Real numbers with polynomial continued fraction expansions

by

J. Mc Laughlin and Nancy J. Wyshinski (Hartford, CT)

1. Introduction. A polynomial continued fraction is a continued fraction $\mathrm{K}_{n=1}^{\infty} a_n/b_n$ whose partial numerators and denominators are integers and where, for some tail of the continued fraction, $a_n = f(n)$, $b_n = g(n)$ for some polynomials $f(x), g(x) \in \mathbb{Z}[x]$.

One reason polynomial continued fractions are of interest is that sometimes a famous number whose regular continued fraction expansion does not have a predictable pattern can be expressed as polynomial continued fractions (so that the partial quotients do have a predictable pattern). For example the first continued fraction giving values for π (due to Lord Brouncker, first published in [11]) is of this type:

$$(1.1) \qquad \frac{4}{\pi} = 1 + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{(2n-1)^2}{2}.$$

Another reason polynomial continued fractions are interesting is that the elements are integers and standard theorems can sometimes be used to decide if a number is irrational. For example, there is the following criterion given by Tietze, extending the famous Theorem of Legendre (see Perron [9, pp. 252–253]):

TIETZE'S CRITERION. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of integers and $\{b_n\}_{n=1}^{\infty}$ be a sequence of positive integers, with $a_n \neq 0$ for any n. If there exists a positive integer N_0 such that*

$$(1.2) \qquad \begin{cases} b_n \geq |a_n|, \\ b_n \geq |a_n| + 1 & \text{for } a_{n+1} < 0 \end{cases}$$

for all $n \geq N_0$ then $\mathrm{K}_{n=1}^{\infty} a_n/b_n$ converges and its limit is irrational.

Here we use the standard notations

$$\underset{n=1}{\overset{N}{K}} \frac{a_n}{b_n} := \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots + \cfrac{a_N}{b_N}}}} = \frac{a_1}{b_1+} \frac{a_2}{b_2+} \frac{a_3}{b_3+} \cdots \frac{a_N}{b_N}.$$

We write A_N/B_N for the above finite continued fraction written as a rational function of the variables $a_1, \ldots, a_N, b_1, \ldots, b_N$. Then A_N is the Nth *canonical numerator* and B_N is the Nth *canonical denominator*.

It is elementary that the A_N and B_N satisfy the following recurrence relations:

$$(1.3) \qquad A_N = b_N A_{N-1} + a_N A_{N-2}, \qquad B_N = b_N B_{N-1} + a_N B_{N-2}.$$

By $\mathrm{K}_{n=1}^{\infty} a_n/b_n$ we mean the limit of the sequence $\{A_n/B_n\}$ as n goes to infinity, if the limit exists.

This type of continued fraction seems to have been studied systematically for the first time in Perron [9], where degrees through two for a_n and degree one for b_n are studied. Lorentzen and Waadeland [7] also study these cases in detail and they evaluate all such continued fractions in terms of hypergeometric series. There is presently no such systematic treatment for cases of higher degree.

Ramanujan also gave several continued fraction identities in which the free parameters can be specialized to integers to give the values of polynomial continued fractions. An example of such a continued fraction is the following entry from Chapter 12 in the second notebook (see [2, p. 119]):

ENTRY 13. *Let a, b and d be complex numbers such that either $d \neq 0$, $b \neq -kd$, where k is a non-negative integer, and $\mathrm{Re}((a-b)/d) < 0$ (^{1}), or $d \neq 0$ and $a = b$, or $d = 0$ and $|a| < |b|$. Then*

$$a = \frac{ab}{a+b+d} - \frac{(a+d)(b+d)}{a+b+3d} - \frac{(a+2d)(b+2d)}{a+b+5d} - \cdots.$$

In [4], the first author and Douglas Bowman examined certain classes of polynomial continued fractions in which the a_n and b_n were of arbitrarily high degree. One phenomenon we observed was the existence of infinite families of in-equivalent polynomial continued fractions in which each continued fraction converged to the same limit. To illustrate this phenomenon, we cite the following example from [4]:

EXAMPLE 1. *Let f_n be a polynomial in n such that $f_n \geq 1$ for $n \geq 1$ and let m be a positive integer. Then*

(^{1}) This condition is incorrectly given as "$\mathrm{Re}((a-b)/d) > 0$" in [2].

$$\operatornamewithlimits{\mathrm{K}}_{n=1}^{\infty} \frac{f_n((n^2+3n+2)nm+1)+2mn^2+6mn+4m-1}{f_n((n^2-1)nm+1)+2(n^2-1)m-2} = 6m+1.$$

One reason to study infinite families of polynomial continued fractions in which each continued fraction converges to the same limit is that hopefully one might arrive at a classification of a "base set" such that every convergent polynomial continued fraction has the same limit as one of the continued fractions in the base set. One might further hope that every convergent polynomial continued fraction could be shown to have the same limit as some polynomial continued fraction which is at most quadratic in the numerator and at most linear in the denominator, since the limits of these types of continued fraction can be determined as ratios of hypergeometric series. It would be interesting to either see a proof of this or produce a counter-example.

In [4], our results were derived from a theorem of Pincherle (Theorem 1 below) and a variant of the Euler transformation (see equation (3.2)). In the present paper, we study this phenomenon in greater depth, considering not only further applications of Pincherle's theorem and the Euler formula, but also an extension of Euler's formula, an extension of the corresponding transformation that equates infinite products and continued fractions, extensions and contractions of continued fractions and the Bauer–Muir transformation.

As special cases of our results, we give the following examples (proofs are found throughout the paper).

EXAMPLE 4. Let c_n be any polynomial in n such that $c_n \geq 2$ for $n \geq -1$. Then

$$1 = \frac{c_0 + c_1^2}{c_1^2} + \frac{c_0(c_1 + c_2^2)}{c_2^2} + \cdots + \frac{c_{n-2}(c_{n-1} + c_n^2)}{c_n^2} + \cdots.$$

EXAMPLE 5. Let A be a positive integer. Then

$$\frac{\pi}{4} = \frac{1}{A} + \frac{1}{1} - \frac{4+A}{4-2A}$$
$$+ \operatornamewithlimits{\mathrm{K}}_{n=3}^{\infty} \frac{(2n-3)(2n-5)[A(2n-7)+4(n-3)][A(2n-3)+4(n-1)]}{2A(2n-5)+4(2n-3)}.$$

EXAMPLE 6. If $\zeta(n)$ is the Riemann zeta function and A is a positive integer, then

$$\zeta(11) = \frac{1}{A} + \frac{2A-1}{2A} - \frac{2A(3A-2^{10})}{3A(1+2^{11})-2^{12}}$$
$$+ \operatornamewithlimits{\mathrm{K}}_{n=3}^{\infty} \frac{-n(n-1)^{21}[A(n-1)-(n-2)^{10}][A(n+1)-n^{10}]}{A(n+1)[(n-1)^{11}+n^{11}]-2n^{11}(n-1)^{10}}.$$

EXAMPLE 7. If A is any integer different from -7, then

$$\sqrt[5]{\frac{12}{7}} = \frac{A+7}{7} + \frac{7(11A-7)}{-4A+56}$$
$$+ \underset{n=3}{\overset{\infty}{K}} \frac{7(5n-11)(n-2)[(12n-37)A-7][(12n-13)A-7]}{-2A(12n^2-31n+16)+14(2+n)}.$$

EXAMPLE 8. For each integer A,

$$\frac{3\sqrt{3}}{2\pi} = 1 + A + \frac{-10A-2}{18} + \frac{2736A+432}{-692A-132}$$
$$+ \underset{n=3}{\overset{\infty}{K}} \frac{9n(1-n)(3n-2)(3n-4)[(1+9A)n+A+1][(1+9A)n-17A-1]}{-8(1+A)+2(5+9A)n+144An^2-18(1+9A)n^3}.$$

EXAMPLE 9. If A is a non-negative integer, then

$$e = 2 + \frac{1}{1+A} + \frac{1-2A(1+A)}{2(1+A)} + \frac{2(1-3A(1+A))}{3-5A-6A^2}$$
$$+ \underset{n=4}{\overset{\infty}{K}} \frac{(n-1)[1-nA(1+A)][1-(n-2)A(1+A)]}{n-(n(n-1)-1)A-n(n-1)A^2}.$$

We begin with some applications of Pincherle's theorem.

2. Pincherle's theorem

THEOREM 1 (Pincherle [10]). *Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{G_n\}_{n=-1}^{\infty}$ be sequences of real or complex numbers satisfying $a_n \neq 0$ for $n \geq 1$ and for all $n \geq 1$,*

$$(2.1) \qquad\qquad G_n = a_n G_{n-2} + b_n G_{n-1}.$$

Let $\{B_n\}_{n=1}^{\infty}$ denote the denominator convergents of the continued fraction $K_{n=1}^{\infty} a_n/b_n$. If $\lim_{n\to\infty} G_n/B_n = 0$ then $K_{n=1}^{\infty} a_n/b_n$ converges and its limit is $-G_0/G_{-1}$.

In [4] we looked for solutions where a_n and b_n were polynomials. We then showed that the corresponding B_n and G_n in the statement of the theorem satisfied $\lim_{n\to\infty} G_n/B_n = 0$ by using some easily deduced facts about the growth of B_n:

(i) Let a_n and b_n be non-constant polynomials in n such that $a_n \geq 1$, $b_n \geq 1$ for $n \geq 1$, and suppose b_n is a polynomial of degree k. If the leading coefficient of b_n is D, then given $\varepsilon > 0$, there exists a positive constant $C_1 = C_1(\varepsilon)$ such that $B_n \geq C_1(|D|/(1+\varepsilon))^n(n!)^k$.

(ii) If a_n and b_n are positive numbers ≥ 1, then there exists a positive constant C_3 such that $B_n \geq C_3\phi^n$ for $n \geq 1$, where ϕ is the golden ratio $(1+\sqrt{5})/2$.

The drawbacks to this approach are, firstly, the difficulty in finding solutions to equation (2.1) where the a_n and b_n are polynomials in n, and secondly, in showing that B_n and G_n do indeed satisfy $\lim_{n \to \infty} G_n/B_n = 0$.

Here we describe a very simple way of using Pincherle's theorem to write down infinite families of continued fractions such that each member of the family converges to the same limit. Moreover, $\lim_{n \to \infty} G_n/B_n = 0$ is satisfied automatically and it is not necessary to have a_n and b_n polynomials in n. Initially the a_n's and b_n's are rational functions but $\mathrm{K}_{n=1}^{\infty} a_n/b_n$ can be converted to a polynomial continued fraction by a similarity transformation.

We first need some notation. If $f_n = g_n/h_n$ is a rational function in n, where g_n and h_n are polynomials in n, we define the *degree* of f_n to be the degree of g_n minus the degree of h_n and we define the *leading coefficient* of f_n to be the quotient of the leading coefficient of g_n and the leading coefficient of h_n.

PROPOSITION 1. *Let H_n and b_n be rational functions in n such that $H_n > 0$ for $n \geq -1$ and $b_n > 0$ for $n \geq 1$. We further assume that b_n has degree greater than 0 or that, if it has degree 0, then its leading coefficient is greater than 1. Then the continued fraction*

$$(2.2) \qquad \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{(H_n + b_n H_{n-1})/H_{n-2}}{b_n}$$

converges and its limit is H_0/H_{-1}.

Proof. Let $G_n = (-1)^n H_n$ and $a_n = (H_n + b_n H_{n-1})/H_{n-2}$. The sequences $\{G_n\}_{n=-1}^{\infty}$, $\{a_n\}_{n=-1}^{\infty}$ and $\{b_n\}_{n=-1}^{\infty}$ are easily seen to satisfy equation (2.1). Furthermore, the recurrence relations at (1.3) and the conditions on the b_n entail that B_n grows at least as fast as either D^n for some fixed $D > 1$, or $(n!)^{\delta}$ for some fixed $\delta > 0$. Since G_n is of polynomial growth, it follows that $\lim_{n \to \infty} G_n/B_n = 0$. Thus the continued fraction at (2.2) converges and its limit is $-G_0/G_{-1} = H_0/H_{-1}$. ∎

As an illustration, we have the following example.

EXAMPLE 2. Let $H_n = n + 2$ and let b_n be any polynomial in n such that $b_n \geq 2$ for $n \geq 1$. Then

$$2 = \frac{3 + 2b_1}{b_1} + \frac{4 + 3b_2}{2b_2} + \frac{2(5 + 4b_3)}{3b_3} + \cdots$$
$$+ \frac{(n-1)(n+2+(n+1)b_n)}{nb_n} + \cdots$$

This follows from Proposition 1, after a similarity transformation to clear denominators.

REMARKS. 1) The restrictions that $H_n > 0$ and $b_n > 0$ are not so severe since, if we restrict to polynomial continued fractions $\mathrm{K}_{n=1}^{\infty} a_n/b_n$ in which

the polynomials a_n and b_n have positive leading coefficients, then some tail of the continued fraction will have all $a_n, b_n > 0$ so that, if one can find the limit of the tail, the continued fraction $\mathrm{K}_{n=1}^{\infty} a_n/b_n$ reduces to a finite continued fraction.

2) The limit of the continued fraction is independent of b_n so that one has an infinite class of continued fractions converging to the same limit.

3) Although we restrict H_n and b_n to be rational functions of n, the result can easily be seen to be true for more general sequences of positive numbers, $\{H_n\}$ and $\{b_n\}$, provided the b_n satisfy certain size or growth conditions.

4) Convergence to the stated limit may also hold for other polynomial sequences $\{b_n\}$ which do not satisfy the conditions of the proposition (for example, the continued fraction in Example 2 converges when b_n is the constant polynomial 1).

We next write the rational functions H_n and b_n in terms of the polynomials defining them to obtain a result about polynomial continued fractions. This allows greater flexibility in deriving the limits of infinite families of polynomial continued fractions.

COROLLARY 1. *Let f_n, g_n, c_n and d_n be polynomials in n such that $f_n, g_n \neq 0$ for $n \geq -1$, f_n and g_n have the same sign for each $n \geq -1$, $c_n, d_n \neq 0$ for $n \geq 0$ and c_n and d_n have the same sign for each $n \geq 0$. Suppose further that the degree of c_n is greater than the degree of d_n or, if the degrees are equal, that the leading coefficient of c_n is greater than the leading coefficient of d_n. Then the continued fraction*

$$(2.3) \quad \frac{g_{-1}(d_1 f_1 g_0 + c_1 f_0 g_1)}{c_1 f_{-1} g_0 g_1} + \frac{d_1 f_{-1} g_0^2 (d_2 f_2 g_1 + c_2 f_1 g_2)}{c_2 f_0 g_2}$$

$$+ \frac{d_2 f_0 g_1 (d_3 f_3 g_2 + c_3 f_2 g_3)}{c_3 f_1 g_3} + \frac{d_3 f_1 g_2 (d_4 f_4 g_3 + c_4 f_3 g_4)}{c_4 f_2 g_4} + \cdots$$

$$+ \frac{d_{n-1} f_{n-3} g_{n-2} (d_n f_n g_{n-1} + c_n f_{n-1} g_n)}{c_n f_{n-2} g_n} + \cdots$$

converges and its limit is $f_0 g_{-1}/(g_0 f_{-1})$.

Proof. In Proposition 1, let $H_n = f_n/g_n$ and $b_n = c_n/d_n$. The continued fraction at (2.2) is equivalent to the continued fraction at (2.3), after a similarity transformation. The conditions of Proposition 1 are satisfied and the result follows. ■

REMARK. The limit is independent of the polynomials c_n and d_n, provided they satisfy the conditions of the corollary.

If we let $f_n = n^2 + 1$ and $d_n = g_n = 1$ we get the following example.

EXAMPLE 3. Let c_n be any polynomials in n such that $c_n \geq 2$ for $n \geq 0$. Then

$$\frac{1}{2} = \frac{c_1 + 2}{2c_1} + \frac{2(2c_2 + 5)}{c_2} + \frac{(5c_3 + 10)}{2c_3} + \frac{2(10c_4 + 17)}{5c_4} + \frac{5(17c_5 + 26)}{10c_5} + \cdots$$

$$+ \frac{(1 + (-3 + n)^2)((1 + (-1 + n)^2)c_n + (1 + n^2))}{(1 + (-2 + n)^2)c_n} + \cdots$$

If we let $f_n = d_n = 1$ and $g_n = c_n$ and cancel a factor of c_{-1}/c_0 from the continued fraction and its limit, we get the following result.

EXAMPLE 4. Let c_n be any polynomial in n such that $c_n \geq 2$ for $n \geq -1$. Then

$$1 = \frac{c_0 + c_1^2}{c_1^2} + \frac{c_0(c_1 + c_2^2)}{c_2^2} + \cdots + \frac{c_{-2+n}(c_{-1+n} + c_n^2)}{c_n^2} + \cdots$$

We next study a generalization of Euler's transformation of a series into a continued fraction.

3. The Euler transformation of an infinite series. In 1775, Daniel Bernoulli [3] proved the following result (see, for example, [6, pp. 11–12]).

PROPOSITION 2. *Let* $\{K_0, K_1, \ldots\}$ *be a sequence of complex numbers such that* $K_i \neq K_{i-1}$ *for* $i = 1, 2, \ldots$. *Then* $\{K_0, K_1, \ldots\}$ *is the sequence of approximants of the continued fraction*

$$(3.1) \quad K_0 + \frac{K_1 - K_0}{1} + \frac{K_1 - K_2}{K_2 - K_0} + \frac{(K_1 - K_0)(K_2 - K_3)}{K_3 - K_1} + \cdots$$

$$+ \frac{(K_{n-2} - K_{n-3})(K_{n-1} - K_n)}{K_n - K_{n-2}} + \cdots$$

In particular, if $\{K_n\}$ is a convergent sequence, one gets a convergent continued fraction.

If one lets $K_n = \sum_{k=0}^n a_k$, one gets Euler's transformation of a series into a continued fraction [5]:

$$(3.2) \quad \sum_{k=0}^{\infty} a_k = a_0 + \frac{a_1}{1} + \frac{-a_2}{a_1 + a_2} + \frac{-a_1 a_3}{a_2 + a_3} + \cdots + \frac{-a_{n-2} a_n}{a_{n-1} + a_n} + \cdots$$

For example, applying Euler's transformation to the well known series for $\pi/4$,

$$(3.3) \quad \frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} + \cdots + \frac{(-1)^{n-1}}{2n - 1} + \cdots,$$

gives Lord Brouncker's continued fraction (1.1) (after inversion and some similarity transformations to clear fractions).

We now give a generalization of the Euler transformation.

PROPOSITION 3. *Let* $\sum_{k=0}^{\infty} a_k$ *be a convergent series and let* $\{b_n\}$ *be any sequence whose limit is zero such that* $a_n + b_n - b_{n-1} \neq 0$ *for* $n \geq 1$. *Then,*

for $N \geq 3$,

$$(3.4) \quad \sum_{i=0}^{N} a_i + b_N = a_0 + b_0 + \frac{a_1 + b_1 - b_0}{1} + \frac{-a_2 + b_1 - b_2}{a_2 + a_1 + b_2 - b_0}$$

$$+ \underset{n=3}{\overset{N}{\mathrm{K}}} \frac{-(a_{n-2} + b_{n-2} - b_{n-3})(a_n + b_n - b_{n-1})}{a_n + a_{n-1} + b_n - b_{n-2}},$$

and

$$(3.5) \quad \sum_{i=0}^{\infty} a_i = a_0 + b_0 + \frac{a_1 + b_1 - b_0}{1} + \frac{-a_2 + b_1 - b_2}{a_2 + a_1 + b_2 - b_0}$$

$$+ \underset{n=3}{\overset{\infty}{\mathrm{K}}} \frac{-(a_{n-2} + b_{n-2} - b_{n-3})(a_n + b_n - b_{n-1})}{a_n + a_{n-1} + b_n - b_{n-2}}.$$

Proof. This follows immediately from Proposition 2, upon letting $K_n = \sum_{k=0}^{n} a_k + b_n$, noting that $\lim_{n \to \infty} K_n = \sum_{i=0}^{\infty} a_i$. ∎

REMARK. It is easy to see that (3.4) holds for $N = 0$, 1, and 2 also, provided the right side is truncated at the zeroth, first and second approximant, respectively.

For our first example, we consider the series (3.3) for $\pi/4$.

EXAMPLE 5. Let f_n be any polynomial which is positive for $n \geq 1$ and, for ease of notation, define $g_n := f_n f_{n-1} + (2n - 1)(f_n + f_{n-1})$. Then

$$\frac{\pi}{4} = \frac{-1}{f_0} + \frac{g_1}{f_1 f_0} + \frac{f_0^2 g_2}{2 f_2 f_0 + 3(f_2 - f_0)}$$

$$+ \underset{n=3}{\overset{\infty}{\mathrm{K}}} \frac{(2n - 3)^2 g_{n-2} g_n}{2 f_n f_{n-2} + (2n - 1)(2n - 3)(f_n - f_{n-2})}.$$

This follows from setting $b_n = (-1)^{n-1}/f_n$ in Proposition 3 and simplifying the continued fraction. If we let $f_n = A(2n - 1)$, where A is a positive integer, we get, after some further simplification,

$$\frac{\pi}{4} = \frac{1}{A} + \frac{1}{1} - \frac{4 + A}{4 - 2A}$$

$$+ \underset{n=3}{\overset{\infty}{\mathrm{K}}} \frac{(2n - 3)(2n - 5)[A(2n - 7) + 4(n - 3)][A(2n - 3) + 4(n - 1)]}{2A(2n - 5) + 4(2n - 3)}.$$

As a second example, we consider the series representation for $\zeta(k)$, k an integer greater than 1:

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}.$$

If the Euler transformation is applied to this series (letting $a_0 = 0$ and

$a_i = 1/i^k$ for $i \geq 1$), one easily gets

$$\zeta(k) = \cfrac{1}{1} - \cfrac{1}{2^k + 1} - \cfrac{2^{2k}}{3^k + 2^k} - \cfrac{3^{2k}}{4^k + 3^k} - \cdots - \cfrac{(n-1)^{2k}}{n^k + (n-1)^k} - \cdots.$$

If we set $b_n = 1/d_n$ in Proposition 3, where d_n is a polynomial of degree at least 1 such that $g_n := d_n d_{n-1} + n^k (d_{n-1} - d_n) \neq 0$ for $n \geq 1$, then

$$\zeta(k) = \frac{1}{d_0} + \cfrac{g_1}{d_0 d_1} - \cfrac{d_0^2 g_2}{d_2 d_0 (1 + 2^k) + 2^k (d_0 - d_2)}$$

$$+ \mathop{\mathrm{K}}_{n=3}^{\infty} \frac{-(n-1)^{2k} g_{n-2} g_n}{d_n d_{n-2}((n-1)^k + n^k) + (n-1)^k n^k (d_{n-2} - d_n)}.$$

We can specialize further to get the following continued fraction.

EXAMPLE 6. If A is a positive integer, then

$$\zeta(k) = \frac{1}{A} + \cfrac{2A - 1}{2A} - \cfrac{2A(3A - 2^{k-1})}{3A(1 + 2^k) - 2^{k+1}}$$

$$+ \mathop{\mathrm{K}}_{n=3}^{\infty} \frac{-n(n-1)^{2k-1}[A(n-1) - (n-2)^{k-1}][A(n+1) - n^{k-1}]}{A(n+1)[(n-1)^k + n^k] - 2n^k(n-1)^{k-1}}.$$

This follows from defining $d_n := A(n+1)$ and simplifying the continued fraction.

For our third example, we consider the binomial series. Let $|x| < 1$ and $\alpha \in \mathbb{R}$. Then

$$(3.6) \quad (1 + x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!} x^2 + \frac{\alpha(\alpha - 1)(\alpha - 2)}{3!} x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} x^n,$$

where $(\alpha)_n = \alpha(\alpha - 1) \cdots (\alpha - n + 1)$ denotes the *falling factorial*. If we let $b_n = r_n x^n (\alpha)_n / n!$, where r_n is any rational function such that

$$g_n := (\alpha - n + 1)x(1 + r_n) - n r_{n-1} \neq 0, \quad n \geq 1,$$

then Proposition 3 gives

$$(3.7) \quad (1 + x)^\alpha = 1 + r_0 + \cfrac{g_1}{1} - \cfrac{\alpha x g_2}{\alpha x[(\alpha - 1)x(1 + r_2) + 2] - 2r_0}$$

$$+ \mathop{\mathrm{K}}_{n=3}^{\infty} \frac{-(n-1)x(\alpha - n + 2)g_{n-2}g_n}{(\alpha - n + 2)x[(\alpha - n + 1)x(1 + r_n) + n] - n(n-1)r_{n-2}}.$$

If we specialize by letting $\alpha = 1/5$, $x = 5/7$ and $r_n = An - 1$ ($A \neq -7$), and then simplifying the continued fraction, we get:

EXAMPLE 7. If A is any integer different from -7, then

$$\sqrt[5]{\frac{12}{7}} = \frac{A+7}{7} + \frac{7(11A-7)}{-4A+56}$$
$$+ \underset{n=3}{\overset{\infty}{K}} \frac{7(5n-11)(n-2)[(12n-37)A-7][(12n-13)A-7]}{-2A(12n^2-31n+16)+14(2+n)}.$$

REMARK. It follows from equation (3.7) that every real number of the form $(p/q)^{r/s}$, where p, q, r and s are integers, can be expanded in infinitely many ways as a polynomial continued fraction.

We next consider a generalization of the formula transforming an infinite product to a continued fraction.

4. The transformation of infinite products to continued fractions. In equation (3.1), if $K_i \neq 0$ for $i \geq 1$, then the continued fraction on the left side can be re-written as

$$(4.1) \quad K_0 + \frac{K_1 - K_0}{1} - \frac{K_1/K_0(K_2/K_1 - 1)}{K_2/K_0 - 1}$$
$$- \frac{K_2/K_1(K_1/K_0 - 1)(K_3/K_2 - 1)}{K_3/K_1 - 1} - \cdots$$
$$- \frac{K_{n-1}/K_{n-2}(K_{n-2}/K_{n-3} - 1)(K_n/K_{n-1} - 1)}{K_n/K_{n-2} - 1} - \cdots.$$

In particular, if $K_0 = 1$ and $K_n = \prod_{i=1}^{n} a_i$ for $n \geq 1$, where $\prod_{i=1}^{\infty} a_i$ is a convergent infinite product with no $a_i = 0$ or 1, one has

$$(4.2) \quad \prod_{i=1}^{\infty} a_i = 1 + \frac{a_1 - 1}{1} - \frac{a_1(a_2 - 1)}{a_2 a_1 - 1} + \underset{n=3}{\overset{\infty}{K}} \frac{-a_{n-1}(a_{n-2} - 1)(a_n - 1)}{a_n a_{n-1} - 1}.$$

This transformation is not so well known as Euler's transformation of an infinite series to a continued fraction. As with the Euler transformation, it is easy to generalize the transformation at (4.2).

PROPOSITION 4. *Let $\prod_{i=1}^{\infty} a_i$ be a convergent infinite product with $a_i \neq 0$ for $i \geq 1$ and let $\{b_n\}_{n=0}^{\infty}$ be any sequence whose limit is 1 such that $a_i b_i - b_{i-1} \neq 0$ for $i \geq 1$. Then*

$$(4.3) \quad \prod_{i=1}^{\infty} a_i = b_0 + \frac{a_1 b_1 - b_0}{1} - \frac{a_1(a_2 b_2 - b_1)}{a_2 a_1 b_2 - b_0}$$
$$+ \underset{n=3}{\overset{\infty}{K}} \frac{-a_{n-1}(a_{n-2} b_{n-2} - b_{n-3})(a_n b_n - b_{n-1})}{a_n a_{n-1} b_n - b_{n-2}}.$$

Proof. This follows from Proposition 2 upon setting $K_0 = b_0$ and, for $n \geq 1$, setting $K_n = b_n \prod_{i=1}^{n} a_i$ and then simplifying the continued fraction. ■

As an example, we consider the following infinite product identity:

$$\frac{\sin \pi x}{\pi x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2} \right).$$

Set $x = 1/m$, where m is a positive integer, and $b_n = 1 + A/(n+1)$, where A is an integer. For ease of notation, let

$$g_n = (m^2 n + 1)A + (n+1),$$
$$h_n = (m^2 n^2 - 1)(m^2(n-1)^2 - 1)(A+n+1) - m^4 n^2(n^2-1)(A+n-1).$$

Proposition 4.3 then shows, after simplifying the continued fraction, that

$$\frac{m \sin \pi/m}{\pi} = 1 + A + \frac{(m^2-1)(A+2) - 2m^2(A+1)}{2m^2}$$
$$+ \frac{2m^2(m^2-1)g_2}{h_2} + \underset{n=3}{\overset{\infty}{\mathrm{K}}} \frac{-n(n-1)m^2(m^2(n-1)^2-1)g_{n-2}g_n}{h_n}.$$

If we specialize further and let $m = 3$, we have

EXAMPLE 8. For each integer A,

$$\frac{3\sqrt{3}}{2\pi} = 1 + A + \frac{-10A-2}{18} + \frac{2736A+432}{-692A-132}$$
$$+ \underset{n=3}{\overset{\infty}{\mathrm{K}}} \frac{9n(1-n)(3n-2)(3n-4)[(1+9A)n+A+1][(1+9A)n-17A-1]}{-8(1+A)+2(5+9A)n+144An^2-18(1+9A)n^3}.$$

5. Extensions, contractions and the Bauer–Muir transformation. We start with the concepts of extensions and contractions of continued fractions. Before coming to details, we borrow some notation from [7] (page 83). A continued fraction $d_0 + \mathrm{K}_{n=1}^{\infty} c_n/d_n$ is said to be a *contraction* of the continued fraction $b_0 + \mathrm{K}_{n=1}^{\infty} a_n/b_n$ if its classical approximants $\{g_n\}$ form a subsequence of the classical approximants $\{f_n\}$ of $b_0 + \mathrm{K}_{n=1}^{\infty} a_n/b_n$. In this case $b_0 + \mathrm{K}_{n=1}^{\infty} a_n/b_n$ is called an *extension* of $d_0 + \mathrm{K}_{n=1}^{\infty} c_n/d_n$.

We call $d_0 + \mathrm{K}_{n=1}^{\infty} c_n/d_n$ a *canonical contraction* of $b_0 + \mathrm{K}_{n=1}^{\infty} a_n/b_n$ if

$$C_k = A_{n_k}, \quad D_k = B_{n_k} \quad \text{for } k = 0, 1, \ldots,$$

where C_n, D_n, A_n and B_n are canonical numerators and denominators of $d_0 + \mathrm{K}_{n=1}^{\infty} c_n/d_n$ and $b_0 + \mathrm{K}_{n=1}^{\infty} a_n/b_n$ respectively.

From [7] (pages 83 and 85) we also have the following theorems.

THEOREM 2. *The canonical contraction of* $b_0 + \mathrm{K}_{n=1}^{\infty} a_n/b_n$ *with*

$$C_k = A_{2k}, \quad D_k = B_{2k} \quad \text{for } k = 0, 1, \ldots$$

exists if and only if $b_{2k} \neq 0$ *for* $K = 1, 2, \ldots$, *and in this case is given by*

$$(5.1) \quad b_0 + \frac{b_2 a_1}{b_2 b_1 + a_2} - \frac{a_2 a_3 b_4/b_2}{a_4 + b_3 b_4 + a_3 b_4/b_2} - \frac{a_4 a_5 b_6/b_4}{a_6 + b_5 b_6 + a_5 b_6/b_4} + \cdots$$

The continued fraction (5.1) is called the *even part* of $b_0 + \mathrm{K}_{n=1}^{\infty} a_n/b_n$.

THEOREM 3. *The canonical contraction of* $b_0 + \mathrm{K}_{n=1}^{\infty} a_n/b_n$ *with* $C_0 = A_1/B_1$,

$$C_k = A_{2k+1}, \quad D_k = B_{2k+1} \quad \text{for } k = 1, 2, \ldots$$

exists if and only if $b_{2k+1} \neq 0$ *for* $K = 0, 1, 2, \ldots$, *and in this case is given by*

$$(5.2) \quad \frac{b_0 b_1 + a_1}{b_1} - \frac{a_1 a_2 b_3/b_1}{b_1(a_3 + b_2 b_3) + a_2 b_3} - \frac{a_3 a_4 b_5 b_1/b_3}{a_5 + b_4 b_5 + a_4 b_5/b_3}$$
$$- \frac{a_5 a_6 b_7/b_5}{a_7 + b_6 b_7 + a_6 b_7/b_5} - \frac{a_7 a_8 b_9/b_7}{a_9 + b_8 b_9 + a_8 b_9/b_7} + \cdots.$$

The continued fraction (5.2) is called the *odd part* of $b_0 + \mathrm{K}_{n=1}^{\infty} a_n/b_n$.

One might expect that if a continued fraction is constructed whose even part is the convergent continued fraction $\mathrm{K}_{n=1}^{\infty} a_n/b_n$, one might have some flexibility in the choice of the elements of the extended continued fraction. This is indeed the case and if, in addition, the extended continued fraction can be constructed so that it too converges, an infinite family of continued fractions with the same limit as $\mathrm{K}_{n=1}^{\infty} a_n/b_n$ can be found.

However, before discussing this we first consider another way of transforming a continued fraction so as to produce an infinite family of continued fractions with the same limit.

DEFINITION ([7, p. 76]). The *Bauer–Muir transform* of a continued fraction $b_0 + K(a_n/b_n)$ with respect to a sequence $\{w_n\}$ from $\mathbb{C}$ is the continued fraction $d_0 + K(c_n/d_n)$ whose canonical numerators C_n and denominators D_n are given by

$$(5.3) \quad C_{-1} = 1, \quad C_n = A_n + w_n A_{n-1}, \quad D_{-1} = 0, \quad D_n = B_n + w_n B_{n-1}$$

for $n = 0, 1, \ldots$, where $\{A_n\}$ and $\{B_n\}$ are the canonical numerators and denominators of $b_0 + K(a_n/b_n)$.

This transformation dates back to the 1870's and the work of Bauer [1] and Muir [8]. From [7, p. 76], there is the following theorem:

THEOREM 4. *The Bauer–Muir transform of* $b_0 + K(a_n/b_n)$ *with respect to* $\{w_n\}$ *from* $\mathbb{C}$ *exists if and only if*

$$(5.4) \quad a_n - w_{n-1}(b_n + w_n) \neq 0 \quad \text{for } n = 1, 2, \ldots.$$

If it exists, then it is given by

$$(5.5) \quad b_0 + w_0 + \frac{a_1 - w_0(b_1 + w_1)}{b_1 + w_1} + \frac{c_2}{d_2} + \frac{c_3}{d_3} + \cdots$$

where

$$\text{(5.6)} \quad \begin{aligned} c_n &= a_{n-1} \frac{a_n - w_{n-1}(b_n + w_n)}{a_{n-1} - w_{n-2}(b_{n-1} + w_{n-1})}, \\ d_n &= b_n + w_n - w_{n-2} \frac{a_n - w_{n-1}(b_n + w_n)}{a_{n-1} - w_{n-2}(b_{n-1} + w_{n-1})}. \end{aligned}$$

We have changed the notation found in [7] slightly. One might expect that, by choosing the sequence $\{w_n\}$ appropriately, one could construct an infinite family of continued fractions with the same limit as the convergent continued fraction $K(a_n/b_n)$.

Curiously, these two methods (extensions and contractions and the Bauer–Muir transform) of producing infinite families of continued fractions in which each continued fraction has the same limit as the convergent continued fraction $K(a_n/b_n)$ are related. We believe the following observation to be new.

THEOREM 5. *Let* $\{w_n\}_{n=0}^{\infty}$ *be a sequence from* $\mathbb{C}$ *such that* $w_0 = 0$ *and* $w_n \neq 0$ *for* $n \geq 1$. *Suppose further that* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *are sequences from* $\mathbb{C}$ *such that* $a_n - w_{n-1}(b_n + w_n) \neq 0$ *for* $n = 1, 2, \ldots$ *. Then the even part of the continued fraction*

$$\text{(5.7)} \quad \frac{a_1}{b_1 + w_1} + \frac{-w_1}{1}$$

$$+ \frac{a_2/w_1}{b_2 + w_2 - a_2/w_1} + \frac{-w_2}{1} + \frac{a_3/w_2}{b_3 + w_3 - a_3/w_2} + \cdots$$

$$+ \frac{-w_{n-1}}{1} + \frac{a_n/w_{n-1}}{b_n + w_n - a_n/w_{n-1}} + \frac{-w_n}{1} + \cdots$$

is $K_{n=1}^{\infty} a_n/b_n$ *and its odd part is equal to the Bauer–Muir transform of* $K_{n=1}^{\infty} a_n/b_n$ *with respect to the sequence* $\{w_n\}_{n=0}^{\infty}$.

REMARK. The theorem says that the odd part of the continued fraction at (5.7) "is equal to" the Bauer–Muir transform of $K_{n=1}^{\infty} a_n/b_n$ with respect to the sequence $\{w_n\}_{n=0}^{\infty}$ rather than saying that it "is" this Bauer–Muir transform, since some transformations need to be applied to the odd part to make it equal the stated Bauer–Muir transform of $K_{n=1}^{\infty} a_n/b_n$.

Proof of Theorem 5. That the even part of (5.7) is $K_{n=1}^{\infty} a_n/b_n$ follows immediately from Theorem 2. From Theorem 3, the odd part of (5.7) is

$$\frac{a_1}{b_1 + w_1} - \frac{a_1(-w_1)\dfrac{b_2 + w_2 - a_2/w_1}{b_1 + w_1}}{(b_1 + w_1)\left[\dfrac{a_2}{w_1} + \left(b_2 + w_2 - \dfrac{a_2}{w_1}\right)\right] - w_1(b_2 + w_2 - a_2/w_1)}$$

$$
-\cfrac{a_2/w_1(-w_2)\dfrac{b_3+w_3-a_3/w_2}{b_2+w_2-a_2/w_1}(b_1+w_1)}{a_3/w_2+(b_3+w_3-a_3/w_2)-w_2\dfrac{b_3+w_3-a_3/w_2}{b_2+w_2-a_2/w_1}}
$$

$$
-\cfrac{a_3/w_2(-w_3)\dfrac{b_4+w_4-a_4/w_3}{b_3+w_3-a_3/w_2}}{a_4/w_3+(b_4+w_4-a_4/w_3)-w_3\dfrac{b_4+w_4-a_4/w_3}{b_3+w_3-a_3/w_2}}-\cdots
$$

$$
-\cfrac{a_{n-1}/w_{n-2}(-w_{n-1})\dfrac{b_n+w_n-a_n/w_{n-1}}{b_{n-1}+w_{n-1}-a_{n-1}/w_{n-2}}}{\dfrac{a_n}{w_{n-1}}+\left(b_n+w_n-\dfrac{a_n}{w_{n-1}}\right)-w_{n-1}\dfrac{b_n+w_n-a_n/w_{n-1}}{b_{n-1}+w_{n-1}-a_{n-1}/w_{n-2}}}-\cdots
$$

$$
=\cfrac{a_1}{b_1+w_1}-\cfrac{a_1\dfrac{a_2-w_1(b_2+w_2)}{(b_1+w_1)^2}}{\dfrac{a_2+b_1(b_2+w_2)}{b_1+w_1}}+\cfrac{a_2\dfrac{a_3-w_2(b_3+w_3)}{a_2-w_1(b_2+w_2)}}{b_3+w_3-w_1\dfrac{a_3-w_2(b_3+w_3)}{a_2-w_1(b_2+w_2)}}
$$

$$
+\cfrac{a_3\dfrac{a_4-w_3(b_4+w_4)}{a_3-w_2(b_3+w_3)}}{b_4+w_4-w_2\dfrac{a_4-w_3(b_4+w_4)}{a_3-w_2(b_3+w_3)}}+\cdots
$$

$$
+\cfrac{a_{n-1}\dfrac{a_n-w_{n-1}(b_n+w_n)}{a_{n-1}-w_{n-2}(b_{n-1}+w_{n-1})}}{b_n+w_n-w_{n-2}\dfrac{a_n-w_{n-1}(b_n+w_n)}{a_{n-1}-w_{n-2}(b_{n-1}+w_{n-1})}}+\cdots
$$

Since $b_0 = w_0 = 0$, all that is necessary to complete the proof is to show that, for α arbitrary,

$$
\frac{a_1}{b_1+w_1}+\frac{a_2-w_1(b_2+w_2)}{b_2+w_2+\alpha}=\frac{a_1}{b_1+w_1}-\cfrac{a_1\dfrac{a_2-w_1(b_2+w_2)}{(b_1+w_1)^2}}{\dfrac{a_2+b_1(b_2+w_2)}{b_1+w_1}+\alpha}.
$$

This is immediate from the identity

$$
\frac{E}{F}+\frac{G}{H+\alpha}=\frac{E}{F}-\frac{EG/F^2}{(FH+G)/F+\alpha}.\quad\blacksquare
$$

We next consider the question of when a continued fraction and its Bauer–Muir transform converge to the same limit. One approach is to apply general convergence theorems to the continued fraction at (5.7). If this

continued fraction converges, then its odd and even parts converge to the same limit. Alternatively, we have the following proposition.

PROPOSITION 5. *Let $\{w_n\}_{n=0}^{\infty}$ be any rational function sequence taking only positive values for $n \geq 1$ and suppose $w_0 = 0$. Suppose further that the polynomial sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ take only positive values for $n \geq 1$, that $a_n - w_{n-1}(b_n + w_n) \neq 0$ for $n = 1, 2, \ldots$ and that the continued fraction $\mathrm{K}_{n=1}^{\infty} a_n/b_n$ converges. Then the Bauer–Muir transform of $\mathrm{K}_{n=1}^{\infty} a_n/b_n$ with respect to $\{w_n\}_{n=0}^{\infty}$ converges to the same limit.*

Proof. Suppose $\mathrm{K}_{n=1}^{\infty} a_n/b_n = L$. Then

$$\lim_{n\to\infty} L - \frac{A_n}{B_n} = \lim_{n\to\infty} \frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n} = 0$$

and

$$\left| L - \frac{A_{n+1} + w_n A_n}{B_{n+1} + w_n B_n} \right| \leq \left| L - \frac{A_n}{B_n} \right| + \left| \frac{A_{n+1} + w_n A_n}{B_{n+1} + w_n B_n} - \frac{A_n}{B_n} \right|$$

$$= \left| L - \frac{A_n}{B_n} \right| + \left| \frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n} \right| \frac{1}{1 + w_n B_n/B_{n+1}}$$

$$\leq \left| L - \frac{A_n}{B_n} \right| + \left| \frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n} \right|.$$

The last inequality follows since $w_n B_n/B_{n+1} \geq 0$. Finally, let $n \to \infty$ to get the result. ∎

This can be restated as follows: If a_n and b_n are polynomials taking only positive values for $n \geq 1$ and w_n is a rational function taking only positive values for $n \geq 1$ and w_0 is defined to be 0, then

$$\mathop{\mathrm{K}}_{n=1}^{\infty} \frac{a_n}{b_n} = \frac{a_1}{b_1 + w_1} + \frac{a_1(a_2 - w_1(b_2 + w_2))}{a_1(b_2 + w_2)}$$

$$+ \mathop{\mathrm{K}}_{n=3}^{\infty} \frac{a_{n-1}[a_n - w_{n-1}(b_n + w_n)][a_{n-2} - w_{n-3}(b_{n-2} + w_{n-2})]}{a_{n-1}(b_n + w_n) - w_{n-2}(a_n + b_{n-1}(b_n + w_n))}.$$

As an example, we consider the well known continued fraction expansion for e:

$$e = 2 + \frac{2}{2} + \frac{3}{3} + \frac{4}{4} + \cdots + \frac{n}{n} + \cdots.$$

If we let A be a non-negative integer and define $w_0 := 0$ and $w_n := A(n+1)$ for $n \geq 1$, we get the example below.

EXAMPLE 9. *If A is a non-negative integer, then*

$$e = 2 + \frac{1}{1 + A} + \frac{1 - 2A(1 + A)}{2(1 + A)} + \frac{2(1 - 3A(1 + A))}{3 - 5A - 6A^2}$$

$$+ \mathop{\mathrm{K}}_{n=4}^{\infty} \frac{(n - 1)[1 - nA(1 + A)][1 - (n - 2)A(1 + A)]}{n - (n(n - 1) - 1)A - n(n - 1)A^2}.$$

6. Conclusion. Let $\mathbb{P}$ denote the set of all polynomial continued fractions (^{2}). It is not difficult to construct an injective map from $\mathbb{P}$ to $\mathbb{N}$, so that $\mathbb{P}$ is a countable set and thus, almost all real numbers do *not* have a polynomial continued fraction expansion. Let $\mathbb{P}'$ denote the set of real numbers which are the limits of convergent continued fractions in $\mathbb{P}$. Trivially, $\mathbb{Q} \in \mathbb{P}'$ and, as is shown by examples from the literature, many algebraic and transcendental numbers also have a polynomial continued fraction expansion. We conclude with a number of questions about polynomial continued fractions which we consider interesting.

1) Is there an equivalent classification of the set $\mathbb{P}'$, perhaps in terms of the partial quotients in the regular continued fraction expansion of its elements or in terms of irrationality measure?

2) Does every real algebraic number belong to $\mathbb{P}'$? If not, exhibit a counter-example. Note that the answer to this second question is yes if every real algebraic number α can be expanded as a hypergeometric series $\alpha = \sum_{i=0}^{\infty} b_i$, where each $b_i \in \mathbb{Q}$ and $b_{n+1}/b_n = r(n)$, where $r(x) \in \mathbb{Q}(x)$. We are not aware if this question has been answered in the literature.

3) Since almost all real numbers do *not* belong to $\mathbb{P}'$, exhibit an example of such a number, or perhaps an infinite family of such numbers.

4) With a little more effort, Proposition 5 can be extended to show that, given any convergent polynomial continued fraction, there is an infinite family of convergent polynomial continued fractions with the same limit. Let $\mathbb{P}_{n,d}$ denote the set of all polynomial continued fractions of degree at most n in the numerator and degree at most d in the denominator. Does there exist a pair of non-negative integers n and d such that every element of $\mathbb{P}$ that converges has the same limit as some element of $\mathbb{P}_{n,d}$? Is this true for $n = 2$ and $d = 1$ (^{3})?

5) Trivially, if $\alpha \neq 0$ and $\alpha \in \mathbb{P}'$, then $1/\alpha \in \mathbb{P}'$. Does $\mathbb{P}'$ have any further algebraic structure? Is it true that if $\alpha, \beta \in \mathbb{P}'$, then $\alpha + \beta \in \mathbb{P}'$? Is it true that if $\alpha, \beta \in \mathbb{P}'$, then $\alpha \times \beta \in \mathbb{P}'$? Even negative answers to these questions would be of interest but it would probably not be easy to find counter-examples.

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　　(^{2}) Of course not every element of $\mathbb{P}$ converges to a real number.

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Department of Mathematics
Trinity College
300 Summit Street
Hartford, CT 06106-3100, U.S.A.
E-mail: james.mclaughlin@trincoll.edu
        nancy.wyshinski@trincoll.edu