac{M(f,r)}{q^r} < \frac{1}{e\ln q\, q^{q/(q-1)}}.$$

Then f is a polynomial in $\mathbb{F}_q(T)[X]$.

Moreover, Car showed that the bound may be improved for linear functions.

Theorem 4 ([4, Theorem 4]). *Let f be an $\mathbb{F}_q$-linear entire function such that*

$$f(\mathbb{F}_q[T]) \subset \mathbb{F}_q[T], \qquad \overline{\lim_{r\to+\infty}} \frac{M(f,r)}{q^r} < \frac{1}{e\ln q}.$$

Then f is a polynomial in $\mathbb{F}_q(T)[X]$.

In order to extend this last result to any entire function, Laurence Delamette proved:

Theorem 5 ([8, Théorème 3]). *Let $\varepsilon > 0$. There exists $q(\varepsilon) > 0$ such that for every finite field $\mathbb{F}_q$ with $q > q(\varepsilon)$ elements, every entire function f*

on Ω such that

$$f(\mathbb{F}_q[T]) \subset \mathbb{F}_q[T], \qquad \varlimsup_{r\to+\infty} \frac{M(f,r)}{q^r} < \frac{1}{e \ln q\, q^{\varepsilon}}$$

is a polynomial in $\mathbb{F}_q(T)[X]$.

In this paper, we prove two results. The first one shows that Car's theorem for entire functions is true with the bound $1/e\ln q$ (and Car gave an example that proves that this constant is optimal).

THEOREM 6. *Let f be an entire function on Ω such that*

$$f(\mathbb{F}_q[T]) \subset \mathbb{F}_q[T], \qquad \varlimsup_{r\to+\infty} \frac{M(f,r)}{q^r} < \frac{1}{e \ln q}.$$

Then f is a polynomial in $\mathbb{F}_q(T)[X]$.

Our second result is an analog for $\mathbb{F}_q[T]$ of Gel'fond's theorem:

THEOREM 7. *Let f be an entire function on Ω and let $H \in \mathbb{F}_q[T]$ be a polynomial of degree $h \geq 1$. If, for all $n \in \mathbb{N}$, $f(H^n) \in \mathbb{F}_q[T]$ and*

$$\varlimsup_{r\to+\infty} \frac{M(f,r)}{r^2} < \frac{1}{4h}$$

then f is a polynomial in $\mathbb{F}_q(T)[X]$. Moreover, the constant $1/4h$ is optimal.

2. Car–Pólya's theorem. Let R be a domain with quotient field K and let E be a subset of R. We denote by $\mathrm{Int}(E,R)$ (or $\mathrm{Int}(R)$ if $E = R$) the R-module formed by the polynomials which take values in R on E:

$$\mathrm{Int}(E,R) = \{P \in K[X] \mid P(E) \subset R\}.$$

Here, we consider the case when $E = R = \mathbb{F}_q[T]$. Denote by $u_0 = 0, u_1, \ldots$ $\ldots, u_{q-1}$ the elements of $\mathbb{F}_q$. Mireille Car defines (see [3]) a one-to-one correspondence between $\mathbb{N}$ and $\mathbb{F}_q[T]$ in the following way: for every $n \in \mathbb{N}$, let $n = \sum_{i=0}^{s} n_i q^i$ be its q-adic expansion. Then, put

$$u_n = \sum_{i=0}^{s} u_{n_i} T^i.$$

We recall that Bhargava's factorials for $\mathbb{F}_q[T]$ are given by (see [2, §7] or [1] for a straightforward proof)

$$n!^{\mathcal{B}}_{\mathbb{F}_q[T]} = \prod_{k=0}^{n-1} (u_n - u_k).$$

Then, the sequence of polynomials

$$\prod_{k=0}^{n-1} \frac{X - u_k}{u_n - u_k}$$

is a basis of the $\mathbb{F}_q[T]$-module $\mathrm{Int}(\mathbb{F}_q[T])$.

We recall the definition of Carlitz factorials (see [7]). For every $i \in \mathbb{N}$, put

$$D_i = \prod_{\substack{P \in \mathbb{F}_q[T] \\ \deg P < i}} (T^i + P).$$

For every $n \in \mathbb{N}$ with q-adic expansion $n = \sum_{i=0}^{s} n_i q^i$, the Carlitz nth factorial is defined by

$$n!^{\mathcal{C}}_{\mathbb{F}_q[T]} = \prod_{i=0}^{s} D_i^{n_i}.$$

We know that (see [2, §7])

$$n!^{\mathcal{C}}_{\mathbb{F}_q[T]} = \xi_n \times n!^{\mathcal{B}}_{\mathbb{F}_q[T]} \quad \text{with } \xi_n \in \mathbb{F}_q^{\times}. \tag{2}$$

Therefore, the sequence of polynomials

$$\binom{X}{n} = \frac{\prod_{k=0}^{n-1}(X - u_i)}{n!^{\mathcal{C}}_{\mathbb{F}_q[T]}}$$

is a basis of the $\mathbb{F}_q[T]$-module $\mathrm{Int}(\mathbb{F}_q[T])$. From now on, for simplicity, the Carlitz nth factorial will be denoted by $n!$. There is no risk of confusion, because it will be the only one used. The degree of $n!$ is

$$\deg(n!) = \sum_{i=0}^{s} i n_i q^i. \tag{3}$$

We have the relation (see [14])

$$\frac{n!}{(n-1)!} = L_{e(n)}, \tag{4}$$

where $e(n)$ denotes the highest power of q dividing n and, for every $m \in \mathbb{N}$, L_m is the polynomial defined by

$$L_m = \prod_{j=1}^{m} (T^{q^j} - T).$$

The degree of L_m is

$$\deg(L_m) = \frac{q^{m+1} - q}{q - 1}. \tag{5}$$

For every $n, k \in \mathbb{N}$, we define the elements $a_{n,k}$ and $b_{n,k}$ of $\mathbb{F}_q[T]$ by

$$X^n = \sum_{k=0}^{n} b_{n,k} \binom{X}{k}, \tag{6}$$

$$n! \binom{X}{n} = \sum_{k=0}^{n} (-1)^{n-k} a_{n,k} X^k. \tag{7}$$

Hence, for all $k > n$ and $k < 0$, we have $b_{n,k} = a_{n,k} = 0$. We see that $b_{n,0} = a_{n,0} = 0$ for all $n \in \mathbb{N}^*$.

LEMMA 8. *The $b_{n,k}$ and $a_{n,k}$ satisfy the recurrence relations*

$$a_{r+1,k} = u_r a_{r,k} + a_{r,k-1}, \tag{8}$$

$$b_{r,k} = L_{e(k)} b_{r-1,k-1} + u_k b_{r-1,k}. \tag{9}$$

Proof. 1) Using (7), we have

$$\begin{aligned}(X-u_0)(X-u_1)\cdots(X-u_r) &= \sum_{k=0}^{r}(-1)^{r-k}a_{r,k}X^{k+1} - u_r\sum_{k=0}^{r}(-1)^{r-k}a_{r,k}X^k \\ &= \sum_{k=1}^{r+1}(-1)^{r-k+1}a_{r,k-1}X^k - u_r\sum_{k=0}^{r}(-1)^{r-k}a_{r,k}X^k\end{aligned}$$

and by identification

$$a_{r+1,k} = a_{r,k-1} + u_r a_{r,k}.$$

2) Using (4) and (6), we may write

$$\begin{aligned}X^r = X^{r-1}X &= \sum_{k=0}^{r-1} b_{r-1,k}\frac{(X-u_0)(X-u_1)\cdots(X-u_{k-1})(X-u_k+u_k)}{k!} \\ &= \sum_{k=0}^{r-1} b_{r-1,k}\binom{X}{k+1}L_{e(k+1)} + \sum_{k=0}^{r-1} b_{r-1,k}u_k\binom{X}{k}. \ \blacksquare\end{aligned}$$

We begin to give upper bounds for the degrees of $b_{r,k}$ and $a_{r,k}$.

LEMMA 9. *Let $r,s,k\in\mathbb{N}$ be such that $r\in\mathbb{N}^*$ and $k\in[q^s,q^{s+1}[$. Then*

$$\deg(b_{r,k}) \le (r-k)s + \sum_{j=1}^{s}\left(\left[\frac{k}{q^j}\right]-\left[\frac{k}{q^{j+1}}\right]\right)\frac{q^{j+1}-q}{q-1}.$$

Proof. We prove the inequality by induction on r. We may assume that $0\le q^s\le r$. Clearly, the lemma is true for $r=1$.

We first assume that $k\in\,]q^s,q^{s+1}[$. By Lemma 8, we have

$$\deg(b_{r,k}) \le \max(\deg(b_{r-1,k-1})+\deg(L_{e(k)}), \deg(u_k)+\deg(b_{r-1,k})).$$

By the induction hypothesis, we have

$$\deg(b_{r-1,k-1}) \le (r-k)s + \sum_{j=1}^{s}\left(\left[\frac{k-1}{q^j}\right]-\left[\frac{k-1}{q^{j+1}}\right]\right)\frac{q^{j+1}-q}{q-1}.$$

For every $n\in\mathbb{N}$, $[n/q^j]-[n/q^{j+1}]$ is the number of integers $\le n$ divisible by q^j and not by q^{j+1}:

$$\left[\frac{n}{q^j}\right]-\left[\frac{n}{q^{j+1}}\right] = \#\{m\in\mathbb{N}\mid 1\le m\le n \text{ and } q^j \parallel m\}$$

where the symbol $\parallel$ means "exactly divisible by". However, if $j\ne e(k)$ then

$$\#\{n \in \mathbb{N} \mid 1 \le n \le k \text{ and } q^j \,\|\, n\} = \#\{n \in \mathbb{N} \mid 1 \le n \le k-1 \text{ and } q^j \,\|\, n\},$$

and if $j = e(k)$ then

$$\#\{n \in \mathbb{N} \mid 1 \le n \le k \text{ and } q^j \,\|\, n\} = 1 + \#\{n \in \mathbb{N} \mid 1 \le n \le k-1 \text{ and } q^j \,\|\, n\}.$$

So, if $j \ne e(k)$ then

$$\left[\frac{k-1}{q^j}\right] - \left[\frac{k-1}{q^{j+1}}\right] = \left[\frac{k}{q^j}\right] - \left[\frac{k}{q^{j+1}}\right],$$

and if $j = e(k)$ then

$$\left[\frac{k-1}{q^{e(k)}}\right] - \left[\frac{k-1}{q^{e(k)+1}}\right] = \left[\frac{k}{q^{e(k)}}\right] - \left[\frac{k}{q^{e(k)+1}}\right] - 1.$$

By (5), we have

$$\deg(L_{e(k)}) = \frac{q^{e(k)+1} - q}{q-1}.$$

It follows that

$$\sum_{j=1}^{s} \left(\left[\frac{k-1}{q^j}\right] - \left[\frac{k-1}{q^{j+1}}\right] \right) \frac{q^{j+1}-q}{q-1} + \deg(L_{e(k)}) = \sum_{j=1}^{s} \left(\left[\frac{k}{q^j}\right] - \left[\frac{k}{q^{j+1}}\right] \right) \frac{q^{j+1}-q}{q-1}.$$

Therefore

$$\deg(b_{r-1,k-1} L_{e(k)}) \le (r-k)s + \sum_{j=1}^{s} \left(\left[\frac{k}{q^j}\right] - \left[\frac{k}{q^{j+1}}\right] \right) \frac{q^{j+1}-q}{q-1}.$$

Since $\deg(u_k) = s$, we have

$$\begin{aligned} \deg(b_{r-1,k}) + \deg(u_k) &\le (r-1-k)s + s + \sum_{j=1}^{s} \left(\left[\frac{k}{q^j}\right] - \left[\frac{k}{q^{j+1}}\right] \right) \frac{q^{j+1}-q}{q-1} \\ &\le (r-k)s + \sum_{j=1}^{s} \left(\left[\frac{k}{q^j}\right] - \left[\frac{k}{q^{j+1}}\right] \right) \frac{q^{j+1}-q}{q-1}. \end{aligned}$$

We now assume that $k = q^s$. Then

$$\deg(b_{r,q^s}) \le \max\left(\deg(b_{r-1,q^s-1}) + \frac{q^{s+1}-q}{q-1}, \deg(b_{r-1,q^s}) + s \right).$$

By the induction hypothesis, we have

$$\deg(b_{r-1,q^s-1}) + \frac{q^{s+1}-q}{q-1} \le (r-q^s)(s-1) + \sum_{j=1}^{s-1} \left(\left[\frac{q^s-1}{q^j}\right] - \left[\frac{q^s-1}{q^{j+1}}\right] \right) \frac{q^{j+1}-q}{q-1} + \frac{q^{s+1}-q}{q-1}.$$

For $1 \le j \le s-1$, we have the equality

$$\#\{n \in \mathbb{N} \mid 1 \le n \le q^s - 1 \text{ and } q^j \parallel n\} = \#\{n \in \mathbb{N} \mid 1 \le n \le q^s \text{ and } q^j \parallel n\},$$

and so

$$\left[\frac{q^s-1}{q^j}\right] - \left[\frac{q^s-1}{q^{j+1}}\right] = \left[\frac{q^s}{q^j}\right] - \left[\frac{q^s}{q^{j+1}}\right].$$

As a consequence,

$$\sum_{j=1}^{s-1}\left(\left[\frac{q^s-1}{q^j}\right] - \left[\frac{q^s-1}{q^{j+1}}\right]\right)\frac{q^{j+1}-q}{q-1} + \frac{q^{s+1}-q}{q-1} = \sum_{j=1}^{s}\left(\left[\frac{q^s}{q^j}\right] - \left[\frac{q^s}{q^{j+1}}\right]\right)\frac{q^{j+1}-q}{q-1}.$$

Since $(r-q^s) \ge 0$, we have $(r-q^s)(s-1) \le (r-q^s)s$, and

$$\deg(b_{r-1,q^s-1}L_{e(q^s)}) \le (r-q^s)s + \sum_{j=1}^{s}\left(\left[\frac{q^s}{q^j}\right] - \left[\frac{q^s}{q^{j+1}}\right]\right)\frac{q^{j+1}-q}{q-1}.$$

Moreover,

$$\deg(b_{r-1,q^s}) + s \le (r-q^s)s + \sum_{j=1}^{s}\left(\left[\frac{q^s}{q^j}\right] - \left[\frac{q^s}{q^{j+1}}\right]\right)\frac{q^{j+1}-q}{q-1}. \blacksquare$$

PROPOSITION 10. *For all $r, k \in \mathbb{N}^*$ such that $1 \le k \le r$, we have*

(10) $$\deg(b_{r,k}) \le r \log_q k.$$

Proof. Let s be the integer such that $k \in [q^s, q^{s+1}[$. We compute

$$\begin{aligned} S &= \sum_{j=1}^{s}\left(\left[\frac{k}{q^j}\right] - \left[\frac{k}{q^{j+1}}\right]\right)\frac{q^{j+1}-q}{q-1} \\ &= \frac{1}{q-1}\Bigg(q\sum_{j=1}^{s}\left[\frac{k}{q^j}\right]q^j - \sum_{j=1}^{s-1}\left[\frac{k}{q^{j+1}}\right]q^{j+1} \\ &\qquad - q\Bigg(\sum_{j=1}^{s}\left[\frac{k}{q^j}\right] - \sum_{j=1}^{s-1}\left[\frac{k}{q^{j+1}}\right]\Bigg)\Bigg) \\ &= \sum_{j=1}^{s}\left[\frac{k}{q^j}\right]q^j \le \sum_{j=1}^{s}\frac{k}{q^j}\,q^j \le ks. \end{aligned}$$

By Lemma 9, we easily deduce that $\deg(b_{r,k}) \le rs - ks + ks \le rs \le r\log_q k$. $\blacksquare$

LEMMA 11. *Let $r, k \in \mathbb{N}^*$, $l, s \in \mathbb{N}$ be such that $r \in [q^l, q^{l+1}[$ and $k \in [q^s, q^{s+1}[$. Then*

$$\deg(a_{r,k}) \leq rl - \frac{q}{q-1}\,(q^l - q^s) - sq^s - (k - q^s)l.$$

Proof. We may assume $s \leq l$. The proof is by induction on r. The lemma is true for $r = q^l$, by the following result of Carlitz [6, Theorem 2.1]:

$$q^l!\binom{X}{q^l} = \sum_{i=0}^{l}(-1)^{r-i}\begin{bmatrix} l \\ i \end{bmatrix} X^{q^i},$$

where

$$\begin{bmatrix} l \\ i \end{bmatrix} = \frac{D_l}{D_i L_{l-i}^{q^i}} = a_{q^l,q^i}.$$

By (3) and (5) (see also [4, Lemma IV.4(ii)]), for all $0 \leq i \leq l$, we have

$$\deg(a_{q^l,q^i}) = lq^l - iq^i - q^i\,\frac{q^{l-i+1} - q}{q-1}$$

and, for every i which is not power of q, $\deg(a_{q^l,i}) = -\infty$. By Lemma 8, we have

$$\deg(a_{r+1,k}) \leq \max(l + \deg(a_{r,k}), \deg(a_{r,k-1})).$$

We first assume that $k \in \,]q^s, q^{s+1}[$. It is a straightforward exercise to verify that

$$\deg(a_{r+1,k}) \leq (r+1)l - \frac{q}{q-1}\,(q^l - q^s) - sq^s - (k - q^s).$$

We now assume that $k = q^s$. Then, by induction,

$$l + \deg(a_{r,q^s}) \leq l + rl - \frac{q}{q-1}\,(q^l - q^s) - sq^s \leq (r+1)l - \frac{q}{q-1}\,(q^l - q^s) - sq^s,$$

$$\deg(a_{r,q^s-1}) \leq rl - \frac{q}{q-1}\,(q^l - q^{s-1}) - (s-1)q^{s-1} - (q^s - q^{s-1} - 1)l.$$

The following inequality holds because $l \geq s - 1$:

$$\frac{q}{q-1}\,q^s - sq^s \geq \frac{q}{q-1}\,q^{s-1} - (s-1)q^{s-1} - (q^s - q^{s-1})l.$$

Hence,

$$\deg(a_{r,q^s-1}) \leq rl - \frac{q}{q-1}\,(q^l - q^s) - sq^s + l. \ \blacksquare$$

From this, we deduce

PROPOSITION 12. *Let $r, k \in \mathbb{N}^*$ be such that $1 \leq k \leq r$. Then*

$$\deg\left(\frac{a_{r,k}}{r!}\right) \leq -\log_q r + \frac{2q-1}{q-1}\,k - k\log_q k.$$

Proof. Let $\sum_{j=0}^{l} r_j q^j$ be the q-adic expansion of r and let $s \in \mathbb{N}$ be such that $k \in [q^s, q^{s+1}[$. By Lemma 11 and (3), we have

$$\begin{aligned}\deg\left(\frac{a_{r,k}}{r!}\right) &\le l\sum_{j=0}^{l} r_j q^j - \frac{q}{q-1}\,(q^l - q^s) - sq^s - (k-q^s)l - \sum_{j=0}^{l} j r_j q^j \\ &\le \sum_{j=0}^{l}(q-1)q^j(l-j) - \frac{q}{q-1}\,(q^l-q^s) - sq^s - (k-q^s)l \\ &\le -\frac{l(q-1)+q}{q-1} + \frac{q}{q-1}\,q^s - ks.\end{aligned}$$

Since $-l \le 1 - \log_q r$ and $-s \le 1 - \log_q k$, we get

$$\begin{aligned}\deg\left(\frac{a_{r,k}}{r!}\right) &\le 1 - \log_q r - \frac{q}{q-1} + \frac{q}{q-1}\,k + k(1-\log_q k) \\ &\le -\log_q r - \frac{1}{q-1} + \frac{2q-1}{q-1}\,k - k\log_q k. \ \blacksquare\end{aligned}$$

PROPOSITION 13. *Let $x \in \Omega$ be of degree δ and $r \in \mathbb{N}^*$. Then*

$$\deg\binom{x}{r} \le -\log_q r + \frac{q^{(2q-1)/(q-1)+\delta}}{e\ln q}. \tag{11}$$

Proof. By Proposition 12, for all $k \in \mathbb{N}^*$ we have

$$\deg\left(\frac{a_{r,k}}{r!}\,x^k\right) \le k\delta - \log_q r + \frac{2q-1}{q-1}\,k - k\log_q k.$$

Moreover, for all $k \ge 1$, it is easy to verify that

$$k\delta + \frac{2q-1}{q-1}\,k - k\log_q k \le \frac{q^{(2q-1)/(q-1)+\delta}}{e\ln q}.$$

Inequality (11) follows from

$$\binom{x}{r} = \sum_{k=0}^{r}(-1)^{r-k}\,\frac{a_{r,k}}{r!}\,x^k. \ \blacksquare$$

Let $g(X) = \sum_{n=0}^{k} c_n X^n$ be a polynomial of $\Omega[X]$ of degree k. We have

$$g(X) = \sum_{n=0}^{k} c_n \sum_{j=0}^{n} b_{n,j}\binom{X}{j} = \sum_{j=0}^{+\infty}\sum_{n=0}^{+\infty} c_n b_{n,j}\binom{X}{j},$$

where $c_n = 0$ when $n > k$ and $b_{n,j} = 0$ when $j > n$. We put

$$\Delta_j(g) = \sum_{n\ge 0} c_n b_{n,j} = \sum_{n\ge j} c_n b_{n,j},$$

and so

$$g(X) = \sum_{j\ge 0} \Delta_j(g)\binom{X}{j}. \tag{12}$$

Let $f(X) = \sum_{n\ge 0} c_n X^n$ be an entire function on Ω. Let j be a non-negative integer. If $\lim_{n\to+\infty} \deg(c_n b_{n,j}) = -\infty$, we put

$$\Delta_j(f) = \sum_{n\ge 0} c_n b_{n,j}.$$

THEOREM 14. *Let f be an entire function on Ω and let*

$$\tau(f) = \overline{\lim_{r\to+\infty}} \frac{M(f,r)}{q^r}.$$

If $\tau(f) < 1/e\ln q$, then

(13) $\Delta_j(f)$ *exists for all* $j \in \mathbb{N}$,

(14) $\sum_{j\ge 0} \Delta_j(f)\binom{x}{j}$ *converges for all* $x \in \Omega$,

(15) $f(x) = \sum_{j\ge 0} \Delta_j(f)\binom{x}{j}$ *for all* $x \in \Omega$.

Proof. Let $\tau \in \mathbb{R}^+$ be such that $\tau(f) < \tau < 1/e\ln q$, and let $j \in \mathbb{N}$. By [4, Proposition III.1], there exists $N_1 \in \mathbb{N}$ such that, for all $n \ge N_1$, we have

$$\deg(c_n) \le n\theta - n\log_q n \tag{16}$$

where $\theta = \log_q(e\tau\ln q) < 0$. Let $j \ge N_1$ and $n \ge j$. By (10) and (16), we have

$$\deg(c_n b_{n,j}) \le n\theta - n\log_q n + n\log_q j.$$

We deduce that $\lim_{n\to+\infty} \deg(c_n b_{n,j}) = -\infty$. This proves (13) and for $j \ge N_1$,

$$\deg(\Delta_j(f)) \le \theta j. \tag{17}$$

Let $x \in \Omega$ be of degree δ. By (11) and (17), we have

$$\deg\left(\Delta_j(f)\binom{x}{j}\right) \le -\log_q j + \frac{q^{(2q-1)/(q-1)+\delta}}{e\ln q} + \theta j,$$

hence $\lim_{j\to+\infty} \deg\left(\Delta_j(f)\binom{x}{j}\right) = -\infty$ and (14) holds.

We put

$$\bar f(X) = \sum_{j\ge 0} \Delta_j(f)\binom{X}{j}, \quad f_N(X) = \sum_{n=0}^{N-1} c_n X^n, \quad \bar f_N(X) = \sum_{j=0}^{N-1} \Delta_j(f)\binom{X}{j}.$$

Let $x \in \Omega$ be of degree δ and $A \in \mathbb{R}^+$. We have $\lim_{N\to+\infty} f(x) - f_N(x) = 0$. Therefore, there exists $N_2 \in \mathbb{N}$ such that

$$\deg(f(x) - f_N(x)) \le -A \quad \text{for all } N \ge N_2. \tag{18}$$

In the same way, there exists $N_3 \in \mathbb{N}$ such that

$$\deg(\overline{f}(x) - \overline{f}_N(x)) \le -A \quad \text{for all } N \ge N_3. \tag{19}$$

Let N be an integer $\ge N_1$. With (12), we have

$$f_N(x) - \overline{f}_N(x) = \sum_{j=0}^{N-1} [\Delta_j(f_N) - \Delta_j(f)] \binom{x}{j}.$$

Clearly, $\Delta_0(f_N) = \Delta_0(f)$ and, for all $1 \le j \le N-1$,

$$\Delta_j(f_N) - \Delta_j(f) = \sum_{n\ge N} c_n b_{n,j}.$$

For all $n \ge N$, $\deg(c_n b_{n,j}) \le n(\theta + \log_q j) - n \log_q n \le N\theta$. Hence,

$$\deg(\Delta_j(f_N) - \Delta_j(f)) \le N\theta,$$
$$\deg\left((\Delta_j(f_N) - \Delta_j(f))\binom{x}{j}\right) \le N\theta - \log_q j + \frac{q^{(2q-1)/(q-1)+\delta}}{e\ln q},$$
$$\deg((f_N - \overline{f}_N)(x)) \le N\theta + \frac{q^{(2q-1)/(q-1)+\delta}}{e\ln q}.$$

Therefore, there exists $N_4 \in \mathbb{N}$ such that

$$\deg(f_N(x) - \overline{f}_N(x)) \le -A \quad \text{for all } N \ge N_4. \tag{20}$$

By (18)–(20), for all $N \ge \max(N_1, N_2, N_3, N_4)$ we have

$$\deg(f(x) - \overline{f}(x)) \le -A.$$

As a consequence, $\deg(f(x) - \overline{f}(x)) = -\infty$ and $f(x) = \overline{f}(x)$, that is,

$$f(x) = \sum_{n\ge 0} \Delta_n(f) \binom{x}{n}. \ \blacksquare$$

THEOREM 15. *Let f be an entire function on Ω such that*

$$f(\mathbb{F}_q[T]) \subset \mathbb{F}_q[T], \qquad \overline{\lim_{r\to+\infty}} \frac{M(f,r)}{q^r} < \frac{1}{e\ln q}.$$

Then f is a polynomial of $\mathbb{F}_q(T)[X]$. Moreover, $1/e\ln q$ is optimal.

Proof. By Theorem 14, we have

$$\forall x \in \Omega, \qquad f(x) = \sum_{j\ge 0} \Delta_j(f) \binom{x}{j}.$$

Consequently, the $\Delta_j(f)$ form the solution of the following linear system:

$$
\begin{aligned}
f(u_0) &= \Delta_0(f),\\
f(u_1) &= \Delta_0(f)\binom{u_1}{0} + \Delta_1(f)\binom{u_1}{1},\\
&\vdots\\
f(u_n) &= \Delta_0(f)\binom{u_n}{0} + \Delta_1(f)\binom{u_n}{1} + \cdots + \Delta_n(f)\binom{u_n}{n},\\
&\vdots
\end{aligned}
$$

For all $n \in \mathbb{N}$, $f(u_n) \in \mathbb{F}_q[T]$, $\binom{u_n}{n} \in \mathbb{F}_q^\times$ and, for all $0 \le j < n$, $\binom{u_n}{j} \in \mathbb{F}_q[T]$. By induction, we deduce that for all $j \in \mathbb{N}$, $\Delta_j(f) \in \mathbb{F}_q[T]$. Moreover, we know that, for j large enough, $\deg(\Delta_j) \le \theta j$. Therefore, for j large enough, $\Delta_j(f)$ is a polynomial of negative degree, that is, $\Delta_j(f) = 0$. As a consequence, f is a polynomial of $\mathbb{F}_q(T)[X]$.

Finally, recall that in [4, §VI], M. Car shows that the exponential type of the entire function

$$
f(z) = \sum_{n\ge 0} \binom{z}{q^n}
$$

is $1/e \ln q$, that $f(\mathbb{F}_q[T]) \subset \mathbb{F}_q[T]$, and that f is not a polynomial. This proves that the upper bound $1/e \ln q$ is optimal. ∎

3. The analog of Gel'fond's theorem. In this section, we show how to modify the proof from the previous section to show the following:

THEOREM 16. *Let $f : \Omega \to \Omega$ be an entire function and let $H \in \mathbb{F}_q[T]$ be of degree $h \ge 1$. If, for all $n \in \mathbb{N}$, $f(H^n) \in \mathbb{F}_q[T]$ and*

$$
\overline{\lim_{r\to+\infty}} \frac{M(f,r)}{r^2} < \frac{1}{4h},
$$

then f is a polynomial in $\mathbb{F}_q(T)[X]$. Moreover, the bound $1/4h$ is optimal.

Let $E = \{H^n \mid n \in \mathbb{N}\}$. As for the case of $\mathrm{Int}(\{t^n \mid n \in \mathbb{N}\}, \mathbb{Z})$ where t is an integer ≥ 2 (see [11, Théorème 3]), one shows that the sequence of polynomials

$$
\binom{X}{n}_E = \prod_{k=0}^{n-1} \frac{X - H^k}{H^n - H^k}
$$

is a basis of the $\mathbb{F}_q[T]$-module $\mathrm{Int}(E, \mathbb{F}_q[T])$. Here

$$
n!_E = \prod_{k=0}^{n-1} (H^n - H^k).
$$

Analogously to Section 2, we define the elements $a_{n,k}$ and $b_{n,k}$ of $\mathbb{F}_q[T]$ with similar formulas

$$X^n = \sum_{k=0}^{n} b_{n,k} \binom{X}{k}_E,$$

$$n! \binom{X}{n}_E = \sum_{k=0}^{n} (-1)^{n-k} a_{n,k} X^k.$$

Lemma 8 is then replaced by

LEMMA 17. *The $a_{r,k}$ and $b_{r,k}$ satisfy the recurrence relations*

$$a_{r+1,k} = H^r a_{r,k} + a_{r,k-1}, \tag{21}$$

$$b_{r,k} = H^{k-1}(H^k - 1) b_{r-1,k-1} + H^k b_{r-1,k}. \tag{22}$$

By induction on r, we prove the following proposition that corresponds to Propositions 10, 12 and 13.

PROPOSITION 18. *For all $r \in \mathbb{N}^*$ and $1 \le k \le r$,*

$$\deg(b_{r,k}) \le rkh, \tag{23}$$

$$\deg(a_{r,k}) = (r-1)(r-k)h, \tag{24}$$

and, for all $x \in \Omega$ and $r \ge 1 + \deg(x)/h$,

$$\deg\left(\binom{x}{r}_E\right) \le -rh. \tag{25}$$

PROPOSITION 19. *Let $f(X) = \sum_{n \ge 0} c_n X^n$ be an entire function on Ω and let $\tau \in \mathbb{R}^+$ be such that*

$$\overline{\lim_{r \to +\infty}} \frac{M(f,r)}{r^2} < \tau.$$

Then there exists $N \in \mathbb{N}$ such that, for all $n \ge N$,

$$\deg(c_n) \le -\frac{n^2}{4\tau}.$$

Proof. By (1), there exists $\varrho \in \mathbb{R}^+$ such that

$$r \ge \varrho \;\Rightarrow\; \sup_{n \in \mathbb{N}} \{nr + \deg(c_n)\} \le \tau r^2.$$

For all $n \in \mathbb{N}$, $\deg(c_n) \le \tau r^2 - nr$ if $r \ge \varrho$. Consequently,

$$\deg(c_n) \le \inf_{r \ge \varrho} \{\tau r^2 - nr\} = -\frac{n^2}{4\tau}$$

if $n \ge 2\tau\varrho$. ∎

As in Section 2, if j is an integer such that $\lim_{n \to +\infty} \deg(c_n b_{n,j}) = -\infty$, we define $\Delta_j^E(f)$ by

$$\Delta_j^E(f) = \sum_{n \ge 0} c_n b_{n,j}.$$

Let

$$\tau \in \left]\overline{\lim_{r\to+\infty}} \frac{M(f,r)}{r^2}, \frac{1}{4h}\right[.$$

For all $x \in \Omega$ and for $j \in \mathbb{N}$ large enough,

$$\deg(\Delta_j^E(f)) \le \left(h - \frac{1}{4\tau}\right) j^2, \tag{26}$$

$$\deg\left(\Delta_j^E(f)\binom{x}{j}\right) \le \left(h - \frac{1}{4\tau}\right) j^2 - hj. \tag{27}$$

As a consequence, $\sum_{j\ge0} \Delta_j^E(f)\binom{x}{j}_E$ converges for every $x \in \Omega$ and we prove as in Theorem 14 that

$$f(x) = \sum_{j\ge0} \Delta_j^E(f)\binom{x}{j}_E.$$

Then we may end the proof of Theorem 16 analogously to Car–Pólya's theorem. Now, we give an example which proves that the upper bound $1/4h$ is optimal.

PROPOSITION 20. *The function* $\Psi(z) = \sum_{n\ge0}\binom{z}{n}_E$ *is an entire function on* Ω *such that*

(1) $\overline{\lim\limits_{r\to+\infty}} \dfrac{M(f,r)}{r^2} = \dfrac{1}{4h}$,
(2) $\Psi(H^n) \in \mathbb{F}_q[T]$ *for all* $n \in \mathbb{N}$,
(3) Ψ *is not a polynomial in* $\Omega[X]$.

Proof. By (25), the function Ψ is well defined on Ω. Clearly, $\Psi(H^m) \in \mathbb{F}_q[T]$ for all $m \in \mathbb{N}$, since for every $m, n \in \mathbb{N}$, we have $\binom{H^m}{n}_E = 0$ when $n > m$ and $\binom{H^m}{n} \in \mathbb{F}_q[T]$ when $n \le m$. This proves the second assertion of the proposition.

Let $z \in \Omega$. We can write

$$\Psi(z) = \sum_{r\ge0}\sum_{k=0}^{r}(-1)^{r-k}\frac{a_{r,k}}{r!}z^k.$$

It follows from (24) that $\sum_{r\ge k}(-1)^{r-k}a_{r,k}/r!$ exists and that

$$\deg\left(\sum_{r\ge k}(-1)^{r-k}\frac{a_{r,k}}{r!}\right) = -k^2h, \tag{28}$$

and so

$$\sum_{k\ge0}\left[\sum_{r\ge k}(-1)^{r-k}\frac{a_{r,k}}{r!}\right]z^k \text{ converges.}$$

Now, we prove that

$$\Psi(z)=\sum_{k\ge 0}\Big[\sum_{r\ge k}(-1)^{r-k}\,\frac{a_{r,k}}{r!}\Big]z^k. \tag{29}$$

For all $r,k\in\mathbb{N}$, we put

$$v_{r,k}=(-1)^{r-k}\,\frac{a_{r,k}}{r!}\,z^k.$$

Of course, $v_{r,k}=0$ if $k>r$. For every $R\in\mathbb{N}$, we may write

$$\begin{aligned}\Psi(z)&-\sum_{k\ge 0}\sum_{r\ge k}v_{r,k}\\ &=\Big(\sum_{r\ge 0}\sum_{k=0}^{r}v_{r,k}-\sum_{r=0}^{R}\sum_{k=0}^{r}v_{r,k}\Big)+\Big(\sum_{r=0}^{R}\sum_{k=0}^{r}v_{r,k}-\sum_{k\ge 0}\sum_{r\ge k}v_{r,k}\Big).\end{aligned}$$

Clearly,

$$\sum_{k\ge 0}\sum_{r\ge k}v_{r,k}-\sum_{r=0}^{R}\sum_{k=0}^{r}v_{r,k}=\sum_{k\ge 0}\sum_{r\ge R}v_{r,k}.$$

Consequently,

$$\Psi(z)-\sum_{k\ge 0}\sum_{r\ge k}v_{r,k}=\sum_{r\ge R}\sum_{k=0}^{r}v_{r,k}-\sum_{k\ge 0}\sum_{r\ge R}v_{r,k}.$$

For R large enough and $r\ge R$, we have $\deg(v_{r,k})\le -Rh$. We see that both sums in the previous difference tend to zero as R tends to infinity. Equality (29) is proved and Ψ is an entire function on Ω. From (1), (28) and (29), we have

$$\overline{\lim_{r\to+\infty}}\,\frac{M(f,r)}{r^2}=\frac{1}{4h}.$$

This proves the first part of the proposition.

Finally, Ψ is not a polynomial of $\Omega[X]$ since the quadratic type of any polynomial of $\Omega[X]$ equals zero. This proves the third assertion. ∎

In [4, Corollary IV.8 and Corollary V.2], Car studied entire functions on Ω which are constant on $\mathbb{F}_q[T]$. In [5, Theorem 3], Thiery showed:

Theorem 21. *Let $f:\Omega\to\Omega$ be an entire function, and let $A\in\Omega$. If*

$$f(\mathbb{F}_q[T])=\{A\},\qquad \overline{\lim_{r\to+\infty}}\,\frac{M(f,r)}{q^r}<\frac{q^{q/(q-1)}}{e\ln q},$$

then f is the constant function A. Moreover, $q^{q/(q-1)}/e\ln q$ is optimal.

Here, we are interested in entire functions which are constant on E, that is, an analog for geometric sequences.

Proposition 22. *Let $f : \Omega \to \Omega$ be an entire function, let $A \in \Omega$ and let $H \in \mathbb{F}_q[T]$ be of degree $h \geq 1$. If*

$$f(H^n) = A \quad \text{for all } n \in \mathbb{N}, \qquad \overline{\lim_{r\to+\infty}} \frac{M(f,r)}{r^2} < \frac{1}{2h},$$

then f is the constant function A. Moreover, $1/2h$ is optimal.

Proof. Following Thiery's proof, we assume $A=0$. By [10, Theorem 2.14], the function

$$F(X) = \prod_{n\geq 0} \left(1 - \frac{X}{H^n}\right)$$

is well defined and is an entire function on Ω. Again by [10, Theorem 2.14], f admits a Weierstrass expansion

$$f(X) = cX^k \prod_{\lambda\in\Lambda} \left(1 - \frac{X}{\lambda}\right)^{n_\lambda}$$

where $c \in \Omega^*$, Λ is the set of non-zero roots of f, k is the order of f in 0 and n_λ is the order of zero λ. Since $\{H^n \mid n \in \mathbb{N}\} \subset \Lambda$, for all $x \in \Omega$ we have

$$f(X) = g(X)F(X),$$

where g is an entire function on Ω. We get

$$\overline{\lim_{r\to+\infty}} \frac{M(f,r)}{r^2} \geq \overline{\lim_{r\to+\infty}} \frac{M(F,r)}{r^2} + \overline{\lim_{r\to+\infty}} \frac{M(g,r)}{r^2}.$$

Hence, the following lemma finishes the proof. ■

Lemma 23. *Let F be the previous entire function, i.e.*

$$F(X) = \prod_{n\geq 0} \left(1 - \frac{X}{H^n}\right).$$

Then

$$\overline{\lim_{r\to+\infty}} \frac{M(F,r)}{r^2} = \frac{1}{2h}.$$

Proof. Let $z \in \Omega$ be of degree r. We have

$$\begin{aligned} \text{(30)} \quad \deg(F(z)) &= \sum_{0\leq n\leq[r/h]} \deg\left(1 - \frac{z}{H^n}\right) \\ &= \sum_{0\leq n<[r/h]} \deg\left(\frac{z}{H^n}\right) + \delta_{[r/h],r/h} \deg\left(1 - \frac{z}{H^{[r/h]}}\right) \end{aligned}$$

where $\delta_{i,j}$ denotes the Kronecker symbol. We conclude that for all $r \geq 0$,

$$\frac{M(f,r)}{r^2} \leq \frac{1}{2h} + \frac{3}{2r}.$$

Let $(r_k)_{k\in\mathbb{N}}$ be the sequence of positive integers defined by $r_k = (k+1)h-1/2$, and let $(z_k)_{k\in\mathbb{N}}$ be a sequence of elements of Ω such that $\deg(z_k) = r_k$ for all $k \in \mathbb{N}$. Then, using (30), we obtain

$$\frac{\deg(F(z_k))}{r_k^2} \sim \frac{1}{2h} \quad \text{as } k \to +\infty.$$

This proves the lemma. ∎

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