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On conjectures of Minkowski and Woods for $n = 8$

by

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1. Introduction. Let $\mathbb{L}$ be a lattice in the Euclidean space $\mathbb{R}^n$. By the reduction theory of quadratic forms introduced by Korkine and Zolotareff [9], a cartesian coordinate system may be chosen in $\mathbb{R}^n$ in such a way that $\mathbb{L}$ has a basis of the form $(A_1, 0, 0, \ldots, 0)$, $(a_{2,1}, A_2, 0, \ldots, 0), \ldots, (a_{n,1}, a_{n,2}, \ldots, a_{n,n-1}, A_n)$, where $A_1, \ldots, A_n$ are all positive and further for each $i = 1, \ldots, n$ any two points of the lattice in $\mathbb{R}^{n-i+1}$ with basis $(A_i, 0, \ldots, 0)$, $(a_{i+1,i}, A_{i+1}, 0, \ldots, 0), \ldots, (a_{n,i}, a_{n,i+1}, \ldots, a_{n,n-1}, A_n)$ are at a distance at least A_i apart. Here we shall be considering the following conjecture of Woods:

Conjecture (Woods). If $A_1 \cdots A_n = 1$ and $A_i \leq A_1$ for each i then any closed sphere in $\mathbb{R}^n$ of radius $\sqrt{n}/2$ contains a point of $\mathbb{L}$.

Woods [11, 12, 13] proved this conjecture for $4 \leq n \leq 6$ (see also Cleaver [3] for $n = 4$). Recently Hans-Gill et al. [6] have given a simpler and unified proof of Woods' Conjecture for $n \leq 6$. The present authors [7] proved it for $n = 7$.

Woods [13] showed that his conjecture implies the following conjecture:

Conjecture I. If Λ is a lattice of determinant 1 and there is a sphere $|X| < R$ which contains no point of Λ other than O and has n linearly independent points of Λ on its boundary then Λ is a covering lattice for the closed sphere of radius $\sqrt{n/4}$. Equivalently every closed sphere of radius $\sqrt{n/4}$ lying in $\mathbb{R}^n$ contains a point of Λ.

It is well known that together with the result of McMullen [10], truth of Conjecture I for a fixed n would imply the following long standing conjecture attributed to Minkowski on the product of n non-homogeneous linear forms in n variables:

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Conjecture (Minkowski). Let $L_i = a_{i1}x_1 + \cdots + a_{in}x_n$, $1 \le i \le n$, be n real linear forms in n variables $x_1, \ldots, x_n$ having determinant $\Delta = \det(a_{ij}) \ne 0$. For any given real numbers $c_1, \ldots, c_n$ there exist integers $x_1, \ldots, x_n$ such that

$$|(L_1 + c_1) \cdots (L_n + c_n)| \le |\Delta|/2^n.$$

Minkowski's Conjecture is known to be true for $n \le 7$. For a more detailed history of Minkowski's Conjecture and related results, see Gruber [4], Gruber and Lekkerkerker [5], Bambah et al. [1] and Hans-Gill et al. [7].

In this paper we shall prove

Theorem. *Woods' Conjecture is true for $n = 8$.*

Conjecture I follows for $n = 8$ from our Theorem. Hence Minkowski's Conjecture is proved for $n = 8$. We use the notations and method of proof of our paper [7]. We include some of the details given there for the convenience of the reader. It may be remarked that one can easily supplement this proof to show that in fact any open sphere with radius $\sqrt{2}$ contains a point of $\mathbb{L}$, except in the case $A_1 = \cdots = A_8 = 1$.

In principle, this method can be used in higher dimensions but the details would become much more involved. Even though a part of the proof (see remarks in Section 4) can be extended easily to all n, the remaining part, particularly corresponding to Section 5, will become much harder to settle with these techniques. In [8], the authors have obtained estimates on Woods' Conjecture and hence on Minkowski's Conjecture for $9 \le n \le 22$. These estimates on Minkowski's Conjecture are better than the known ones.

2. Preliminary lemmas. Let $d(\Lambda)$ denote the determinant of a lattice Λ. For the unit sphere S_n with centre O in $\mathbb{R}^n$, the *critical determinant* is defined as $\Delta(S_n) = \inf\{d(\Lambda) : \Lambda \text{ has no non-zero point in the interior of } S_n\}$. Let $\mathbb{L}$ be a lattice in $\mathbb{R}^n$ reduced in the sense of Korkine and Zolotareff. Let $A_1, \ldots, A_n$ be as defined in Section 1.

We state below some preliminary lemmas. Lemmas 1 and 2 are due to Woods [11] while Lemma 3 is due to Korkine and Zolotareff [9]. In Lemma 4, the case $n = 3$ is a classical result of Gauss, $n = 4, 5$ are due to Korkine and Zolotareff [9] while $n = 6, 7, 8$ are due to Blichfeldt [2].

Lemma 1. *If $2\Delta(S_{n+1})A_1^n \ge d(\mathbb{L})$ then any closed sphere of radius*

$$R = A_1\{1 - (A_1^n \Delta(S_{n+1})/d(\mathbb{L}))^2\}^{1/2}$$

in $\mathbb{R}^n$ contains a point of $\mathbb{L}$.

Lemma 2. *For a fixed integer i with $1 \le i \le n-1$, denote by $\mathbb{L}_1$ the lattice in $\mathbb{R}^i$ with the reduced basis*

$$(A_1, 0, \ldots, 0), (a_{2,1}, A_2, 0, \ldots, 0), \ldots, (a_{i,1}, a_{i,2}, \ldots, a_{i,i-1}, A_i)$$

and denote by $\mathbb{L}_2$ the lattice in $\mathbb{R}^{n-i}$ with the reduced basis

$$(A_{i+1}, 0, \ldots, 0), (a_{i+2,i+1}, A_{i+2}, 0, \ldots, 0), \ldots, (a_{n,i+1}, a_{n,i+2}, \ldots, a_{n,n-1}, A_n).$$

If any sphere in $\mathbb{R}^i$ of radius r_1 contains a point of $\mathbb{L}_1$ and if any sphere in $\mathbb{R}^{n-i}$ of radius r_2 contains a point of $\mathbb{L}_2$ then any sphere in $\mathbb{R}^n$ of radius $(r_1^2 + r_2^2)^{1/2}$ contains a point of $\mathbb{L}$.

LEMMA 3. *For all relevant i, $A_{i+1}^2 \geq \frac{3}{4} A_i^2$ and $A_{i+2}^2 \geq \frac{2}{3} A_i^2$.*

LEMMA 4. $\Delta(S_n) = 1/\sqrt{2}$, $1/2$, $1/2\sqrt{2}$, $\sqrt{3}/8$, $1/8$, $1/16$ *for* $n = 3, 4, 5, 6, 7, 8$ *respectively.*

3. Plan of the proof. We assume that Woods' Conjecture is false for $n = 8$ and derive a contradiction. Let $\mathbb{L}$ be a lattice satisfying the hypothesis of the conjecture for $n = 8$. Suppose that there exists a closed sphere of radius $\sqrt{2}$ in $\mathbb{R}^8$ that contains no point of $\mathbb{L}$. Write $A = A_1^2$, $B = A_2^2$, $C = A_3^2$, $D = A_4^2$, $E = A_5^2$, $F = A_6^2$, $G = A_7^2$ and $H = A_8^2$. As $A_1 \cdots A_n = 1$ we have $ABCDEFGH = 1$.

We give some examples of inequalities that arise. Let $\mathbb{L}_i$, $1 \leq i \leq 5$, be lattices in $\mathbb{R}^1$ with basis (A_i) and $\mathbb{L}_6$ be a lattice in $\mathbb{R}^3$ with basis $(A_6, 0, 0)$, $(a_{7,6}, A_7, 0)$, $(a_{8,6}, a_{8,7}, A_8)$. Applying Lemma 2 repeatedly and using Lemma 1, we see that if $2\Delta(S_4)A_6^3 \geq A_6 A_7 A_8$ then any closed 8-sphere of radius

$$\left(\frac{1}{4}A_1^2 + \frac{1}{4}A_2^2 + \frac{1}{4}A_3^2 + \frac{1}{4}A_4^2 + \frac{1}{4}A_5^2 + A_6^2 - \frac{A_6^8 \Delta(S_4)^2}{A_6^2 A_7^2 A_8^2}\right)^{1/2}$$

contains a point of $\mathbb{L}$. By the initial hypothesis this radius exceeds $\sqrt{2}$. Since $\Delta(S_4) = 1/2$ and $A_1 \cdots A_8 = 1$, this results in the conditional inequality

$$\text{(3.1)} \quad \text{if} \quad F^2 \geq GH \quad \text{then} \quad A + B + C + D + E + 4F - F^4 ABCDE > 8.$$

We call this inequality $(1, 1, 1, 1, 1, 3)$, since it corresponds to the ordered partition $(1, 1, 1, 1, 1, 3)$ of 8 for the purpose of applying Lemma 2. Similarly the conditional inequality $(1, 1, 1, 1, 1, 1, 2)$ corresponding to the ordered partition $(1, 1, 1, 1, 1, 1, 2)$ is

$$\text{(3.2)} \qquad \text{if} \quad 2G \geq H \quad \text{then} \quad A + B + C + D + E + F + 4G - \frac{2G^2}{H} > 8.$$

Since $4G - 2G^2/H \leq 2H$, the second inequality in (3.2) gives

$$\text{(3.3)} \qquad A + B + C + D + E + F + 2H > 8.$$

Using $ABCDEFGH = 1$, the second inequality in (3.2) can also be written as

$$\text{(3.4)} \qquad A + B + C + D + E + F + 4G - 2G^3 ABCDEF > 8.$$

Inequality $(4,1,1,1,1)$ is

$$\text{(3.5)} \quad \text{if} \quad A^4EFGH \geq 2 \quad \text{then} \quad 4A - \frac{1}{2}A^5EFGH + E + F + G + H > 8.$$

In general, if $(\lambda_1, \ldots, \lambda_s)$ is an ordered partition of n, then the conditional inequality arising from it, by using Lemmas 1 and 2, is also denoted by $(\lambda_1, \ldots, \lambda_s)$. If the conditions in an inequality $(\lambda_1, \ldots, \lambda_s)$ are satisfied then we say that $(\lambda_1, \ldots, \lambda_s)$ holds.

Sometimes, instead of Lemma 1, we are able to use the fact that Woods' Conjecture is true for dimensions less than or equal to 6. The use of this is indicated by putting an asterisk on the corresponding part of the partition. For example, the inequality $(6^*, 2)$ is

$$\text{(3.6)} \quad \text{if} \quad 2G \geq H \quad \text{then} \quad 6(ABCDEF)^{1/6} + 4G - 2G^2/H > 8,$$

the hypothesis of the conjecture in six variables being satisfied.

Throughout the paper we shall use the following notation: $a = A - 1$, $b = |B - 1|$, $c = |C - 1|$, $d = |D - 1|$, $e = |E - 1|$, $f = |F - 1|$, $g = |G - 1|$, $h = |H - 1|$. We can assume $A > 1$, because if $A \leq 1$, we must have $A = B = C = D = E = F = G = H = 1$. In this case Woods' Conjecture can be seen to be true using inequality $(1,1,1,1,1,1,1,1)$. Also the lattice $\mathbb{L}$ has no point in the interior of the sphere of radius A_1 centred at the origin. Therefore $\Delta(A_1S_8) \leq 1$. As $\Delta(S_8) = 1/16$, we get $A^8 \leq 256$, which implies $A \leq 2$.

Each of $B, C, \ldots, H$ can be either > 1 or ≤ 1. This gives rise to $2^7 = 128$ cases which are listed in Table 1. Case 1 does not arise because $ABCDEFGH = 1$. For the remaining cases we give the proposition in which each case is considered. We also indicate the inequalities used to get a contradiction in 113 easy cases. These are discussed in Section 4. The remaining 14 cases which have no inequality indicated need a more intricate analysis of available inequalities. Out of these cases, five are somewhat less difficult and have been dealt with in Propositions 12–16. The remaining nine difficult cases are dealt with separately in Section 5.

We would like to remark that in many cases there are alternative ways to get a contradiction. We have chosen to describe the method which we find convenient. The following observations help us to check that the conditions in certain inequalities are satisfied:

Since $A_1 \geq A_i$ for $2 \leq i \leq 8$, we have $\sqrt{2}A_1 \geq A_2$ and $A_1^2 \geq A_2A_3$. Thus $(2,1,1,1,1,1,1)$ and $(3,1,1,1,1,1)$ hold. Using Lemma 3, we get $2B \geq C$, $2C \geq D$, $2D \geq E$. Thus $(1,2,1,1,1,1,1)$, $(1,2,2,1,1,1)$, $(3,2,1,1,1)$ etc. always hold. If $A_i > 1$ then $A_{i+3}^2 \geq \frac{3}{4}A_{i+2}^2 \geq \frac{3}{4} \cdot \frac{2}{3}A_i^2 > \frac{1}{2}$. Thus if we also have $A_{i+4} \leq 1$, then $\sqrt{2}A_{i+3} > A_{i+4}$.

We also observe that for positive real numbers $X_1, \dots, X_k$ we have $X_1 + \dots + X_k \le (k-1) + X_1 \cdots X_k$ if either all $X_i \le 1$ or all $X_i > 1$. This we shall use several times without referring to it.

In this paper we frequently need to maximize functions of several variables. While doing this we shall find it convenient to name the function involved as $\phi(x), \psi(y)$ etc. to indicate that it is being regarded as a function of that variable and other variables are kept fixed. When we say that a given function of several variables in $x, y, \dots$ is an increasing/decreasing function of $x, y, \dots$, it means that the relevant property holds when the function is considered as a function of one variable at a time, all other variables being fixed. Sometimes the same name is given to different functions in the proof of a proposition. We think it causes no confusion since in the proof of a particular claim we have taken care to give different names to different functions.

Almost all inequalities in the proofs have been checked using calculus except those specifically mentioned.

4. Easy cases. Here we illustrate how contradiction is obtained in the easy cases. Some of the lemmas that we use are obvious generalizations of the lemmas that we have proved in [7], so we shall omit proofs of these. Since the corresponding cases can be dealt with in the same manner, we state these without any illustration. It may be remarked that these lemmas generalize to dimension n and imply the conclusions for the corresponding cases there.

PROPOSITION 1. *Cases which have $G > 1$ and $H \le 1$ do not arise.*

Proof. Note that inequality $(6^*, 2)$ together with $ABCDEFGH = 1$ gives $6(GH)^{-1/6} + 2H > 8$. The left side of this inequality is less than $6H^{-1/6} + 2H$, which is an increasing function of H for $H > \frac{3}{4}$ (the lower bound on H follows from Lemma 3). Since $H \le 1$, we get a contradiction.

PROPOSITION 2. *Cases which have $F > 1$, $G \le 1$ and $H \le 1$ do not arise.*

Proof. Observe that inequality $(5^*, 3)$ together with $ABCDEFGH = 1$ gives $5(FGH)^{-1/5} + 4F - \frac{F^3}{GH} > 8$, i.e. $5(Fx)^{-1/5} + 4F - F^3x^{-1} > 8$, where $x = GH \le 1$. The left side is an increasing function of x, so we can replace x by 1 to get $5F^{-1/5} + 4F - F^3 > 8$, which is not true for $F > 1$.

REMARK 1. Proposition 1 settles 32 cases and Proposition 2 settles 16 cases. Both these propositions can be proved in general, settling $2^{n-3} + 2^{n-4}$ cases in dimension n.

PROPOSITION 3. *Cases in which $B \le 1$ and either of the following holds, do not arise:*

(i) *at most two out of* C, D, E, F, G, H *are greater than* 1,

(ii) *any three out of* C, D, E, F, G, H *are greater than* 1 *and* $A < 1.196$.

Proof. We illustrate a case when exactly two out of C, D, E, F, G, H are greater than 1. Consider Case (125) where $G > 1$, $H > 1$ and C, D, E, F are all ≤ 1. Inequality $(2, 1, 1, 1, 1, 1, 1)$ gives $4A - 2A^3CDEFGH + C + D + E + F + G + H > 8$, i.e. $4A - 2A^3xGH + 3 + x + G + H > 8$, where $x = CDEF \ge \frac{1}{AGH}$. We can successively replace x by $\frac{1}{AGH}$, G by A and H by A to get $6A - 2A^2 + \frac{1}{A^3} > 5$, which is not true for $1 < A \le 2$. The proof is similar if three or one (or none) out of C, D, E, F, G, H are greater than 1. The case when none of C, D, E, F, G, H are greater than 1 can also be seen directly by inequality $(2, 1, \ldots, 1)$, which gives $2B + C + D + E + F + G + H > 8$.

REMARK 2. Proposition 3(i) settles 22 cases (many of these have already been settled by Propositions 1 and 2). The new cases settled are (88), (96), (104), (111), (112), (119), (120), (123), (125), (127) and (128). Proposition 3(ii) will be used to settle Case (121) in Proposition 7. This proposition can also be proved in general.

LEMMA 5. *Let* $X_1, \ldots, X_8$ *be positive real numbers, each* ≤ 2, *satisfying* $X_1 > 1$ *and* $X_1 \cdots X_8 = 1$. *Then the following hold:*

(i) *If* $X_i > 1$ *for* $3 \le i \le 8$, *then*

$$\mathfrak{S}_1 = 4X_1 - \frac{2X_1^2}{X_2} + X_3 + \cdots + X_8 \le 8.$$

(ii) *If* $X_i > 1$ *for* $i = 3, 5, 6, 7, 8$, *then*

$$\mathfrak{S}_2 = 4X_1 - \frac{2X_1^2}{X_2} + 4X_3 - \frac{2X_3^2}{X_4} + X_5 + \cdots + X_8 \le 8.$$

(iii) *If* $X_i > 1$ *for* $i = 3, 5, 7, 8$, *then*

$$\mathfrak{S}_3 = 4X_1 - \frac{2X_1^2}{X_2} + 4X_3 - \frac{2X_3^2}{X_4} + 4X_5 - \frac{2X_5^2}{X_6} + X_7 + X_8 \le 8.$$

(iv) *If* $X_i > 1$ *for* $i = 4, 7, 8$ *and* $X_7 < X_1$, $X_8 < X_1$, *then*

$$\mathfrak{S}_4 = 4X_1 - \frac{X_1^3}{X_2X_3} + 4X_4 - \frac{X_4^3}{X_5X_6} + X_7 + X_8 \le 8.$$

(v) *If* $X_i > 1$ *for* $i = 3, 5, 8$, $X_2 \le X_1$, $X_4 \le X_3$, $X_8 \le X_1X_5$, *then*

$$\mathfrak{S}_5 = 4X_1 - \frac{2X_1^2}{X_2} + 4X_3 - \frac{2X_3^2}{X_4} + 4X_5 - \frac{X_5^3}{X_6X_7} + X_8 \le 8.$$

(vi) *If* $X_i > 1$ *for* $i = 4, 6, 7, 8$, $X_i \le X_1X_4$ *for* $i = 6, 7, 8$, $X_5 \le X_4$, *then*

$$\mathfrak{S}_6 = 4X_1 - \frac{X_1^3}{X_2X_3} + 4X_4 - \frac{2X_4^2}{X_5} + X_6 + X_7 + X_8 \le 8.$$

Table 1

Case	A	B	C	D	E	F	G	H	Proposition	Inequalities
1	$>$	$>$	$>$	$>$	$>$	$>$	$>$	$>$	–	$ABCDEFGH = 1$
2	$>$	$>$	$>$	$>$	$>$	$>$	$>$	$\leq$	1	$(6^*, 2)$
3	$>$	$>$	$>$	$>$	$>$	$>$	$\leq$	$>$	4(i)	$(1, 1, 1, 1, 1, 2, 1)$
4	$>$	$>$	$>$	$>$	$>$	$>$	$\leq$	$\leq$	2	$(5^*, 3)$
5	$>$	$>$	$>$	$>$	$>$	$\leq$	$>$	$>$	4(i)	$(1, 1, 1, 1, 2, 1, 1)$
6	$>$	$>$	$>$	$>$	$>$	$\leq$	$>$	$\leq$	1	$(6^*, 2)$
7	$>$	$>$	$>$	$>$	$>$	$\leq$	$\leq$	$>$	5(i)	$(1, 1, 1, 1, 3, 1)$
8	$>$	$>$	$>$	$>$	$>$	$\leq$	$\leq$	$\leq$	17	–
9	$>$	$>$	$>$	$>$	$\leq$	$>$	$>$	$>$	4(i)	$(1, 1, 1, 2, 1, 1, 1)$
10	$>$	$>$	$>$	$>$	$\leq$	$>$	$>$	$\leq$	1	$(6^*, 2)$
11	$>$	$>$	$>$	$>$	$\leq$	$>$	$\leq$	$>$	4(ii)	$(1, 1, 1, 2, 2, 1)$
12	$>$	$>$	$>$	$>$	$\leq$	$>$	$\leq$	$\leq$	2	$(5^*, 3)$
13	$>$	$>$	$>$	$>$	$\leq$	$\leq$	$>$	$>$	5(i)	$(1, 1, 1, 3, 1, 1)$
14	$>$	$>$	$>$	$>$	$\leq$	$\leq$	$>$	$\leq$	1	$(6^*, 2)$
15	$>$	$>$	$>$	$>$	$\leq$	$\leq$	$\leq$	$>$	18	–
16	$>$	$>$	$>$	$>$	$\leq$	$\leq$	$\leq$	$\leq$	19	–
17	$>$	$>$	$>$	$\leq$	$>$	$>$	$>$	$>$	4(i)	$(1, 1, 2, 1, 1, 1, 1)$
18	$>$	$>$	$>$	$\leq$	$>$	$>$	$>$	$\leq$	1	$(6^*, 2)$
19	$>$	$>$	$>$	$\leq$	$>$	$>$	$\leq$	$>$	4(ii)	$(1, 1, 2, 1, 2, 1)$
20	$>$	$>$	$>$	$\leq$	$>$	$>$	$\leq$	$\leq$	2	$(5^*, 3)$
21	$>$	$>$	$>$	$\leq$	$>$	$\leq$	$>$	$>$	4(ii)	$(1, 1, 2, 2, 1, 1)$
22	$>$	$>$	$>$	$\leq$	$>$	$\leq$	$>$	$\leq$	1	$(6^*, 2)$
23	$>$	$>$	$>$	$\leq$	$>$	$\leq$	$\leq$	$>$	4(v)	$(2, 2, 3, 1)$
24	$>$	$>$	$>$	$\leq$	$>$	$\leq$	$\leq$	$\leq$	16	–
25	$>$	$>$	$>$	$\leq$	$\leq$	$>$	$>$	$>$	5(i)	$(1, 1, 3, 1, 1, 1)$
26	$>$	$>$	$>$	$\leq$	$\leq$	$>$	$>$	$\leq$	1	$(6^*, 2)$
27	$>$	$>$	$>$	$\leq$	$\leq$	$>$	$\leq$	$>$	4(v)	$(2, 3, 2, 1)$
28	$>$	$>$	$>$	$\leq$	$\leq$	$>$	$\leq$	$\leq$	2	$(5^*, 3)$
29	$>$	$>$	$>$	$\leq$	$\leq$	$\leq$	$>$	$>$	20	–
30	$>$	$>$	$>$	$\leq$	$\leq$	$\leq$	$>$	$\leq$	1	$(6^*, 2)$
31	$>$	$>$	$>$	$\leq$	$\leq$	$\leq$	$\leq$	$>$	21	–
32	$>$	$>$	$>$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	22	–
33	$>$	$>$	$\leq$	$>$	$>$	$>$	$>$	$>$	4(i)	$(1, 2, 1, 1, 1, 1, 1)$
34	$>$	$>$	$\leq$	$>$	$>$	$>$	$>$	$\leq$	1	$(6^*, 2)$
35	$>$	$>$	$\leq$	$>$	$>$	$>$	$\leq$	$>$	4(ii)	$(1, 2, 1, 1, 2, 1)$
36	$>$	$>$	$\leq$	$>$	$>$	$>$	$\leq$	$\leq$	2	$(5^*, 3)$
37	$>$	$>$	$\leq$	$>$	$>$	$\leq$	$>$	$>$	4(ii)	$(1, 2, 1, 2, 1, 1)$
38	$>$	$>$	$\leq$	$>$	$>$	$\leq$	$>$	$\leq$	1	$(6^*, 2)$

Table 1 (cont.)

Case	A	B	C	D	E	F	G	H	Proposition	Inequalities
39	$>$	$>$	$\leq$	$>$	$>$	$\leq$	$\leq$	$>$	4(iv)	$(3,1,3,1)$
40	$>$	$>$	$\leq$	$>$	$>$	$\leq$	$\leq$	$\leq$	9	$(1,2,2,1,2),(3,1,3,1)$
41	$>$	$>$	$\leq$	$>$	$\leq$	$>$	$>$	$>$	4(ii)	$(1,2,2,1,1,1)$
42	$>$	$>$	$\leq$	$>$	$\leq$	$>$	$>$	$\leq$	1	$(6^*,2)$
43	$>$	$>$	$\leq$	$>$	$\leq$	$>$	$\leq$	$>$	4(iii)	$(1,2,2,2,1)$
44	$>$	$>$	$\leq$	$>$	$\leq$	$>$	$\leq$	$\leq$	2	$(5^*,3)$
45	$>$	$>$	$\leq$	$>$	$\leq$	$\leq$	$>$	$>$	4(iv)	$(3,3,1,1)$
46	$>$	$>$	$\leq$	$>$	$\leq$	$\leq$	$>$	$\leq$	1	$(6^*,2)$
47	$>$	$>$	$\leq$	$>$	$\leq$	$\leq$	$\leq$	$>$	14	–
48	$>$	$>$	$\leq$	$>$	$\leq$	$\leq$	$\leq$	$\leq$	5(ii)	$(1,2,2,1,1,1),(3,1,\ldots,1)$
49	$>$	$>$	$\leq$	$\leq$	$>$	$>$	$>$	$>$	5(i)	$(1,3,1,1,1,1)$
50	$>$	$>$	$\leq$	$\leq$	$>$	$>$	$>$	$\leq$	1	$(6^*,2)$
51	$>$	$>$	$\leq$	$\leq$	$>$	$>$	$\leq$	$>$	12	–
52	$>$	$>$	$\leq$	$\leq$	$>$	$>$	$\leq$	$\leq$	2	$(5^*,3)$
53	$>$	$>$	$\leq$	$\leq$	$>$	$\leq$	$>$	$>$	13	–
54	$>$	$>$	$\leq$	$\leq$	$>$	$\leq$	$>$	$\leq$	1	$(6^*,2)$
55	$>$	$>$	$\leq$	$\leq$	$>$	$\leq$	$\leq$	$>$	11	$(1,3,3,1),(2,2,2,1,1),(3,1,\ldots,1)$
56	$>$	$>$	$\leq$	$\leq$	$>$	$\leq$	$\leq$	$\leq$	5(ii)	$(1,2,1,2,1,1),(1,3,1,\ldots,1)$
57	$>$	$>$	$\leq$	$\leq$	$\leq$	$>$	$>$	$>$	23	–
58	$>$	$>$	$\leq$	$\leq$	$\leq$	$>$	$>$	$\leq$	1	$(6^*,2)$
59	$>$	$>$	$\leq$	$\leq$	$\leq$	$>$	$\leq$	$>$	15	–
60	$>$	$>$	$\leq$	$\leq$	$\leq$	$>$	$\leq$	$\leq$	2	$(5^*,3)$
61	$>$	$>$	$\leq$	$\leq$	$\leq$	$\leq$	$>$	$>$	24	–
62	$>$	$>$	$\leq$	$\leq$	$\leq$	$\leq$	$>$	$\leq$	1	$(6^*,2)$
63	$>$	$>$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$>$	25	–
64	$>$	$>$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	5(ii)	$(1,2,1,\ldots,1),(3,1,\ldots,1)$
65	$>$	$\leq$	$>$	$>$	$>$	$>$	$>$	$>$	4(i)	$(2,1,\ldots,1)$
66	$>$	$\leq$	$>$	$>$	$>$	$>$	$>$	$\leq$	1	$(6^*,2)$
67	$>$	$\leq$	$>$	$>$	$>$	$>$	$\leq$	$>$	4(ii)	$(2,1,1,1,2,1)$
68	$>$	$\leq$	$>$	$>$	$>$	$>$	$\leq$	$\leq$	2	$(5^*,3)$
69	$>$	$\leq$	$>$	$>$	$>$	$\leq$	$>$	$>$	4(ii)	$(2,1,1,2,1,1)$
70	$>$	$\leq$	$>$	$>$	$>$	$\leq$	$>$	$\leq$	1	$(6^*,2)$
71	$>$	$\leq$	$>$	$>$	$>$	$\leq$	$\leq$	$>$	4(vi)	$(2,1,1,3,1)$
72	$>$	$\leq$	$>$	$>$	$>$	$\leq$	$\leq$	$\leq$	10	$(2,1,1,2,2),(2,1,1,3,1)$
73	$>$	$\leq$	$>$	$>$	$\leq$	$>$	$>$	$>$	4(ii)	$(2,1,1,2,1,1)$
74	$>$	$\leq$	$>$	$>$	$\leq$	$>$	$>$	$\leq$	1	$(6^*,2)$
75	$>$	$\leq$	$>$	$>$	$\leq$	$>$	$\leq$	$>$	4(iii)	$(2,1,2,1,2)$
76	$>$	$\leq$	$>$	$>$	$\leq$	$>$	$\leq$	$\leq$	2	$(5^*,3)$

Table 1 (cont.)

Case	A	B	C	D	E	F	G	H	Proposition	Inequalities
77	$>$	$\leq$	$>$	$>$	$\leq$	$\leq$	$>$	$>$	4(vi)	$(2, 1, 3, 1, 1)$
78	$>$	$\leq$	$>$	$>$	$\leq$	$\leq$	$>$	$\leq$	1	$(6^*, 2)$
79	$>$	$\leq$	$>$	$>$	$\leq$	$\leq$	$\leq$	$>$	10	$(2, 1, 2, 2, 1), (2, 1, 3, 1, 1)$
80	$>$	$\leq$	$>$	$>$	$\leq$	$\leq$	$\leq$	$\leq$	5(ii)	$(2, 1, 2, 1, 1, 1), (3, 1, \ldots, 1)$
81	$>$	$\leq$	$>$	$\leq$	$>$	$>$	$>$	$>$	4(ii)	$(2, 2, 1, 1, 1, 1)$
82	$>$	$\leq$	$>$	$\leq$	$>$	$>$	$>$	$\leq$	1	$(6^*, 2)$
83	$>$	$\leq$	$>$	$\leq$	$>$	$>$	$\leq$	$>$	4(iii)	$(2, 2, 1, 2, 1)$
84	$>$	$\leq$	$>$	$\leq$	$>$	$>$	$\leq$	$\leq$	2	$(5^*, 3)$
85	$>$	$\leq$	$>$	$\leq$	$>$	$\leq$	$>$	$>$	4(iii)	$(2, 2, 2, 1, 1)$
86	$>$	$\leq$	$>$	$\leq$	$>$	$\leq$	$>$	$\leq$	1	$(6^*, 2)$
87	$>$	$\leq$	$>$	$\leq$	$>$	$\leq$	$\leq$	$>$	5(ii)	$(2, 2, 2, 1, 1), (3, 1, \ldots, 1)$
88	$>$	$\leq$	$>$	$\leq$	$>$	$\leq$	$\leq$	$\leq$	3(i)	$(2, 1, \ldots, 1)$
89	$>$	$\leq$	$>$	$\leq$	$\leq$	$>$	$>$	$>$	4(vi)	$(2, 3, 1, 1, 1)$
90	$>$	$\leq$	$>$	$\leq$	$\leq$	$>$	$>$	$\leq$	1	$(6^*, 2)$
91	$>$	$\leq$	$>$	$\leq$	$\leq$	$>$	$\leq$	$>$	5(ii)	$(2, 2, 1, 2, 1), (3, 1, \ldots, 1)$
92	$>$	$\leq$	$>$	$\leq$	$\leq$	$>$	$\leq$	$\leq$	2	$(5^*, 3)$
93	$>$	$\leq$	$>$	$\leq$	$\leq$	$\leq$	$>$	$>$	8	$(2, 2, 2, 2), (2, 3, 1, 1, 1)$
94	$>$	$\leq$	$>$	$\leq$	$\leq$	$\leq$	$>$	$\leq$	1	$(6^*, 2)$
95	$>$	$\leq$	$>$	$\leq$	$\leq$	$\leq$	$\leq$	$>$	5(ii)	$(2, 2, 1, 1, 1, 1), (3, 1, \ldots, 1)$
96	$>$	$\leq$	$>$	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	3(i)	$(2, 1, 1, 1, 1, 1, 1)$
97	$>$	$\leq$	$\leq$	$>$	$>$	$>$	$>$	$>$	5(i)	$(3, 1, \ldots, 1)$
98	$>$	$\leq$	$\leq$	$>$	$>$	$>$	$>$	$\leq$	1	$(6^*, 2)$
99	$>$	$\leq$	$\leq$	$>$	$>$	$>$	$\leq$	$>$	4(vi)	$(3, 1, 1, 2, 1)$
100	$>$	$\leq$	$\leq$	$>$	$>$	$>$	$\leq$	$\leq$	2	$(5^*, 3)$
101	$>$	$\leq$	$\leq$	$>$	$>$	$\leq$	$>$	$>$	4(vi)	$(3, 1, 2, 1, 1)$
102	$>$	$\leq$	$\leq$	$>$	$>$	$\leq$	$>$	$\leq$	1	$(6^*, 2)$
103	$>$	$\leq$	$\leq$	$>$	$>$	$\leq$	$\leq$	$>$	4(iv)	$(3, 1, 3, 1)$
104	$>$	$\leq$	$\leq$	$>$	$>$	$\leq$	$\leq$	$\leq$	3(i)	$(2, 1, 1, 1, 1, 1, 1)$
105	$>$	$\leq$	$\leq$	$>$	$\leq$	$>$	$>$	$>$	4(vi)	$(3, 2, 1, 1, 1)$
106	$>$	$\leq$	$\leq$	$>$	$\leq$	$>$	$>$	$\leq$	1	$(6^*, 2)$
107	$>$	$\leq$	$\leq$	$>$	$\leq$	$>$	$\leq$	$>$	4(v)	$(3, 2, 2, 1)$
108	$>$	$\leq$	$\leq$	$>$	$\leq$	$>$	$\leq$	$\leq$	2	$(5^*, 3)$
109	$>$	$\leq$	$\leq$	$>$	$\leq$	$\leq$	$>$	$>$	4(iv)	$(3, 3, 1, 1)$
110	$>$	$\leq$	$\leq$	$>$	$\leq$	$\leq$	$>$	$\leq$	1	$(6^*, 2)$
111	$>$	$\leq$	$\leq$	$>$	$\leq$	$\leq$	$\leq$	$>$	3(i)	$(2, 1, 1, 1, 1, 1, 1)$
112	$>$	$\leq$	$\leq$	$>$	$\leq$	$\leq$	$\leq$	$\leq$	3(i)	$(2, 1, 1, 1, 1, 1, 1)$
113	$>$	$\leq$	$\leq$	$\leq$	$>$	$>$	$>$	$>$	6	$(2, 2, 1, \ldots, 1), (3, 1, \ldots, 1), (4, 1, \ldots, 1)$
114	$>$	$\leq$	$\leq$	$\leq$	$>$	$>$	$>$	$\leq$	1	$(6^*, 2)$

Table 1 (cont.)

Case	A	B	C	D	E	F	G	H	Proposition	Inequalities
115	$>$	$\le$	$\le$	$\le$	$>$	$>$	$\le$	$>$	5(ii)	$(2,2,1,2,1),(3,1,\dots,1)$
116	$>$	$\le$	$\le$	$\le$	$>$	$>$	$\le$	$\le$	2	$(5^*,3)$
117	$>$	$\le$	$\le$	$\le$	$>$	$\le$	$>$	$>$	5(ii)	$(2,2,2,1,1),(3,1,\dots,1)$
118	$>$	$\le$	$\le$	$\le$	$>$	$\le$	$>$	$\le$	1	$(6^*,2)$
119	$>$	$\le$	$\le$	$\le$	$>$	$\le$	$\le$	$>$	3(i)	$(2,1,\dots,1)$
120	$>$	$\le$	$\le$	$\le$	$>$	$\le$	$\le$	$\le$	3(i)	$(2,1,1,1,1,1,1)$
121	$>$	$\le$	$\le$	$\le$	$\le$	$>$	$>$	$>$	7	$(2,1,2,1,1,1),(4,1,\dots,1),(2,1,\dots,1)$
122	$>$	$\le$	$\le$	$\le$	$\le$	$>$	$>$	$\le$	1	$(6^*,2)$
123	$>$	$\le$	$\le$	$\le$	$\le$	$>$	$\le$	$>$	3(i)	$(2,1,\dots,1)$
124	$>$	$\le$	$\le$	$\le$	$\le$	$>$	$\le$	$\le$	2	$(5^*,3)$
125	$>$	$\le$	$\le$	$\le$	$\le$	$\le$	$>$	$>$	3(i)	$(2,1,1,1,1,1,1)$
126	$>$	$\le$	$\le$	$\le$	$\le$	$\le$	$>$	$\le$	1	$(6^*,2)$
127	$>$	$\le$	$\le$	$\le$	$\le$	$\le$	$\le$	$>$	3(i)	$(2,1,1,1,1,1,1)$
128	$>$	$\le$	$\le$	$\le$	$\le$	$\le$	$\le$	$\le$	3(i)	$(2,1,1,1,1,1,1)$

Proof. The proofs of (i) to (iv) are simple extensions of Lemmas 5, 6, 7 and 9 of [7] to the case of eight variables. For the proof of (v) one notices that, using the AM-GM inequality,

$$\begin{aligned}\mathfrak{S}_5 &\le 4X_1 - \frac{X_1^2}{X_2} + 4X_3 - \frac{X_3^2}{X_4} + 4X_5 - 3X_1X_3X_5^{4/3}X_8^{1/3} + X_8\\ &\le 3X_1 + 3X_3 + 4X_5 - 3X_1X_3X_5^{4/3} + 1\\ &\le 3X_1 + 3X_3 + 5 - 3X_1X_3 \le 8.\end{aligned}$$

The proof of (vi) is similar.

It may be noticed that this lemma can be easily extended to n variables.

PROPOSITION 4. *The following cases do not arise:*

(i) (3), (5), (9), (17), (33), (65)*;*
(ii) (11), (19), (21), (35), (37), (41), (67), (69), (73), (81)*;*
(iii) (43), (75), (83), (85)*;*
(iv) (39), (45), (103), (109)*;*
(v) (23), (27), (107)*;*
(vi) (71), (77), (89), (99), (101), (105).

Proof. It is easy to see that each part of Proposition 4 follows immediately from the corresponding part of Lemma 5, after selecting a suitable inequality. The inequalities used are mentioned in Table 1.

LEMMA 6. *Let X_i be positive real numbers for $1 \le i \le 8$ satisfying $X_1 > 1$, $X_1X_2X_3X_4X_5X_6X_7X_8 = 1$. Let*

$$x_i = |X_i - 1|, \qquad \gamma = \sum_{\substack{4\le i\le 8\\ X_i\le 1}} x_i, \qquad \delta = \sum_{\substack{4\le i\le 8\\ X_i>1}} x_i.$$

Suppose that either

(i) $X_i > 1$ *for each* i, $4 \le i \le 8$, *or*
(ii) $\gamma \le x_1 \le 0.5$, *or*
(iii) $\gamma \le \frac{2}{3}x_1$ *and* $x_1 \le 1$, *or*
(iv) $\gamma \le \delta/2$ *and* $\delta \le 4x_1$ *with* $x_1 \le 0.226$, *or*
(v) $\delta \ge 2\gamma$ *and* $\gamma \le 2x_1$ *with* $x_1 \le 0.226$, *or*
(vi) $\delta \ge \frac{4}{3}\gamma$ *and* $\gamma \le 2x_1$ *with* $x_1 \le 0.175$.

Then

$$\mathfrak{S}_7 = 4X_1 - X_1^4X_4\cdots X_8 + X_4 + \cdots + X_8 \le 8.$$

The simple proof similar to that given in Lemmas 8 and 10 of [7] is omitted.

PROPOSITION 5. *The following cases do not arise:*

(i) (7), (13), (25), (49), (97);
(ii) (48), (56), (64), (80), (87), (91), (95), (115), (117).

Proof. This follows immediately from Lemma 6(i)&(ii) after selecting a suitable inequality. The inequalities used are mentioned in Table 1.

LEMMA 7. *Let $X_i > 1$ be real numbers for $1 \le i \le 5$.*

(i) *If* $X_1^5 \ge 2$, *then*

$$\mathfrak{S}_8 = 4X_1 - \frac{1}{2}X_1^5X_2X_3X_4X_5 + X_2 + X_3 + X_4 + X_5 \le 8.$$

(ii) *If* $X_i \le X_1 = A$ (*say*) *for* $2 \le i \le 5$, $A \le 2$, *then*

$$\mathfrak{S}_9 = A + 4X_2 - \frac{1}{2}X_2^5X_3X_4X_5A + X_3 + X_4 + X_5 \le 8.$$

Proof. The proof of (i) is a simple extension of Lemma 11 of [7] to the case of eight variables. For the proof of (ii) we notice that $\mathfrak{S}_9$ is a linear function of X_i for each i, $3 \le i \le 5$. The coefficient of X_5 in $\mathfrak{S}_9 = \phi(X_3, X_4, X_5)$ (say) may be positive or negative, so its maximum occurs either at $X_5 = 1$ or at $X_5 = A$. A similar argument holds for X_3 and X_4. Symmetry of $\phi(X_3, X_4, X_5)$ in X_3, X_4 and X_5 gives $\phi(X_3, X_4, X_5) \le \max\{\phi(1,1,1), \phi(1,1,A), \phi(1,A,A), \phi(A,A,A)\}$. One can easily prove that the right side is at most 8 for $1 < X_2 \le A$.

PROPOSITION 6. *Case* (113), *i.e.* $A > 1$, $B \le 1$, $C \le 1$, $D \le 1$, $E > 1$, $F > 1$, $G > 1$, $H > 1$, *does not arise.*

Proof. Assume first that $A^4 \geq 2$. This gives $A^4EFGH > A^4 \geq 2$, therefore $(4,1,1,1,1)$ holds. That is, $4A - \frac{1}{2}A^5EFGH + E + F + G + H > 8$. Using Lemma 7(i) with $X_1 = A$, $X_2 = E$, $X_3 = F$, $X_4 = G$ and $X_5 = H$ we get a contradiction. So we have $A^4 < 2$, which implies $A < 1.19$. From inequality $(2,2,1,1,1,1)$, we get $\delta = e + f + g + h > 2d$. Now we get a contradiction using Lemma 6(iv) with $\gamma = d$ and $\delta \leq 4a = 4x_1$, $x_1 < 0.19$.

Proposition 7. *Case* (121), *i.e.* $A > 1$, $B \leq 1$, $C \leq 1$, $D \leq 1$, $E \leq 1$, $F > 1$, $G > 1$, $H > 1$, *does not arise.*

Proof. Here $a \leq \frac{1}{3}$ by Lemma 3. Using inequality $(2,1,2,1,1,1)$, we have $2B + C + 2E + F + G + H > 8$. This gives $e < \frac{f+g+h}{2} = \frac{k}{2}$, say. Therefore $EFGH \geq (1-e)(1+k) > (1-\frac{k}{2})(1+k) \geq 1$ for $k \leq 3a \leq 1$. Suppose first that $A^4 \geq 2$; then $A^4EFGH > A^4 \geq 2$. Therefore $(4,1,1,1,1)$ holds, i.e. $4A - \frac{1}{2}A^5EFGH + E + 2 + FGH > 8$. The left side is a decreasing function of E and $E > 1 - \frac{k}{2}$. So we can replace E by $1 - \frac{k}{2}$. Further $FGH \geq 1 + f + g + h = 1 + k$. This gives $\phi(k) = 4a + \frac{k}{2} - \frac{1}{2}(1+a)^5(1+k)(1-\frac{k}{2}) > 0$. As $\phi''(k) > 0$, we have $\phi(k) \leq \max\{\phi(0), \phi(3a)\}$, which is negative for $0 < a \leq \frac{1}{3}$, giving thereby a contradiction.

Hence we must have $A^4 < 2$. So $a < 0.19$. By Proposition 3(ii), this case does not arise.

Proposition 8. *Case* (93), *i.e.* $A > 1$, $B \leq 1$, $C > 1$, $D \leq 1$, $E \leq 1$, $F \leq 1$, $G > 1$, $H > 1$, *does not arise.*

Proof. Here $2G > H$, since $2G \leq H \leq A \leq \frac{4}{3}$ gives $G < 1$. So $(2,2,2,2)$ holds and we get $2B + 2D + 2F + 2H > 8$, which gives $F > 1 - h$. Using the AM-GM inequality in $(2,3,1,1,1)$ we get $4A - \frac{A^2}{B} + 4C + F + G + H - 2C^2A^{3/2}F^{1/2}G^{1/2}H^{1/2} > 8$. Since $B \leq 1$ and $F > 1 - h$, we see that $4A - A^2 + 4C + G - 2C^2A^{3/2}(1-h)^{1/2}G^{1/2}H^{1/2} > 6$. Since $h \leq a$, we have $4A - A^2 + 4C + G - 2C^2A^{3/2}(1-a^2)^{1/2}G^{1/2} > 6$. Further the left side is a decreasing function of G as well as of C for $1 \leq G \leq A$, $1 \leq C \leq A$ and $1 \leq A \leq \frac{4}{3}$. On replacing G and C by 1 we get $4A - A^2 - 2A^{3/2}(1-a^2)^{1/2} > 1$, which can be easily seen to be false.

Proposition 9. *Case* (40), *i.e.* $A > 1$, $B > 1$, $C \leq 1$, $D > 1$, $E > 1$, $F \leq 1$, $G \leq 1$, $H \leq 1$, *does not arise.*

Proof. Here $a \leq \frac{1}{2}$ and $e \leq \frac{1}{3}$ by Lemma 3. As $2G > H$, $(1,2,2,1,2)$ holds, therefore we get $A + 2C + 2E + F + 2H > 8$, which gives $H > 1 - e - \frac{a}{2}$. After using the AM-GM inequality in $(3,1,3,1)$ we get $4A + 4E + D + H - 2A^2E^2D^{1/2}H^{1/2} > 8$. The left side is a decreasing function of H as well as of D, so we can successively replace H by $1 - e - \frac{a}{2}$ and D by 1 to get $\phi(e) = 2 + 3.5a + 3e - 2(1+a)^2(1+e)^2(1-e-\frac{a}{2})^{1/2} > 0$. One easily verifies that

$\phi''(e) > 0$ for $0 < e \le \min\{a, \frac{1}{3}\}$. Therefore $\phi(e) \le \max\{\phi(0), \phi(\min(a, \frac{1}{3}))\}$, which is negative for $a \le \frac{1}{2}$. This gives a contradiction.

PROPOSITION 10. *Cases* (72) *and* (79) *do not arise.*

Proof. First we consider Case (72), i.e. $A>1$, $B\le 1$, $C>1$, $D>1$, $E>1$, $F \le 1$, $G \le 1$, $H \le 1$. Here $a \le \frac{1}{3}$ by Lemma 3. Notice that $2G > 1 > H$. Using inequality $(2,1,1,2,2)$ we get $-2b+c+d-2f-2h > 0$, which gives $h < \frac{c+d}{2}$. Inequality $(2,1,1,3,1)$ holds, i.e. $4A - \frac{2A^2}{B} + C + D + 4E - E^4ABCDH + H > 8$. Applying the AM-GM inequality to $-\frac{A^2}{B} - E^4ABCDH$ and using $B \le 1$ we get $4A - A^2 + C + D + 4E - 2A^{3/2}E^2C^{1/2}D^{1/2}H^{1/2} + H > 8$. As the left side is a decreasing function of H and $H > 1 - \frac{c+d}{2}$, we get

$$4A - A^2 + C + D + 4E - 2A^{3/2}E^2C^{1/2}D^{1/2}\left(1 - \frac{c+d}{2}\right)^{1/2} + 1 - \frac{c+d}{2} > 8.$$

Again the left side is a decreasing function of E. Replacing E by 1 and simplifying we get $\phi(x) = 2+2a-a^2+\frac{x}{2} - 2(1+a)^{3/2}(1+x)^{1/2}(1-\frac{x}{2})^{1/2} > 0$ where $x = c+d$. One verifies that $\phi''(x) > 0$ and $0 < x \le 2a$, so $\phi(x) \le \max\{\phi(0), \phi(2a)\}$, which is non-positive for $0 < a \le \frac{1}{3}$, giving thereby a contradiction.

Now consider Case (79), i.e. $A > 1$, $B \le 1$, $C > 1$, $D > 1$, $E \le 1$, $F \le 1$, $G \le 1$, $H > 1$. Using $(2,1,2,2,1)$ and $(2,1,3,1,1)$ and proceeding as in Case (72), replacing D, E, H by H, D, G respectively we get a contradiction.

PROPOSITION 11. *Case* (55), *i.e.* $A > 1$, $B > 1$, $C \le 1$, $D \le 1$, $E > 1$, $F \le 1$, $G \le 1$, $H > 1$, *does not arise.*

Proof. Here $a \le \frac{1}{2}$ and $b \le \frac{1}{3}$ by Lemma 3. As $B^2 > CD$ and $E^2 > FG$ therefore $(1,3,3,1)$ holds. After using the AM-GM inequality, we get

$$A + 4B + 4E - 2B^2E^2(HA)^{1/2} + H > 8.$$

Suppose first that $A < B^4E^4H$. Then the left side is a decreasing function of A and $A \ge H$. So we get $2H + 4B + 4E - 2B^2E^2H > 8$. Again left side is a decreasing function of B and of E. Replacing B and then E by 1 we get a contradiction. Now let $A \ge B^4E^4H$. This gives $a > 4(b+e) + h \ge 2b + h$. From inequality $(2,2,2,1,1)$ we get $2b - 2d - 2f - g + h > 0$, which gives $d + f + g < 2b + h < a$. Now using $(3,1,1,1,1,1)$ and applying Lemma 6(ii) with $\gamma = d + f + g$ and $x_1 = a$ we get a contradiction.

PROPOSITION 12. *Case* (51), *i.e.* $A > 1$, $B > 1$, $C \le 1$, $D \le 1$, $E > 1$, $F > 1$, $G \le 1$, $H > 1$, *does not arise.*

Proof. Here $a \le \frac{1}{2}$, $b \le \frac{1}{3}$, $e \le \frac{1}{2}$ by Lemma 3. Using inequality $(2,2,1,2,1)$ we get $2b - 2d + e - 2g + h > 0$, which gives $d < \frac{e+h}{2}$.

CLAIM (i). $FG \le 1$.

Suppose that $FG > 1$. As $B^2 > CD$, we can use $(1,3,1,2,1)$ to get $A + 4B - B^4EFGHA + E + 2G + H > 8$. This implies that $A + 4B - B^4EHA + E + 2 + H > 8$. As the coefficient of H is negative we can replace H by 1. Similarly we can replace E by 1 to get $A + 4B - B^4A + 4 > 8$, which is not possible.

Claim (ii). $g \geq b$.

Suppose that $g < b$. Using $(1,3,1,1,1,1)$ and applying Lemma 6(ii) with $\gamma = g$, $x_1 = b \leq \frac{1}{3}$, we get a contradiction.

Final contradiction. Here $E^2 > 1 \geq FG$. Therefore $(3,1,3,1)$ holds. After using the AM-GM inequality we get $4A + D + 4E - 2A^2E^2(DH)^{1/2} + H > 8$. The left side is a decreasing function of D and $D > 1 - \frac{e+h}{2}$, so replacing D by $1 - \frac{e+h}{2}$ we get $\phi(h) = 2 + 4a + \frac{7e}{2} + \frac{h}{2} - 2(1+a)^2 \times (1+e)^2(1 - \frac{e+h}{2})^{1/2}(1+h)^{1/2} > 0$. Now $\phi''(h) > 0$ and $0 < h \leq a$. Therefore $\phi(h) \leq \max\{\phi(0), \phi(a)\}$, which is non-positive for $e \leq \frac{1}{2}$ and $0 < a \leq \frac{1}{2}$, giving thereby a contradiction.

Proposition 13. *Case* (53)*, i.e.* $A > 1$*,* $B > 1$*,* $C \leq 1$*,* $D \leq 1$*,* $E > 1$*,* $F \leq 1$*,* $G > 1$*,* $H > 1$*, does not arise.*

Proof. Here $a \leq \frac{1}{2}$ and $b \leq \frac{1}{3}$ by Lemma 3. Working as in Claims (i) and (ii) of Proposition 12 (Case (51)) and replacing G by F, F by E, and E by G, we can suppose that $EF \leq 1$ and $f \geq b$.

Using inequality $(2,2,2,1,1)$ we have $2b - 2d - 2f + g + h > 0$, which gives $d < \frac{g+h}{2}$. Now $(3,1,2,1,1)$ holds, i.e. $4A - A^4DEFGH + D + 4E - \frac{2E^2}{F} + G + H > 8$. Using the AM-GM inequality and $F \leq 1$ we get $4A + D + 4E - E^2 - 2A^2E^{3/2}(DGH)^{1/2} + G + H > 8$. The left side is a decreasing function of E as well as of D, therefore replacing E by 1 and D by $1 - \frac{g+h}{2}$ we get

$$2 + 4a + \frac{g+h}{2} - 2(1+a)^2\left(1 - \frac{g+h}{2}\right)^{1/2}(1+g+h)^{1/2} > 0.$$

Further the left side is a decreasing function of a and $a \geq \frac{g+h}{2} = \frac{x}{2}$ (say). So we get $2 + \frac{5x}{2} - 2(1+\frac{x}{2})^2(1-\frac{x}{2})^{1/2}(1+x)^{1/2} > 0$. This is not possible for $x \leq 1$.

Proposition 14. *Case* (47)*, i.e.* $A > 1$*,* $B > 1$*,* $C \leq 1$*,* $D > 1$*,* $E \leq 1$*,* $F \leq 1$*,* $G \leq 1$*,* $H > 1$*, does not arise.*

Proof. Here $a \leq \frac{1}{2}$ and $b \leq \frac{1}{3}$ by Lemma 3. As $2F > G$, $(1,2,2,2,1)$ holds. Therefore we get $A + 2C + 2E + 2G + H > 8$, which gives $g < \frac{a+h}{2}$.

Claim (i). $a > 0.386$.

Suppose $a \le 0.386$. Using the AM-GM inequality in $(3,3,1,1)$ we get $4A+4D+G+H-2A^2D^2G^{1/2}H^{1/2} > 8$. The left side is a decreasing function of D as well as of G. So we can replace D by 1 and G by $1-\frac{a+h}{2}$ and get $\phi(h) = 2+3.5a+0.5h-2(1+a)^2(1+h)^{1/2}(1-\frac{a+h}{2})^{1/2} > 0$. One easily verifies that $\phi''(h) > 0$ for $0 < h \le a$, therefore $\phi(h) \le \max\{\phi(0), \phi(a)\}$, which is negative for $a \le 0.386$. This gives a contradiction.

Claim (ii). $d < 0.0265$.

Suppose $d \ge 0.0265$. Working as in the previous claim and replacing D by 1.0265 instead of 1 we get a contradiction for $a \le \frac{1}{2}$.

Claim (iii). $A < 1.4$.

Suppose $A \ge 1.4$. Then $A^4EFGH > \frac{A^3}{BD} > \frac{3(1.4)^3}{4\times 1.0265} > 2$. Therefore inequality $(4,2,1,1)$ holds. Using the AM-GM inequality we get $4A+4E+G+H-2A^{5/2}E^{3/2}G^{1/2}H^{1/2} > 8$. The left side is a decreasing function of E as well as of G. Replacing E by $\frac{3}{4}$ and G by $\frac{1}{2}$ we get $\phi(H) = 4A+3.5+H-2\sqrt{\frac{27}{128}}A^{5/2}H^{1/2} > 8$. As $\phi''(H) > 0$, we have $\phi(H) \le \min\{\phi(1), \phi(A)\}$, which is negative for $1.4 \le A \le 1.5$. This gives a contradiction.

Claim (iv). $B^3 \le 2D$.

Suppose $B^3 > 2D$. Then as $CE \le 1$, we have $B^4FGHA \ge \frac{B^3}{D} > 2$. Therefore $(1,4,1,1,1)$ holds. Using $F+G \le 1+FG$, we get $A+4B-\frac{1}{2}B^5FGHA+1+FG+H > 8$. The left side is a decreasing function of FG and $FG > \frac{1}{ABDH} > \frac{0.974}{ABH}$, therefore we get $A+4B-\frac{1}{2}B^4(0.974)+1+\frac{0.974}{ABH}+H > 8$. We can replace H by A to get $2A+4B-0.487B^4+\frac{0.974}{A^2B} > 7$, which is not true for $A < 1.4$ and $B^3 > 2$.

Final contradiction. Using Claims (iv) and (ii) we have $AEFGH > \frac{1}{BD} > \frac{1}{D(2D)^{1/3}} > 0.766$, hence $A^4EFGH > 0.766A^3 > 2$ for $A > 1.386$. Now $(4,1,1,1,1)$ holds, which gives $4A-\frac{1}{2}A^5EFGH+2+EFG+H > 8$. The coefficient of EFG is negative so we replace EFG by $\frac{0.766}{AH}$. Then the resulting function is an increasing function of H and $H \le A$, so we get $5A-\frac{1}{2}\cdot 0.766A^4+\frac{0.766}{A^2} > 6$, which is not true for $1.386 < A < 1.4$.

Proposition 15. *Case* (59), *i.e.* $A > 1$, $B > 1$, $C \le 1$, $D \le 1$, $E \le 1$, $F > 1$, $G \le 1$, $H > 1$, *does not arise.*

Proof. Here $a \le \frac{1}{2}$ and $b \le \frac{1}{3}$ by Lemma 3. Using inequalities $(1,2,2,2,1)$, $(2,2,1,2,1)$ and $(2,1,2,2,1)$ we get

$$a-2c-2e-2g+h > 0, \tag{4.1}$$

$$2b-2d-e-2g+h > 0, \tag{4.2}$$

$$2b-c-2e-2g+h > 0. \tag{4.3}$$

Claim (i). $A < 1.226$.

Suppose $A \geq 1.226$. From (4.1), we have $e+g < \frac{a+h}{2}$. Therefore $AEGH > (1+a+h)(1-\frac{a+h}{2}) > 1$ for $a+h \leq 2a \leq 1$. Also $A^4EFGH > A^4EGH > (1+a)^4(1-\frac{a+h}{2})(1+h) \geq \min\{(1+a)^4(1-\frac{a}{2}),(1+a)^5(1-a)\} > 2$. So $(4,1,1,1,1)$ holds. This gives $4A - \frac{1}{2}A^5EFGH + EG + F + H > 7$. As the coefficient of EG on the left side is negative and $EG > (AH)^{-1}$, it follows that $4A - \frac{1}{2}A^4F + (AH)^{-1} + F + H > 7$. The left side is an increasing function of H and a decreasing function of F and $H \leq A$, $F > 1$. So we get $5A - \frac{1}{2}A^4 + A^{-2} > 6$, which is not true for $1 < A \leq 3/2$.

Claim (ii). $e + g > 2b$.

Suppose that $e + g \leq 2b$. We use $(1,3,1,1,1,1)$ and apply Lemma 6(v) with $\gamma = e+g$, $\delta = a+f+h$. We have $\gamma < \frac{1}{2}\delta$ from (4.1). Since $b \leq a < 0.226$ we get a contradiction.

Final contradiction. Using Claim (ii) and inequality (4.2), we get $d < \frac{h}{2}$. Also from (4.3), we have $e + g < b + \frac{h}{2}$. Using these together with $F > 1$ in inequality $(1,2,1,1,1,1,1)$ we get $A+4B-2B^3HA(1-b-\frac{h}{2})(1-\frac{h}{2})-b > 3$, i.e. $\phi(h) = 2+a+3b-2(1+b)^3(1+a)(1+h)(1-b-\frac{h}{2})(1-\frac{h}{2}) > 0$. As $\phi''(h) > 0$ and $0 < h \leq a$, we get $\phi(h) \leq \min\{\phi(0),\phi(a)\}$, which can be easily verified to be negative for $0 < b \leq a < 0.226$. This gives a contradiction.

Proposition 16. *Case* (24), *i.e.* $A > 1$, $B > 1$, $C > 1$, $D \leq 1$, $E > 1$, $F \leq 1$, $G \leq 1$, $H \leq 1$, *does not arise.*

Proof. Here $a \leq 1$, $b \leq \frac{1}{2}$ and $c \leq \frac{1}{3}$ by Lemma 3. Also $2G > 1 \geq H$. Using inequalities $(2,2,2,2)$ and $(2,2,2,1,1)$ we get

$$2b - 2d - 2f - 2h > 0, \tag{4.4}$$

$$2b - 2d - 2f - g - h > 0. \tag{4.5}$$

Claim (i). $B > C$.

Suppose $B \leq C$. Using $(2,2,3,1)$ we have $4A - \frac{2A^2}{B} + 4C - \frac{2C^2}{D} + 4E - E^4HABCD + H > 8$. Applying the AM-GM inequality to $-\frac{A^2}{B} - \frac{C^2}{D} - E^4HABCD$ and using $B \leq A$, $D \leq 1$ we get $3A + 4C - C^2 + 4E - 3E^{4/3}H^{1/3}AC + H > 8$. As the left side is a decreasing function of H, and $h < b$ from (4.4), we can replace H by $1-b$ and get $3A + 4C - C^2 + 4E - 3E^{4/3}(1-b)^{1/3}AC + (1-b) > 8$. Again the left hand side is a decreasing function of C and $C \geq B$, therefore we get $3A + 4B - B^2 + 4E - 3E^{4/3}(1-b)^{1/3}AB + 1 - b > 8$. Similarly replacing E by 1 we get $3A + 4B - B^2 - 3(1-b)^{1/3}AB - b > 3$. This is not possible for $A \geq B$ and $1 < B \leq C \leq \frac{4}{3}$.

Claim (ii). $a < 3b$.

Suppose $a \geq 3b$. Then from (4.5), $d + f + g + h < 2b \leq \frac{2a}{3}$. Using $(3, 1, 1, 1, 1, 1)$ and applying Lemma 6(iii) with $\gamma = d+f+g+h$, $x_1 = a \leq 1$, we get a contradiction.

Claim (iii). $b > 0.386$.

Suppose $b \leq 0.386$. From Claim (i) we have $B^2 > CD$, therefore $(1, 3, 3, 1)$ holds. Applying the AM-GM inequality we get $A + 4B + 4E + H - 2B^2E^2A^{1/2}H^{1/2} > 8$. As the left side is a decreasing function of H and of E, we can replace successively H by $1 - b$ and E by 1 to get $A + 4B - b - 2B^2A^{1/2}(1-b)^{1/2} > 3$. Further using Claim (ii), $A < 1 + 3b < B^4(1-b)$ for $b \leq \frac{1}{2}$, so the left side is a decreasing function of A. Replacing A by B we get $2 + 4b - 2(1+b)^{5/2}(1-b)^{1/2} > 0$, which is not possible for $b \leq 0.386$.

Claim (iv). $E^4ABCD < 2$.

Assume $E^4ABCD \geq 2$. Therefore $(2, 2, 4)$ holds, i.e. $2B+4C-\frac{2C^2}{D}+4E-\frac{1}{2}E^5ABCD > 8$. Applying the AM-GM inequality and then using $A \geq B$, we get $\phi(C, E) = 2B+4C+4E-2E^{5/2}C^{3/2}B > 8$. As $\phi(C, E)$ is a decreasing function of C and E for $B > 1.386$, we have $\phi(C, E) \leq \phi(1, 1) = 8$, which gives a contradiction.

Final contradiction. From Claim (iv) we get $FGH > \frac{1}{2}$, which gives $B^4FGHA > \frac{1}{2}B^5 > 2$ for $B > 1.386$. Therefore $(1, 4, 1, 1, 1)$ holds, i.e. $A + 4B - \frac{1}{2}B^5FGHA + F + G + H > 8$. This implies that $A + 4B - \frac{1}{2}B^5FGHA + 2 + FGH > 8$. As the coefficient of FGH on the left side is negative, replacing FGH by $\frac{1}{2}$ we get $A + 4B - \frac{1}{4}B^5A + \frac{5}{2} > 8$, which is not possible for $A \geq B$ and $B > 1.386$.

5. Difficult cases

5.1. Case (8)

Proposition 17. *Case* (8)*, i.e.* $A > 1$*,* $B > 1$*,* $C > 1$*,* $D > 1$*,* $E > 1$*,* $F \leq 1$*,* $G \leq 1$*,* $H \leq 1$*, does not arise.*

Proof.

Claim (i). $E^4ABCD \leq 2$*,* $E < 1.149$ *and* $FGH > 1/2$.

Suppose $E^4ABCD > 2$. Inequality $(1, 1, 1, 1, 4)$ is $A + B + C + D + 4E - \frac{1}{2}E^5ABCD > 8$. This is not true by Lemma 7(ii) with $X_2 = E$, $X_3 = B$, $X_4 = C$, $X_5 = D$. Therefore $E^4ABCD \leq 2$. This implies $E^5 \leq 2$, which gives $E < 1.149$. Also $FGH = \frac{1}{ABCDE} \geq \frac{E^3}{2} > \frac{1}{2}$.

Claim (ii). $A^4EFGH \leq 2$ *and* $A \leq \sqrt{2}$.

Suppose $A^4EFGH > 2$. Then inequality $(4, 1, 1, 1, 1)$ holds, i.e. $4A - \frac{1}{2}A^5EFGH + E + F + G + H > 8$. This implies $\phi(y) = 4A - \frac{1}{2}A^5Ey + E +$

$2+y>8$, where $y=FGH>1/2$ by Claim (i). As the coefficient of y in $\phi(y)$ is negative, we can replace y by $\frac{1}{2}$ to get

$$\psi(E)=4A-\frac{1}{4}A^5E+E+2.5>8. \tag{5.1.1}$$

We see that $\psi(E)\le\max\{\psi(1),\psi(A)\}$, which can be easily seen to be less than 8, contradicting (5.1.1). This proves $A^4EFGH\le 2$.

Now $A^4<A^4E\le\frac{2}{FGH}\le 4$ gives $A\le\sqrt{2}$.

Claim (iii). *$C<1.22$; $C<1.1$ if $A>1.38$.*

Suppose $C\ge 1.22$. Using $(2,2,4^*)$ and the AM-GM inequality we get $\phi(x)=4A+4C-4A^{3/2}C^{3/2}x^{1/2}+4x^{1/4}>8$, where $x=EFGH$. Since $x>FGH>\frac{1}{2}$ and $\phi(x)$ is a decreasing function of x, we have $\phi(x)\le\phi(\frac{1}{2})$, which can be easily verified to be less than 8 for $A\ge C\ge 1.22$. This gives a contradiction. Further if $A>1.38$, then $\phi(\frac{1}{2})<8$ for $C\ge 1.1$, a contradiction.

Final contradiction. Let

$$\lambda=\begin{cases} A & \text{if } A\le 1.22,\\ 1.22 & \text{if } 1.22<A\le 1.38,\\ 1.1 & \text{if } 1.38<A\le\sqrt{2}.\end{cases} \tag{5.1.2}$$

Using $(3,1,3,1)$ and the AM-GM inequality we get $4A+4E+D+H-2A^2E^2\sqrt{D}\sqrt{H}>8$. The left side of this inequality is a quadratic in $\sqrt{H}$. Since $A^4E^4D-4A-4E-D+8>0$, we have

$$\sqrt{H}<A^2E^2\sqrt{D}-(A^4E^4D-4A-4E-D+8)^{1/2}=\alpha \quad \text{(say)}. \tag{5.1.3}$$

Using the AM-GM inequality in $(2,2,2,2)$, we get $4A+2D+4E+4G-6AEGC^{1/3}D^{1/3}>8$, which gives $G<(2D+4A+4E-8)(6AEC^{1/3}D^{1/3}-4)^{-1}$. Substituting this upper bound of G in inequality $(1,2,2,2,1)$, we get

$$H>8-A-2C-2E-2\left\{\frac{2D+4A+4E-8}{6AEC^{1/3}D^{1/3}-4}\right\}=\beta \quad \text{(say)}. \tag{5.1.4}$$

From (5.1.3) and (5.1.4) we have $\beta<\alpha^2$. On simplifying we get

$$\begin{aligned}\phi(C)=\left\{A^4E^4D-\frac{3A}{2}-E-\frac{D}{2}+C+\frac{2D+4A+4E-8}{6AEC^{1/3}D^{1/3}-4}\right\}\\ -A^2E^2\{A^4E^4D^2-4AD-4ED-D^2+8D\}^{1/2}>0.\end{aligned} \tag{5.1.5}$$

One can see that $\phi(C)$ is an increasing function of C. From Claim (iii), we have $C\le\lambda$. Therefore $\phi(C)\le\phi(\lambda)$, which gives

$$\begin{aligned}\psi(D)=\left\{A^4E^4D-\frac{3A}{2}-E-\frac{D}{2}+\lambda+\frac{2D+4A+4E-8}{6EA\lambda^{1/3}D^{1/3}-4}\right\}\\ -A^2E^2\{A^4E^4D^2-4AD-4ED-D^2+8D\}^{1/2}>0.\end{aligned} \tag{5.1.6}$$

$$\psi'(D) = A^4E^4 - \frac{1}{2} + \frac{2EA\lambda^{1/3}D^{1/3} - 2 - (2A+2E-4)EA\lambda^{1/3}D^{-2/3}}{(3EA\lambda^{1/3}D^{1/3}-2)^2} - \frac{A^2E^2\{A^4E^4D - 2A - 2E - D + 4\}}{\{A^4E^4D^2 - 4AD - 4ED - D^2 + 8D\}^{1/2}}.$$

We first prove that $\psi'(D) < 0$ for $1 < D \le A$, $1 < A \le \sqrt{2}$ and $1 < E < 1.149$. Let

$$(5.1.7)\qquad (\mu_1, \mu_2) = \begin{cases} (0.42, 0.08) & \text{if } A \le 1.22, \\ (0.44, 0.06) & \text{if } 1.22 < A \le 1.38, \\ (0.45, 0.05) & \text{if } 1.38 < A \le \sqrt{2}. \end{cases}$$

Let $\psi'(D) = P + Q$ where

$$P = A^4E^4 - \mu_1 - \frac{A^2E^2\{A^4E^4D - 2A - 2E - D + 4\}}{\{A^4E^4D^2 - 4AD - 4ED - D^2 + 8D\}^{1/2}},$$
$$Q = \frac{2EA\lambda^{1/3}D^{1/3} - 2 - (2A+2E-4)EA\lambda^{1/3}D^{-2/3}}{(3EA\lambda^{1/3}D^{1/3}-2)^2} - \mu_2.$$

To prove $\psi'(D) < 0$, we show that $P < 0$ and $Q < 0$ for $1 < D \le A$, $1 < A \le \sqrt{2}$ and $1 < E < 1.149$.

Now $P < 0$ if $\theta_1(D) = (A^4E^4 - \mu_1)^2(A^4E^4D^2 - 4AD - 4ED - D^2 + 8D) - A^4E^4\{A^4E^4D - 2A - 2E - D + 4\}^2 < 0$. As $\theta_1(D)$ is an increasing function of D, so $\theta_1(D) \le \theta_1(A)$ which is a function in two variables A and E and can be shown to be negative by plotting its 3-dimensional surface graph for $1 < E < 1.149$ and μ_1 as given in (5.1.7) (using the package Mathematica 5.1).

Further $Q < 0$ if $\theta_2(D, E) = EA\lambda^{1/3}D^{1/3} - 1 - (A+E-2)EA\lambda^{1/3}D^{-2/3} - \frac{1}{2}\mu_2(3EA\lambda^{1/3}D^{1/3} - 2)^2 < 0$. One finds that $\theta_2(D, E)$ is a decreasing function of E and an increasing function of D, therefore $\theta_2(D, E) \le \theta_2(A, 1) = \lambda^{1/3}A^{1/3} - \frac{1}{2}\mu_2(3\lambda^{1/3}A^{4/3} - 2)^2 - 1$, which is negative for λ and μ_2 as given in (5.1.2) and (5.1.7) respectively.

Thus $\psi(D)$ is a decreasing function of D, therefore $\psi(D) \le \psi(1)$, where

$$\psi(1) = \left\{A^4E^4 - \frac{3A}{2} - E - \frac{1}{2} + \lambda + \frac{4A + 4E - 6}{6EA\lambda^{1/3} - 4}\right\} - A^2E^2\{A^4E^4 - 4A - 4E + 7\}^{1/2}.$$

This is again a function in two variables A and E and can be shown to be negative by plotting its 3-dimensional surface graph for $1 < E < 1.149$ and λ as given in (5.1.2) (using Mathematica 5.1), which gives a contradiction to (5.1.6).

5.2. Case (15)

PROPOSITION 18. *Case* (15), *i.e.* $A > 1$, $B > 1$, $C > 1$, $D > 1$, $E \le 1$, $F \le 1$, $G \le 1$, $H > 1$, *does not arise.*

Proof. The proof of this case is similar to that of Case (8).

Claim (i). $D^4HABC \le 2$, $D < 1.149$ *and* $EFG > 1/2$.

Suppose $D^4HABC > 2$. We use inequality $(1,1,1,4,1)$ and proceed as in Claim (i) of Case (8) to get the desired result.

Claim (ii). $A^4EFGH \le 2$, $A \le \sqrt{2}$ *and* $H < 1.31951$.

Suppose $A^4EFGH > 2$. Then we use inequality $(4,1,1,1,1)$ and proceed as in Claim (ii) of Case (8). For the bound on H we notice that $H^5 \le A^4H \le \frac{2}{EFG} \le 4$ gives $H < 1.31951$.

Claim (iii). $B^4FGHA \le 2$, $B^4HA < 4$ *and* $B < 1.31951$.

Suppose $B^4FGHA > 2$. Then inequality $(1,4,1,1,1)$ holds, i.e. $A+4B-\frac{1}{2}B^5FGHA + F + G + H > 8$. Let $z = FG$. This gives $\phi(z) = A + 4B - \frac{1}{2}B^5zHA + 1 + z + H > 8$. By Claim (i), $z > EFG > \frac{1}{2}$. As the coefficient of z is negative, we can replace z by $\frac{1}{2}$ to get

$$\psi(H) = A + 4B - \frac{1}{4}B^5HA + H + 1.5 > 8. \tag{5.2.1}$$

We have $\psi(H) \le \max\{\psi(1), \psi(A)\}$. Let $\psi(1) = A + 4B - \frac{1}{4}B^5A + 2.5 = \varphi(B)$. One finds that $\varphi(B)$ has a maximum at $B = (\frac{16}{5A})^{1/4}$ and therefore $\varphi(B) \le \varphi((\frac{16}{5A})^{1/4})$, which is less than 8 for $1 < A \le \sqrt{2}$. Let $\psi(A) = 2A + 4B - \frac{1}{4}B^5A^2 + 1.5 = \vartheta(B)$. It is easily seen that $\vartheta(B) \le \vartheta((\frac{16}{5A^2})^{1/4})$, which is less than 8 for $1 < A \le \sqrt{2}$. This contradicts (5.2.1).

Now $B^5 \le B^4AH \le \frac{2}{FG} < 4$ gives $B < 4^{1/5} < 1.31951$.

Final contradiction. Let

$$\lambda = \begin{cases} A & \text{if } A \le 1.284 \text{ and } 1 < H \le A, \\ 1.31951 & \text{if } A > 1.284 \text{ and } 1 < H \le 1.15, \\ 1.284 & \text{if } A > 1.284 \text{ and } 1.15 < H \le 1.25, \\ 1.257 & \text{if } A > 1.284 \text{ and } 1.25 < H \le 1.31951. \end{cases} \tag{5.2.2}$$

Since $B \le A$ and $B^4HA < 4$ by Claim (iii), we get $B \le \lambda$.

Using $(3,3,1,1)$ and the AM-GM inequality we get $4A + 4D + G + H - 2A^2D^2\sqrt{G}\sqrt{H} > 8$. The left side of this inequality is a quadratic in $\sqrt{G}$. Since $A^4D^4H - 4A - 4D - H + 8 > 0$, we have

$$\sqrt{G} < A^2D^2\sqrt{H} - (A^4D^4H - 4A - 4D - H + 8)^{1/2} = \alpha \quad \text{(say)}. \tag{5.2.3}$$

Also inequality $(1,2,2,2,1)$ on using the AM-GM inequality gives $A + 4B + 4D + 4F + H - 6BDFA^{1/3}H^{1/3} > 8$, which gives $F < (A + 4B + 4D + H - 8)(6BDA^{1/3}H^{1/3} - 4)^{-1}$. Substituting this upper bound of F in inequality $(2,2,2,1,1)$, we get

$$G > 8 - 2B - 2D - H - 2\left\{\frac{A + 4B + 4D + H - 8}{6BDA^{1/3}H^{1/3} - 4}\right\} = \beta \text{ (say)}. \tag{5.2.4}$$

From (5.2.3) and (5.2.4), we have $\beta < \alpha^2$. On simplifying we get

$$\begin{aligned}(5.2.5)\qquad \phi(B) &= \left\{A^4D^4H - 2A - D + B + \frac{A + 4B + 4D + H - 8}{6BDA^{1/3}H^{1/3} - 4}\right\}^2 \\ &\quad - A^4D^4\{A^4D^4H^2 - 4AH - 4DH - H^2 + 8H\} > 0,\\ \phi'(B) &= \left\{A^4D^4H - 2A - D + B + \frac{A + 4B + 4D + H - 8}{6BDA^{1/3}H^{1/3} - 4}\right\} \\ &\quad \times \left\{1 + \frac{-16 + 6DA^{1/3}H^{1/3}(8 - A - 4D - H)}{(6BDA^{1/3}H^{1/3} - 4)^2}\right\}.\end{aligned}$$

One finds that $\phi'(B) > 0$ if $\psi(A, D, H) = 6DA^{1/3}H^{1/3} - A - 4D - H > 0$. Now $\psi(A, D, H)$ is an increasing function of D, H and A, therefore $\psi(A, D, H) > \psi(1, 1, 1) = 0$. Hence $\phi(B)$ is an increasing function of B. Since $B \leq \lambda$, we have $\phi(B) \leq \phi(\lambda)$ where λ is as given in (5.2.2). From (5.2.5), we get

$$\begin{aligned}(5.2.6)\qquad \varphi(H) &= \left\{A^4D^4H - 2A - D + \lambda + \frac{A + 4\lambda + 4D + H - 8}{6\lambda DA^{1/3}H^{1/3} - 4}\right\}^2 \\ &\quad - A^4D^4\{A^4D^4H^2 - 4AH - 4DH - H^2 + 8H\} > 0.\end{aligned}$$

Write $\varphi(H) = (\chi_1(H))^2 - \chi_2(H)$, which gives

$$\varphi''(H) = 2(\chi_1'(H))^2 + 2\chi_1(H)\chi_1''(H) - \chi_2''(H),$$

where

$$\begin{aligned}\chi_1(H) &= A^4D^4H - 2A - D + \lambda + \frac{A + 4\lambda + 4D + H - 8}{6\lambda DA^{1/3}H^{1/3} - 4},\\ \chi_2(H) &= A^4D^4\{A^4D^4H^2 - 4AH - 4DH - H^2 + 8H\}.\end{aligned}$$

For $1 < D < 1.149$ and λ as defined in (5.2.2), one can show that $\varphi''(H) > 0$ by proving that $2(\chi_1'(H))^2 - \chi_2''(H) > 0$, $\chi_1(H) > 0$ and $\chi_1''(H) > 0$. Let

$$(5.2.7)\qquad (\mu_1, \mu_2) = \begin{cases} (1, A) & \text{if } A \leq 1.284 \text{ and } \lambda = A,\\ (1, 1.15) & \text{if } A > 1.284 \text{ and } \lambda = 1.31951,\\ (1.15, 1.25) & \text{if } A > 1.284 \text{ and } \lambda = 1.284,\\ (1.25, 1.31951) & \text{if } A > 1.284 \text{ and } \lambda = 1.257.\end{cases}$$

From (5.2.2) and (5.2.7), we have $\mu_1 \leq H \leq \mu_2$. Therefore $\varphi(H) \leq \max\{\varphi(\mu_1), \varphi(\mu_2)\}$. Now $\varphi(\mu_1)$ and $\varphi(\mu_2)$ are functions in two variables A and D and can be shown to be negative in each of the cases, by plotting their 3-dimensional surface graphs for $1 < D < 1.149$ and $1 < A \leq \sqrt{2}$ (using Mathematica 5.1), which gives a contradiction to (5.2.6).

5.3. Case (16)

PROPOSITION 19. *Case* (16), *i.e.* $A > 1$, $B > 1$, $C > 1$, $D > 1$, $E \leq 1$, $F \leq 1$, $G \leq 1$, $H \leq 1$, *does not arise.*

Proof. Here $c \le \frac{1}{2}$, $d \le \frac{1}{3}$, $f \le \frac{1}{3}$, $g \le \frac{1}{2}$ by Lemma 3. Also $H \ge \frac{4}{9}D$, $F \ge \frac{4}{9}B$, $H \ge \frac{8B}{27}$ and $G \ge \frac{4}{9}C$. Since the lattice generated by $(A_2, 0, 0, \ldots, 0)$, $(a_{3,2}, A_3, 0, \ldots, 0), \ldots, (a_{8,2}, a_{8,3}, \ldots, a_{8,7}, A_8)$ in $\mathbb{R}^7$ has no point in the interior of the sphere with radius A_2 centred at the origin, it follows that $\Delta(A_2 S_7) \le A_2 A_3 \cdots A_8$, which gives $B^7 \Delta^2(S_7) \le BCDEFGH = \frac{1}{A} \le \frac{1}{B}$. Hence $B^8 \le 64$, which gives $b < 0.69$. Also $2E > 1 \ge F$, $2F > 1 \ge G$ and $2G > 1 \ge H$. Using inequalities $(2,2,2,2), (1,2,2,2,1), (1,2,2,1,2)$, $(2,1,2,1,2), (2,2,2,1,1)$, and $(2,1,2,2,1)$ we get

$$2b + 2d - 2f - 2h > 0, \tag{5.3.1}$$

$$a + 2c - 2e - 2g - h > 0, \tag{5.3.2}$$

$$a + 2c - 2e - f - 2h > 0, \tag{5.3.3}$$

$$2b + c - 2e - f - 2h > 0, \tag{5.3.4}$$

$$2b + 2d - 2f - g - h > 0, \tag{5.3.5}$$

$$2b + c - 2e - 2g - h > 0. \tag{5.3.6}$$

First we prove that if $d \ge \frac{1}{8}$ and λ and μ are positive functions of d such that $b \le \lambda$ and $a \ge \mu$ then

$$D^4 HABC \ge \min \begin{cases} D^4(1-\mu)(1+\mu)^2, \\ D^4(1-\lambda)(1+\lambda)^2, \\ \frac{4}{9}D^5(1+\mu)^2(\frac{19}{9} - \frac{8d}{9} - 2\mu), \\ \frac{4}{9}D^5(1+\lambda)^2(\frac{19}{9} - \frac{8d}{9} - 2\lambda). \end{cases} \tag{5.3.7}$$

Since $h \le \frac{5}{9} - \frac{4d}{9}$ and also $h \le b + \frac{c}{2}$ from (5.3.4), we divide the proof of (5.3.7) into two cases.

If $c \le \frac{10}{9} - 2b - \frac{8d}{9}$, i.e. $b + \frac{c}{2} \le \frac{5}{9} - \frac{4d}{9}$ then $D^4 HABC > D^4(1 - b - \frac{c}{2})ABC = \phi(c)$, say. As $\phi''(c) < 0$, we have $D^4 HABC > \min\{\phi(0), \phi(\frac{10}{9} - 2b - \frac{8d}{9})\}$. Now $\phi(0) = D^4(1-b)(1+a)(1+b)$ and $b \le \min(a, \lambda)$, $a \ge \mu$, therefore we find that $\phi(0) \ge \min\{D^4(1-\mu)(1+\mu)^2, D^4(1-\lambda)(1+\lambda)^2\}$. Further $\phi(\frac{10}{9} - 2b - \frac{8d}{9}) = \frac{4}{9}D^5 AB(\frac{19}{9} - \frac{8d}{9} - 2b) = \psi(b)$, say. As $\psi'(b) < 0$ for $d \ge \frac{1}{8}$ we get $\psi(b) \ge \psi(\min(a, \lambda))$, which gives the desired result for $a \ge \mu$.

If $c > \frac{10}{9} - 2b - \frac{8d}{9}$ then $D^4 HABC > \frac{4}{9}D^5 AB(\frac{19}{9} - \frac{8d}{9} - 2b) = \psi(b)$, which has already been dealt with. This proves (5.3.7).

Claim (i). $d < 0.155$.

Suppose $d \ge 0.155$. We first show that $D^4 HABC > 2$. If $B > \frac{3}{2}$, we get this by using $H \ge \frac{4}{9}D, A \ge B$ and $C > 1$. For $B \le \frac{3}{2}$ we get this by (5.3.7) with $\lambda = \frac{1}{2}$, $\mu = d \ge 0.155$. So $(3,4,1)$ holds. Therefore $4A - A^4 DEFGH + 4D - \frac{1}{2}D^5 HABC + H > 8$. Using the AM-GM inequality we get $4A + 4D - \sqrt{2}A^2 D^{5/2} H^{1/2} + H > 8$. As the left side is a decreasing function of H and $H \ge \frac{4}{9}D$, we get $4A + \frac{40}{9}D - \frac{2\sqrt{2}}{3}A^2 D^3 > 8$. Further, as a function of A, the left side has maximum at $A = \frac{3}{\sqrt{2D^3}}$, therefore we get $\frac{40}{9}D + \frac{6}{\sqrt{2D^3}} > 8$. This inequality does not hold for $1.155 \leq D \leq \frac{4}{3}$. For future reference we notice that in fact it does not hold for $D > 1.12$: see Claims (v) and (vii).

CLAIM (ii). $C < 1.322$ *and* $A < 1.983$.

Assume $C \geq 1.322$. Then since $EF < 1$, we get $C^4GHAB > \frac{C^3}{D} > 2$ for $D < 1.155$. Therefore $(2, 4, 2)$ holds, i.e. $4A - \frac{2A^2}{B} + 4C - \frac{1}{2}C^5GHAB + 4G - \frac{2G^2}{H} > 8$. Using the AM-GM inequality we get $4A - \frac{A^2}{B} + 4C + 4G - 3AGC^{5/3} > 8$. The left side is a decreasing function of G and $G \geq \frac{4C}{9}$, therefore we get $4A - \frac{A^2}{B} + \frac{52C}{9} - \frac{4}{3}AC^{8/3} > 8$. Further the left side is a decreasing function of C as well as of A. Therefore we can replace C by 1.322 and A by B to get $3B + \frac{52\times 1.322}{9} - \frac{4}{3}B(1.322)^{8/3} > 8$, which is not possible for $B < 1.69$. So we must have $C < 1.322$. Since $A \leq \frac{3}{2}C$ we get $A < 1.983$.

CLAIM (iii). $B < 1.4509$.

Suppose $B \geq 1.4509$. Using Claims (i) and (ii) we get $B^4FGHA \geq \frac{B^3}{CD} > 2$. Therefore $(1, 4, 2, 1)$ holds, i.e. $A + 4B - \frac{1}{2}B^5FGHA + 4F - \frac{2F^2}{G} + H > 8$. Using the AM-GM inequality we get

$$A + 4B + 4F + H - 2B^{5/2}F^{3/2}H^{1/2}A^{1/2} > 8. \tag{5.3.8}$$

Now the left side of (5.3.8) is a decreasing function of F as well as of H. We shall use different lower bounds of F and of H for different ranges of B.

CASE 1: $B > 1.586$. Here we replace H by $\frac{8B}{27}$ and F by $\frac{4B}{9}$ to get $A + \frac{164B}{27} - 2(\frac{8B}{27})^{1/2}(\frac{4B}{9})^{3/2}B^{5/2}A^{1/2} > 8$, which is not true for $1.586 < B \leq A < 1.983$.

CASE 2: $1.47 < B \leq 1.586$.

SUBCASE 2.1: $H \geq \frac{5.8}{9}$. Here we replace H by $\frac{5.8}{9}$, F by $\frac{4B}{9}$ and get $A + \frac{52B}{9} + \frac{5.8}{9} - 2(\frac{4B}{9})^{3/2}(\frac{5.8}{9})^{1/2}B^{5/2}A^{1/2} > 8$, which is not possible for $1.47 < B \leq 1.586$ and $B \leq A < 1.983$.

SUBCASE 2.2: $H < \frac{5.8}{9}$, i.e. $\frac{3.2}{9} < h \leq \frac{5}{9}$. From (5.3.1) and Claim (i), we have $F > 1 - b - d + h > 1 - b - 0.155 + h$. Here we replace F by $1-b-0.155+h$ to get $\phi(h) = A+4B+4(1-b-0.155+h)+H-2(1-b-0.155+h)^{3/2}H^{1/2}B^{5/2}A^{1/2} > 8$. As $\phi''(h) > 0$, we have $\phi(h) < \max\{\phi(\frac{3.2}{9}), \phi(\frac{5}{9})\}$. A simple calculation shows that $\phi(\frac{3.2}{9}) < 8$ and $\phi(\frac{5}{9}) < 8$ for $1.47 \leq B \leq 1.586$ and $B \leq A < 1.983$, which gives a contradiction.

CASE 3: $1.4509 \leq B \leq 1.47$. In this case we work as in Case 2 above partitioning the interval for H into the subcases $H \geq \frac{6.5}{9}$ and $H < \frac{6.5}{9}$.

In Claims (iv), (v), (xvi), (xxi) and final contradiction we shall divide the discussion into two cases $g+h \le \alpha a + \beta d$ and $g+h > \alpha a + \beta d$ for different choices of α and β.

Claim (iv). $a > 0.245$.

Suppose $a \le 0.245$.

Case 1: $g+h \le 1.6d + 0.52a$. Using inequality $(3,3,1,1)$ we have $4A - A^4DEFGH + 4D - D^4GHABC + G + H > 8$. Using the AM-GM inequality we get $4A + 4D - 2A^2D^2\sqrt{GH} + G + H > 8$. This implies that

$$2 + 4a + 4d - (g+h) - 2(1+a)^2(1+d)^2(1-g-h)^{1/2} > 0. \tag{5.3.9}$$

As the left side is an increasing function of $g+h$ and $g+h \le 1.6d + 0.52a$, we get $\phi(d) = 2 + 4a + 4d - (1.6d + 0.52a) - 2(1+a)^2(1+d)^2(1 - 1.6d - 0.52a)^{1/2} > 0$. As $\phi''(d) > 0$ and $0 < d \le \min(a, 0.155)$, we have $\phi(d) \le \max\{\phi(0), \phi(\min(a, 0.155))\}$. Now one can easily check that $\phi(0) < 0$ for $0 < a \le 0.245$; $\phi(a) < 0$ for $0 < a \le 0.155$; and $\phi(0.155) < 0$ for $0.155 < a \le 0.245$. This gives a contradiction.

Case 2: $g+h > 1.6d + 0.52a$. Using (5.3.5) we get $f < b + 0.2d - 0.26a$. Inequality $(1,2,2,2,1)$ after using AM-GM gives $A + 4B + 4D + 4F - 6BDFA^{1/3}H^{1/3} + H > 8$. As the left side is a decreasing function of H and $H > 1 - b - d + f$, we get

$$\phi(f) = 6 + a + 3b + 3d - 3f - 6BDFA^{1/3}(1 - b - d + f)^{1/3} > 0. \tag{5.3.10}$$

Since $\phi''(f) > 0$ and $0 < f < b + 0.2d - 0.26a$, we have $\phi(f) < \max\{\phi(0), \phi(b + 0.2d - 0.26a)\}$. Let $\phi(0) = 6 + a + 3b + 3d - 6BDA^{1/3}(1 - b - d)^{1/3} = \psi(a)$. As $\psi'(a) < 0$ and $a \ge b$, we have $\psi(a) \le \psi(b) = 6 + 4b + 3d - 6DB^{4/3}(1 - b - d)^{1/3}$, which can be verified to be negative for $0 < b \le 0.4509$ and $0 < d \le 0.155$. Thus for b, d lying in these intervals and $a \ge b$ we always have $\phi(0) < 0$. At the other end point of f, let $\phi(b + 0.2d - 0.26a) = 6 + 1.78a + 2.4d - 6BD(1 - b - 0.2d + 0.26a)A^{1/3}(1 - 0.8d - 0.26a)^{1/3} = \vartheta(b)$. As $\vartheta''(b) > 0$ and $0 < b \le a$, we have $\vartheta(b) \le \max(\vartheta(0), \vartheta(a))$. One can easily check that $\vartheta(0)$ and $\vartheta(a)$ are negative for $0 < a \le 0.245$ and $0 < d \le 0.155$. This gives a contradiction.

Claim (v). $a > 0.285$.

Suppose $a \le 0.285$. Under this assumption we first see that $d < 0.144$. If $d \ge 0.144$, we get $D^4HABC > 2$ by (5.3.7) with $\lambda = 0.285$, $\mu = 0.245$. So inequality $(3,4,1)$ gives a contradiction as in Claim (i).

Case 1: $g+h \le 1.53d + 0.69a$. Using inequality $(3,3,1,1)$ and proceeding as in Case 1 of Claim (iv), we get $\phi(d) = 2 + 4a + 4d - (1.53d + 0.69a) - 2(1+a)^2(1+d)^2(1 - 1.53d - 0.69a)^{1/2} > 0$, which is not possible for $0.245 < a \le 0.285$ and $0 < d < 0.144$.

Case 2: $g + h > 1.53d + 0.69a$. From (5.3.5) we get $f < b + 0.235d - 0.345a$. Using inequality $(1, 2, 2, 2, 1)$ and proceeding as in Case 2 of Claim (iv), we just need to check that (5.3.10) is not true at the end point $f = b + 0.235d - 0.345a$. A simple calculation shows that $\phi(b + 0.235d - 0.345a) < 0$ for $0 < b \leq a$, $0 < d < 0.144$ and $0.245 < a \leq 0.285$. This gives a contradiction.

Claim (vi). $b > 0.178$.

Inequality $(2, 1, 5^*)$ gives

$$\phi(C) = 4A - \frac{2A^2}{B} + C + \frac{5}{(ABC)^{1/5}} > 8. \tag{5.3.11}$$

As $\phi(C)$ is an increasing function of C and $C \leq \min(A, 1.322)$, we get $\phi(C) \leq \phi(\min(A, 1.322))$. When $1 \leq B \leq 1.178$, one finds that $\phi(A) < 8$ for $1.285 < A \leq 1.322$ and $\phi(1.322) < 8$ for $1.322 < A < 1.983$. This contradicts (5.3.11).

Claim (vii). $d < 0.147$.

Taking $\lambda = 0.4509$, $\mu = 0.285$ in (5.3.7) we get $D^4HABC > 2$ for $d \geq 0.147$. Now inequality $(3, 4, 1)$ gives a contradiction as in Claim (i).

Claim (viii). $F < 0.82$.

Suppose $F \geq 0.82$, i.e. $f \leq 0.18$. Using $(1, 2, 2, 2, 1)$ and proceeding as in Case 2 of Claim (iv), we just need to check that (5.3.10) is not true at the end point $f = 0.18$. Now $\phi(0.18) = \psi(a)$ is a decreasing function of a and $a \geq \max\{b, 0.285\}$, therefore $\psi(a) \leq \psi(\max(b, 0.285))$, which is negative for $0 < d < 0.147$.

Claim (ix). $C < 1.2345$.

Suppose $C \geq 1.2345$. Then using bounds on D and F from Claims (vii) and (viii) we have

$$C^4GHAB = \frac{C^3}{DEF} \geq \frac{C^3}{DF} > \frac{C^3}{1.147 \times 0.82} > 2. \tag{5.3.12}$$

Case 1: $G > 0.59$. Inequality $(2, 4, 2)$ holds. Applying the AM-GM inequality we get

$$4A - \frac{2A^2}{B} + 4C + 4G - 2C^{5/2}G^{3/2}A^{1/2}B^{1/2} > 8.$$

The left hand side is a decreasing function of both G and A, therefore replacing G by 0.59 and A by B we get $2B + 4C + 2.36 - 2C^{5/2}(0.59)^{3/2}B > 8$, which is not true for $1.178 < B < 1.4509$ and $1 < C < 1.322$.

CASE 2: $G \leq 0.59$. Inequality $(2,4,1,1)$ holds, therefore we have

$$4A - \frac{2A^2}{B} + 4C - \frac{1}{2}C^5GHAB + G + H > 8.$$

As $H > 1 - 2b - c + 2g$ from (5.3.6) and the left hand side is a decreasing function of H, we get $\psi(g) = 4A - \frac{2A^2}{B} + 4C - \frac{1}{2}C^5(1-g)(1-2b-c+2g)AB + 2 - 2b - c + g > 8$. As $\psi''(g) > 0$ and $0.41 \leq g < \frac{1}{2}$, we have $\psi(g) \leq \max\{\psi(0.41), \psi(\frac{1}{2})\}$, which can be easily verified to be less than 8 for $0.178 < b < 0.4509$ and $0 < c < 0.322$. This gives a contradiction.

CLAIM (x). $b < 0.415$.

Suppose $b \geq 0.415$. Then using Claims (ix) and (vii) we obtain $B^4FGHA > \frac{B^3}{CD} > 2$ and therefore $(1,4,2,1)$ holds, which yields inequality (5.3.8). The left side of (5.3.8) is a decreasing function of H. Also from (5.3.1) and Claim (vii), we have $H > 1 - b - d + f > 1 - b - 0.147 + f$. So we replace H by $1 - b - 0.147 + f$ to get $\phi(f) = 2 + a + 3b - 0.147 - 3f - 2(1 - b - 0.147 + f)^{1/2}(1-f)^{3/2}B^{5/2}A^{1/2} > 0$. As $\phi''(f) > 0$ we have $\phi(f) < \max\{\phi(0.18), \phi(\frac{1}{3})\}$. A simple calculation shows that $\phi(0.18) < 0$ and $\phi(\frac{1}{3}) < 0$ for $0.415 \leq b < 0.4509$ and $b \leq a < 0.983$, which gives a contradiction. We note that inequality $(1,4,2,1)$ gives a contradiction even for $0.27 \leq b \leq 0.4509$ and $0.285 < a < 0.983$.

CLAIM (xi). $F < 0.811$.

Suppose $F \geq 0.811$, i.e. $f \leq 0.189$. Working as in Claim (viii) we get a contradiction. Now we use $0.178 < b < 0.415$ in place of $0.178 < b < 0.4509$.

CLAIM (xii). $C < 1.23$.

If $C \geq 1.23$, using upper bounds on D and F from Claims (vii) and (xi) in (5.3.12) and working as in Claim (ix) we get a contradiction.

CLAIM (xiii). $D^4HABC \leq 2$ *and* $G > 0.64$.

Suppose $D^4HABC > 2$. Then $(2,1,4,1)$ holds, i.e.

$$4A - \frac{2A^2}{B} + C + 4D - \frac{1}{2}D^5HABC + H > 8. \tag{5.3.13}$$

From (5.3.4) we have $H > 1 - b - \frac{c}{2}$. The left hand side of (5.3.13) is a decreasing function of both H and A, therefore we can replace H by $1-b-\frac{c}{2}$ and A by B to get $\phi(c) = b + \frac{c}{2} + 4d - \frac{1}{2}(1+d)^5(1-b-\frac{c}{2})(1+b)^2(1+c) > 0$. Now $\phi''(c) > 0$ and $0 < c < 0.23$, therefore $\phi(c) < \max\{\phi(0), \phi(0.23)\}$, which is negative for $0.178 < b < 0.415$ and $0 < d < 0.147$. This gives a contradiction. Therefore $D^4HABC \leq 2$.

This gives $EFG \geq \frac{1}{2}$. Therefore $\frac{1}{2} \leq EFG < \frac{3}{2}G(0.811)G$ and hence $G > 0.64$.

Claim (xiv). $A < 1.587$.

Suppose $A \geq 1.587$. Then $A^4EFGH = \frac{A^3}{BCD} > \frac{A^3}{1.415\times 1.23\times 1.147} > 2$. Therefore $(4,2,2)$ holds. Applying the AM-GM inequality we have

$$4A + 4E + 4G - 3\cdot 2^{1/3}A^{5/3}EG > 8.$$

The left hand side is a decreasing function of both G and E. Therefore we can replace G by 0.64 and E by 0.75 to get $4A+3+2.56-3\cdot 2^{1/3}A^{5/3}(0.48) > 8$, which is not true for $A \geq 1.587$; in fact, it is not true even for $A \geq 1.54$.

Claim (xv). $d < 0.14312$.

Taking $\lambda = 0.415$, $\mu = 0.285$ in (5.3.7) we get $D^4HABC > 2$ for $d \geq 0.14312$, which contradicts Claim (xiii).

Claim (xvi). $b > 0.27$.

Suppose $b \leq 0.27$.

Case 1: $g+h \leq 1.45d+0.6a$. Using inequality $(3,3,1,1)$ and proceeding as in Case 1 of Claim (iv), we get $\phi(d) = 2 + 4a + 4d - (1.45d + 0.6a) - 2(1+a)^2(1+d)^2(1-1.45d-0.6a)^{1/2} > 0$, which is not possible for $0.285 < a \leq 0.587$ and $0 < d < 0.14312$.

Case 2: $g+h > 1.45d+0.6a$. From (5.3.5) we get $f < b+0.275d-0.3a$. Using inequality $(1,2,2,2,1)$ and proceeding as in Case 2 of Claim (iv), we just need to check that (5.3.10) is not true at the end point $f = b+0.275d-0.3a$. A simple calculation shows that $\phi(b+0.275d-0.3a) < 0$ for $b \leq 0.27$, $0 < d < 0.14312$ and $0.285 < a < 0.587$. This gives a contradiction.

Claim (xvii). $e + g > 0.34a + 0.46c$.

Suppose $e+g \leq 0.34a+0.46c$. Using inequality $(2,2,2,2)$ and applying the AM-GM inequality we have

$$2 + a + c - (e+g) - 2(1+a)^{3/4}(1+c)^{3/4}(1-e-g)^{3/4} > 0.$$

The left side is an increasing function of $e+g$ so replacing $e+g$ by $0.34a + 0.46c$ we get $2+a+c-(0.34a+0.46c) - 2(1+a)^{3/4}(1+c)^{3/4}(1-0.34a-0.46c)^{3/4} > 0$, which is not true for $0.285 < a < 0.587$ and $0 < c < 0.23$.

Claim (xviii). $d < 0.1083$.

From Claim (xvii) and (5.3.2), we have $h < 0.32a + 1.08c$. If $d \geq 0.1083$, then $D^4HABC > D^4(1-0.32a-1.08c)ABC > D^4(1-0.32\times 0.285-1.08\times 0.23)\times 1.285\times 1.27\times 1.23 > 2$. This is a contradiction to Claim (xiii).

Claim (xix). $C < 1.21592$.

If $C \geq 1.21592$, using upper bounds on D and F from Claims (xviii) and (xi) in (5.3.12) and working as in Claim (ix) we get a contradiction.

Claim (xx). $B^4FGHA \leq 2$, $B < 1.392$ *and* $A < 1.554$.

Suppose $B^4FGHA>2$. Then $(1,4,2,1)$ holds. Proceeding as in Claim (x) we get a contradiction for $b>0.27$. Now $2\geq B^4FGHA>\frac{B^3}{CD}>\frac{B^3}{1.21592\times 1.1083}$ implies $B<1.392$. Similarly if $A\geq 1.554$, we have $A^4EFGH=\frac{A^3}{BCD}>\frac{A^3}{1.392\times 1.21592\times 1.1083}>2$; then proceeding as in Claim (xiv), inequality $(4,2,2)$ gives a contradiction.

Claim (xxi). $a>0.384$.

Suppose $a\leq 0.384$.

Case 1: $g+h<1.5d+0.7a$. Using $(3,3,1,1)$ and proceeding as in Case 1 of Claim (iv) we get a contradiction for $0.285<a\leq 0.384$ and $0<d<0.1083$.

Case 2: $g+h\geq 1.5d+0.7a$. From (5.3.5) we get $f<b+0.25d-0.35a$. Inequality $(1,2,2,2,1)$ with the AM-GM inequality gives $A+4B-\frac{2B^2}{C}+4D+4F+H-4(DF)^{3/2}(HABC)^{1/2}>8$. As the left hand side is a decreasing function of H and an increasing function of C, we can replace H by $1-b-d+f$ and C by 1.21592 to get

$$\phi(f)=6+a+3b+3d-3f-\frac{2B^2}{1.21592}-4(DF)^{3/2}\{1.21592AB(1-b-d+f)\}^{1/2}>0.$$

One easily checks that $\phi''(f)>0$, therefore $\phi(f)<\max\{\phi(0.189),\phi(b+0.25d-0.35a)\}$. Let $\phi(0.189)=\psi(a)$. It is a decreasing function of a. Replacing a by b one can easily verify that $\phi(0.189)=\psi(a)\leq\psi(b)<0$ for $0.27<b<0.384$ and $0<d<0.1083$. Let $\phi(b+0.25d-0.35a)=\theta(b)$. A simple calculation shows that $\theta(b)$ is an increasing function of b. Therefore $\phi(b+0.25d-0.35a)=\theta(b)\leq\theta(a)$, which is negative for $0.285<a<0.384$ and $0<d<0.1083$. This gives a contradiction.

Claim (xxii). $e+g>0.34a+0.5c$.

Suppose $e+g\leq 0.34a+0.5c$. Proceeding as in Claim (xvii) and using inequality $(2,2,2,2)$ we get a contradiction for $0.384<a<0.554$ and $0<c<0.21592$.

Claim (xxiii). $d<0.09072$, $c<0.21$, $b<0.3821$ *and* $a<0.54$.

From (5.3.2) and Claim (xxii) we get $h<0.32a+c$. Proceeding as in Claim (xviii), we find that $D^4HABC>2$ for $d\geq 0.09072$, which is a contradiction to Claim (xiii). Thus we have $d<0.09072$. Using this improved bound of d and working as in Claim (xix) we get $C<1.21$. This in turn gives $b<0.3821$ as $B^3<2CD$ from Claim (xx). Now $A^4EFGH>2$ for $a\geq 0.54$. Therefore $(4,2,2)$ holds and proceeding as in Claim (xiv) we get a contradiction.

Final contradiction. If $g + h < 1.46d + 0.66a$ then using $(3, 3, 1, 1)$ and proceeding as in Case 1 of Claim (xxi) we get a contradiction for $0.384 < a < 0.54$ and $0 < d < 0.09072$. If $g + h \geq 1.46d + 0.66a$, we get from (5.3.5) that $f < b+0.27d-0.33a$. Now working as in Case 2 of Claim (xxi) we have

$$\phi(f) = 6+a+3b+3d-3f-\frac{2B^2}{1.21}-4(DF)^{3/2}\{1.21AB(1-b-d+f)\}^{1/2} > 0.$$

Since $\phi''(f) > 0$, we obtain $\phi(f) < \max\{\phi(0.189), \phi(b + 0.27d - 0.33a)\}$. As in Claim (xxi), one can easily verify that $\phi(0.189) < 0$ for $a \geq b$, $0.27 < b < 0.3821$ and $0 < d < 0.09072$. Let $\phi(b + 0.27d - 0.33a) = \theta(b)$. As before $\theta(b)$ is an increasing function of b and $b < 0.3821$. Therefore $\phi(b + 0.27d - 0.33a) = \theta(b) < \theta(0.3821)$, which is negative for $0.384 < a < 0.54$ and $0 < d < 0.09072$. This gives a contradiction.

5.4. Case (29)

PROPOSITION 20. *Case* (29), *i.e.* $A > 1$, $B > 1$, $C > 1$, $D \leq 1$, $E \leq 1$, $F \leq 1$, $G > 1$, $H > 1$, *does not arise.*

Proof. Here $b \leq \frac{1}{2}$, $c \leq \frac{1}{3}$, $e \leq \frac{1}{3}$ by Lemma 3. Also $2D > 1 \geq E$, $2E \geq 1 > F$ and $2F > 1 \geq G$. Using inequalities $(1, 1, 2, 2, 1, 1)$, $(2, 2, 2, 1, 1)$, $(1, 2, 1, 2, 1, 1)$ and $(1, 2, 2, 1, 1, 1)$ we get

$$a + b - 2d - 2f + g + h > 0, \tag{5.4.1}$$

$$2b - 2d - 2f + g + h > 0, \tag{5.4.2}$$

$$a + 2c - d - 2f + g + h > 0, \tag{5.4.3}$$

$$a + 2c - 2e - f + g + h > 0. \tag{5.4.4}$$

CLAIM (i). $a < 0.588$.

Suppose $a \geq 0.588$. Then $A^6GH > A^6 > 16$. Therefore inequality $(6, 1, 1)$ holds. That is, $4A - \frac{1}{16}A^7GH + G + H > 8$. As the left hand side is a decreasing function of G and of H, we can replace G as well as H by 1 to get $4A - \frac{1}{16}A^7 > 6$, which is not true for $a \geq 0.588$.

CLAIM (ii). $C^4GHAB < 2$ *and* $c < 0.149$.

Suppose $C^4GHAB \geq 2$. Therefore $(1, 1, 4, 1, 1)$ holds, i.e. $A + B + 4C - \frac{1}{2}C^5GHAB + G + H > 8$, which is not true, by Lemma 7(ii) with $X_2 = C$, $X_3 = B$, $X_4 = G$, $X_5 = H$.

Now $C^4GHAB < 2$ implies $C^5 < 2$ and so $c < 0.149$.

CLAIM (iii). $b < 0.322$.

Suppose $b \geq 0.322$. Then $B^5GHA \geq B^6 > \frac{16}{3}$. Therefore $(1, 5, 1, 1)$ holds, i.e. $A + 4B - \frac{3}{16}B^6GHA + G + H > 8$. As usual we can replace G, H by 1 and A by B to get $5B - \frac{3}{16}B^7 > 6$, which is not true for $B \geq 1.322$.

CLAIM (iv). $b < 0.202$.

Suppose $b \geq 0.202$. We first show that $B^4FGHA > 2$. Let $g + h = k$. We consider the following cases:

Case 1: $k < a$. Here using $f < b + \frac{k}{2}$ from (5.4.2) we have $B^4FGHA > (1+b)^4(1+a)(1-b-\frac{k}{2})(1+k) = \phi(k) \geq \min\{\phi(0), \phi(a)\} > 2$ for $b \leq a < 0.588$ and $0.202 \leq b < 0.322$.

Case 2: $k \geq a$, $a \geq 0.4$. Here using $F > \frac{1}{2}$, we have $B^4FGHA > \frac{1}{2}(1.202)^4(1.4)^2 > 2$.

Case 3: $k \geq a$, $a < 0.4$. $B^4FGHA > (1+b)^4(1+a)(1-b-\frac{k}{2})(1+k) = \phi(k) \geq \min\{\phi(a), \phi(2a)\} > 2$ for $b \leq a < 0.4$ and $0.202 \leq b < 0.322$.

Therefore inequality $(1,4,1,1,1)$ holds, i.e. $A + 4B - \frac{1}{2}B^5FGHA + F + G + H > 8$. Since the left side of this inequality is a decreasing function of B, we can replace B by 1.202 to get

$$A + 4.808 - \frac{1}{2}(1.202)^5 AFGH + F + G + H > 8. \tag{5.4.5}$$

We have $2F \geq C$ and $F \geq \frac{2}{3}D$, i.e. $1 - c \geq 2f$ and $\frac{d}{2} \geq \frac{3f-1}{4}$. Also from (5.4.3), $c + \frac{a+g+h}{2} > f + \frac{d}{2}$. Adding all these we get $f < \frac{1}{3} + \frac{2(a+g+h)}{15}$. Now the coefficient of F on the left hand side of (5.4.5) is negative so we can replace F by $1 - \frac{1}{3} - \frac{2(a+g+h)}{15}$ to get

$$\begin{aligned} \chi(g,h) = 0.475 + \frac{13(a+g+h)}{15} \\ - (1.254)(1+a)(1+g)(1+h)\left\{\frac{2}{3} - \frac{2(a+g+h)}{15}\right\} > 0. \end{aligned} \tag{5.4.6}$$

One can easily check that the second derivative of the function χ firstly with respect to g and then with respect to h is positive. The function being symmetric in g and h, we find that $\chi(g,h) \leq \max\{\chi(0,0), \chi(a,0), \chi(a,a)\}$ which is non-positive for $0 < a < 0.588$. This contradicts (5.4.6).

Claim (v). *$f > 2c$ and $c < 0.097$.*

Assume $f \leq 2c$. From (5.4.1) we have $f < \frac{1}{2}(a+b+g+h)$. Using Lemma 6(v) with inequality $(1,1,3,1,1,1)$, taking $\gamma = f < \frac{1}{2}(a+b+g+h) = \frac{1}{2}\delta$ for $c < 0.149$, we get a contradiction. Hence $f > 2c$.

If $c \geq 0.097$, we get $C^4GHAB > C^4(1+a+b+g+h) > C^4(1+2f) > C^4(1+4c) > 2$, which contradicts Claim (ii).

Claim (vi). *$a < 0.3816$.*

Suppose $a \geq 0.3816$. Then $A^4EFGH > \frac{A^3}{BC} > \frac{(1.3816)^3}{1.202 \times 1.097} > 2$. Therefore $(4,2,1,1)$ holds. Using the AM-GM inequality we get $4A + 4E - 2A^{5/2}E^{3/2}G^{1/2}H^{1/2} + 1 + GH > 8$. The left side is a decreasing function of E for $E > \frac{2}{3}$ and $A > 1.3$. So replacing E by $\frac{2}{3}$ we get $\phi(x) =$

$4A + \frac{8}{3} - 2A^{5/2}(\frac{2}{3})^{3/2}x^{1/2} + 1 + x > 8$, where $1 < x = GH \leq A^2$. As $\phi''(x) > 0$, we have $\phi(x) \leq \max\{\phi(1), \phi(A^2)\}$. One can easily check that $\phi(1)$ and $\phi(A^2)$ are less than 8 even for $A > 1.3$; a contradiction.

Claim (vii). $a > 0.275$.

Suppose $a \leq 0.275$.

Case 1: $f < 1.4c + 0.635(g + h)$. Using inequality $(2, 3, 1, 1, 1)$ we have $2B + 4C - C^4FGHAB + F + G + H > 8$. As the coefficient of B, namely $2-C^4FGHA$, is positive by Claim (ii), we can replace b by $\lambda = \min(a, 0.202)$ and then F by $1 - 1.4c - 0.635(g + h)$ to get

$$(5.4.7) \quad \phi(g, h) = 1 + 2\lambda + 2.6c + 0.365(g + h) - (1 + c)^4(1 + a) \times \{1 - 1.4c - 0.635(g + h)\}(1 + g)(1 + h)(1 + \lambda) > 0.$$

One finds that the second derivative of the function ϕ first with respect to g and then with respect to h is positive, therefore $\phi(g, h) \leq \max\{\phi(0, 0), \phi(a, 0), \phi(a, a)\}$, which is non-positive for $\lambda = a$ if $a < 0.202$, and $\lambda = 0.202$ if $0.202 \leq a \leq 0.275$, and $0 < c < 0.097$. This contradicts (5.4.7).

Case 2: $f \geq 1.4c + 0.635(g + h)$. From (5.4.4) we get $e < 0.5a + 0.3c + 0.1825(g + h)$. Using inequality $(2, 2, 2, 1, 1)$ and applying the AM-GM inequality we have $4A + 4C + 4E - 6ACEG^{1/3}H^{1/3} + G + H > 8$. The left hand side is a decreasing function of E so replacing E by $1 - 0.5a - 0.3c - 0.1825(g + h)$ and simplifying we get

$$(5.4.8) \quad \psi(g, h) = 6 + 2a + 2.8c + 0.27(g + h) - 6(1 + a)(1 + c) \times \{1 - 0.5a - 0.3c - 0.1825(g + h)\}(1 + g)^{1/3}(1 + h)^{1/3} > 0.$$

Again the second derivative of the function ψ first with respect to g and then with respect to h is positive, therefore $\psi(g, h) \leq \max\{\psi(0, 0), \psi(a, 0), \psi(a, a)\}$, which is non-positive for $0 < a \leq 0.275$ and $0 < c < 0.097$. This contradicts (5.4.8).

Claim (viii). $c < 0.079$.

For if $c \geq 0.079$ then $C^4GHAB > C^4A(1 + b + g + h) > C^4 \cdot 1.275(1 + 2c) > 2$, as $b + g + h \geq b + \frac{g}{2} + \frac{h}{2} > f > 2c$ from inequality (5.4.2). This contradicts Claim (ii).

Claim (ix). $b < 0.175$ *and* $a < 0.364$.

Suppose $b \geq 0.175$. Then proceeding as in Claim (iv) we get $B^4FGHA > (1 + b)^4(1 + a)(1 - b - \frac{k}{2})(1 + k) = \phi(k) \geq \min\{\phi(0), \phi(a), \phi(2a)\} > 2$ for $0.275 < a < 0.3816$ and $0.175 \leq b < 0.202$. Now using inequality $(1, 4, 1, 1, 1)$ and working as in Claim (iv) we get a contradiction.

Further if $a \geq 0.364$, then $A^4EFGH > \frac{A^3}{BC} > \frac{A^3}{1.175 \times 1.079} > 2$. Now using inequality $(4, 2, 1, 1)$ and working as in Claim (vi), we get a contradiction.

Claim (x). *$f \geq 1.6c + 0.57(g+h)$ and $a > 0.306$.*

Suppose $f < 1.6c + 0.57(g+h)$. We use inequality $(2,3,1,1,1)$ and work as in Case 1 of Claim (vii) to get a contradiction for $b < 0.175$, $0 < c < 0.079$ and $0.275 < a < 0.364$. So we must have $f \geq 1.6c + 0.57(g+h)$. This gives $e < 0.5a + 0.2c + 0.215(g+h)$.

Suppose $a \leq 0.306$. Using inequality $(2,2,2,1,1)$ and working again as in Case 2 of Claim (vii), we get a contradiction for $0.275 < a \leq 0.306$ and $0 < c < 0.079$.

Claim (xi). *$g + h > a$.*

Suppose if possible $k = g + h \leq a$. From (5.4.4) and Claim (x) we get $e < 0.5a + 0.2c + 0.215k$. Also from (5.4.3) we have $d + f < b + \frac{k}{2} < 0.175 + \frac{k}{2}$. Using inequality $(2,1,1,1,1,1,1)$ we have $4A - 2A^3C(1-d-f)(1-e)GH + 3 - d - f - e + G + H > 8$. Replacing $d + f$ by $0.175 + \frac{k}{2}$ and e by $0.5a + 0.2c + 0.215k$ we get

$$\begin{aligned}(5.4.9)\qquad \psi(k) = 1.825 &+ 3.5a + 0.8c + 0.285k - 2(1+a)^3(1+c)\\ &\times \left(0.825 - \frac{k}{2}\right)(1 - 0.5a - 0.2c - 0.215k)(1+k) > 0.\end{aligned}$$

Again one finds that $\psi''(k) > 0$, therefore $\psi(k) \leq \max\{\psi(0), \psi(a)\}$, which is non-positive for $0.306 \leq a \leq 0.364$ and $0 < c < 0.079$. This gives a contradiction.

Claim (xii). *$b < 0.134$.*

Suppose $b \geq 0.134$. Here using $f < b + \frac{g+h}{2}$ from (5.4.2) we have $B^4FGHA > (1+b)^4(1+a)(1 - b - \frac{g+h}{2})(1+g)(1+h) = \phi(g)$, say. Since $a - h < g \leq a$ we have $\phi(g) \geq \min\{\phi(a-h), \phi(a)\} > \min\{(1+b)^4(1 - b - \frac{a}{2})(1+a)^2, (1+b)^4(1-b-a)(1+a)^3\} > 2$ for $0.306 < a < 0.364$ and $0.134 \leq b < 0.175$. Therefore $B^4FGHA > 2$. Now working as in Claim (iv), inequality $(1,4,1,1,1)$ gives a contradiction.

Claim (xiii). *$c < 0.041$ and $a < 0.3316$.*

If $c \geq 0.041$ then $C^4GHAB > C^4(1+a)^2 > 2$ for $a > 0.306$, which contradicts Claim (ii).

Further if $a \geq 0.3316$, then $A^4EFGH > \frac{A^3}{BC} > \frac{A^3}{1.134 \times 1.041} > 2$. Now using inequality $(4,2,1,1)$ and working as in Claim (vi), we get a contradiction.

Final contradiction. If $f < 2c + 0.61(g+h)$, we use inequality $(2,3,1,1,1)$ and work as in Case 1 of Claim (vii) to get a contradiction for $b < 0.134$, $0 < c < 0.041$ and $0.306 < a < 0.3316$. If $f \geq 2c + 0.61(g+h)$, we find $e < 0.5a + 0.195(g+h)$. Again using inequality $(2,2,2,1,1)$ and working as in Case 2 of Claim (vii), we get a contradiction for $0.306 < a \leq 0.3316$ and $0 < c < 0.041$.

5.5. Case (31)

Proposition 21. *Case* (31), *i.e.* $A > 1$, $B > 1$, $C > 1$, $D \leq 1$, $E \leq 1$, $F \leq 1$, $G \leq 1$, $H > 1$, *does not arise.*

Proof. Here $b \leq \frac{1}{2}$, $c \leq \frac{1}{3}$, $e \leq \frac{1}{3}$ and $G \geq \frac{4C}{9}$, i.e. $g \leq \frac{5}{9} - \frac{4c}{9}$. Also $2D > 1 \geq E$, $2E \geq 1 > F$ and $2F > 1 \geq G$. Using inequalities $(2,2,1,2,1)$, $(1,2,2,1,1,1)$, $(1,2,2,2,1)$ and $(2,2,2,1,1)$ we get

$$2b - 2d - e - 2g + h > 0, \tag{5.5.1}$$

$$a + 2c - 2e - f - g + h > 0, \tag{5.5.2}$$

$$a + 2c - 2e - 2g + h > 0, \tag{5.5.3}$$

$$2b - 2d - 2f - g + h > 0. \tag{5.5.4}$$

Claim (i). $C < 1.155$, $A \leq 1.7325$.

Suppose $C \geq 1.155$. Proceeding as in (5.3.7) and Claim (i) of Proposition 19, replacing d by c, h by g and c by h, we find that $C^4GHAB > 2$.

Therefore $(2,4,1,1)$ holds, i.e.

$$4A - \frac{2A^2}{B} + 4C - \frac{1}{2}C^5GHAB + G + H > 8. \tag{5.5.5}$$

As the coefficient of G in (5.5.5) is negative and $G \geq \frac{4C}{9}$, we can replace G by $\frac{4C}{9}$. Further if $C^6AB < \frac{9}{2}$ we can replace H by A and then B by A to get $\phi(A) = 3A + \frac{40}{9}C - \frac{2}{9}C^6A^3 > 8$. Now $\phi(A)$ has its maximum value at $\frac{3}{\sqrt{2C^3}}$ where it is less than 8 for $C \geq 1.155$, giving thereby a contradiction. If $C^6AB \geq \frac{9}{2}$, we can replace H by 1 and A by B to get $2B + \frac{40}{9}C - \frac{2}{9}C^6B^2 > 7$, which is not true for $1 < B \leq \frac{3}{2}$ and $C \geq 1.155$; again a contradiction.

Now $A \leq \frac{3}{2}C$ implies $A \leq 1.7325$ for $C < 1.155$.

Claim (ii). $B < 1.322$.

Suppose $B \geq 1.322$. Then using Claim (i), $B^4FGHA > \frac{B^3}{C} > 2$ and therefore $(1,4,2,1)$ holds, which by using the AM-GM inequality yields

$$A + 4B + 4F + H - 2B^{5/2}F^{3/2}H^{1/2}A^{1/2} > 8. \tag{5.5.6}$$

The left side of (5.5.6) is a decreasing function of F for $F \geq \frac{4}{9}B$, therefore we can replace F by $\frac{4}{9}B$ to get $\psi(H) = A + \frac{52}{9}B + H - \frac{16}{27}B^4H^{1/2}A^{1/2} > 8$. Now $\psi''(H) > 0$, therefore $\psi(H) \leq \max\{\psi(1), \psi(A)\}$, which can be shown to be less than 8 for $B \geq 1.322$ and $B \leq A \leq 1.7325$. This gives a contradiction.

Claim (iii). $C^4GHAB \leq 2$.

Suppose $C^4GHAB > 2$. Then $(2,4,1,1)$, i.e. inequality (5.5.5), holds. As the coefficient of G in (5.5.5) is negative, we replace G by $\frac{2}{C^4HAB}$. Further it is an increasing function of both H and B, so we can replace H by A,

B by 1.322 to get $5A - \frac{2A^2}{1.322} + 3C + \frac{2}{C^4A^2\times 1.322} > 8$, which is not true for $1 < A \leq 1.7325$ and $1 < C < 1.155$.

Claim (iv). $A < 1.4509$.

Suppose $A \geq 1.4509$. Using Claims (i) and (ii) we have $A^4EFGH > \frac{A^3}{BC} > 2$. Therefore $(4,2,1,1)$ holds, which using the AM-GM inequality yields

$$4A + 4E + G + H - 2A^{5/2}E^{3/2}G^{1/2}H^{1/2} > 8. \tag{5.5.7}$$

The left side of (5.5.7) is a decreasing function of both G and E within the given ranges specified in each case.

Case 1: $A > 1.63$. Here we replace G by $\frac{8A}{27}$ and E by $\frac{4A}{9}$ in the left side of (5.5.7) to get $H + \frac{164A}{27} - 2(\frac{8A}{27})^{1/2}(\frac{4A}{9})^{3/2}A^{5/2}H^{1/2} > 8$, which is not true for $H > 1$ and $1.63 < A < 1.7325$.

Case 2: $1.4509 \leq A \leq 1.63$.

Subcase 2.1: $G \geq \frac{5.35}{9}$. So we replace G by $\frac{5.35}{9}$ and E by $\frac{4A}{9}$ in (5.5.7) and get $\frac{52A}{9} + \frac{5.35}{9} + H - 2(\frac{4A}{9})^{3/2}(\frac{5.35}{9})^{1/2}A^{5/2}H^{1/2} > 8$, which is not possible for $1.4509 \leq A \leq 1.63$ and $1 < H \leq A$.

Subcase 2.2: $G < \frac{5.35}{9}$, i.e. $g > \frac{3.65}{9}$. Also $g \leq \frac{5}{9}$. From (5.5.2) and Claim (i), we have $E > 1 - c - \frac{a+h}{2} + g > 1 - 0.155 - \frac{a+h}{2} + g$. So we replace E by this lower bound in (5.5.7) to get $\phi(g) = 4A + 4(1 - 0.155 - \frac{a+h}{2} + g) + G + H - 2(1 - 0.155 - \frac{a+h}{2} + g)^{3/2}G^{1/2}A^{5/2}H^{1/2} > 8$. As $\phi''(g) > 0$ we have $\phi(g) \leq \max\{\phi(\frac{3.65}{9}), \phi(\frac{5}{9})\}$. A simple calculation shows that $\phi(\frac{3.65}{9}) < 8$ and $\phi(\frac{5}{9}) < 8$ for $1.4509 \leq A \leq 1.63$ and $1 < H \leq A$, which gives a contradiction.

Claim (v). $A > 1.3$.

Suppose $A \leq 1.3$.

Case 1: $f + g \leq 1.6c + 0.37(a+h)$. Using inequality $(2,3,1,1,1)$ we have $2B + 4C - C^4FGHAB + F + G + H > 8$. Since by Claim (iii), the coefficient of B is positive, we can replace B by A to get $1 + 2a + 4c + h - (f+g) - C^4(1-f-g)HA^2 > 0$. Further as the coefficient of $f+g$ is positive and $f + g \leq 1.6c + 0.37(a+h)$, we get

$$\phi(h) = 1 + 1.63a + 0.63h + 2.4c - C^4(1 - 1.6c - 0.37(a+h))HA^2 > 0. \tag{5.5.8}$$

As $\phi''(h) > 0$ and $0 < h \leq a$, it follows that $\phi(h) \leq \max\{\phi(0), \phi(a)\}$. Let $\phi(0) = \psi(c)$ and $\phi(a) = \vartheta(c)$. Since $\psi''(c) > 0$ and $0 < c \leq \min(a, 0.155)$, we have $\psi(c) \leq \{\psi(0), \psi(\min(a, 0.155))\}$. Now one can easily check that $\psi(0) < 0$ for $0 < a \leq 0.3$; $\psi(a) < 0$ for $0 < a \leq 0.155$; and $\psi(0.155) < 0$ for $0.155 < a \leq 0.3$. Similarly one can prove that $\vartheta(c) < 0$ for $0 < c \leq \min(a, 0.155)$. This gives a contradiction to (5.5.8).

CASE 2: $f + g > 1.6c + 0.37(a + h)$. Using (5.5.2) we get $e < 0.2c + 0.315(a+h)$. Inequality $(2, 2, 2, 1, 1)$ after using the AM-GM inequality gives $4A + 4C + 4E - 6ACEG^{1/3}H^{1/3} + G + H > 8$. As the left hand side is a decreasing function of G and $G > 1 - c - \frac{a+h}{2} + e$ from (5.5.3), we get

$$\begin{aligned}\phi(e) &= 6 + 3.5a + 3c + 0.5h - 3e \\ &\quad - 6ACE\left(1 - c - \frac{a+h}{2} + e\right)^{1/3} H^{1/3} > 0.\end{aligned} \tag{5.5.9}$$

Since $\phi''(e) > 0$ and $0 \le e < 0.2c + 0.315(a + h)$, it follows that $\phi(e) \le \max\{\phi(0), \phi(0.2c + 0.315(a + h))\}$. Let $\phi(0) = \psi(h)$. As $\psi''(h) > 0$ and $0 < h \le a$, we have $\psi(h) \le \max\{\psi(0), \psi(a)\}$, which can be verified to be negative for $0 < a \le 0.3$ (in fact it is so for $a \le 0.4509$) and $0 < c \le 0.155$. Let $\phi(0.2c + 0.315(a + h)) = \vartheta(h)$. As $\vartheta''(h) > 0$ and $0 < h \le a$, we have $\vartheta(h) \le \max(\vartheta(0), \vartheta(a))$. One can easily check that $\vartheta(0)$ and $\vartheta(a)$ are negative for $0 < a \le 0.3$ and $0 < c \le 0.155$. This gives a contradiction to (5.5.9).

CLAIM (vi). $B > 1.185$.

Using inequality $(2, 5^*, 1)$ we get

$$\phi(H) = 4A - \frac{2A^2}{B} + \frac{5}{(ABH)^{1/5}} + H > 8. \tag{5.5.10}$$

As $\phi(H)$ is an increasing function of H and $H \le A$, we have $5A - \frac{2A^2}{B} + \frac{5}{(A^2B)^{1/5}} > 8$, which is not possible for $1 < B \le 1.185$ and $1.3 < A < 1.4509$.

CLAIM (vii). $E < 0.84$.

Suppose $E \ge 0.84$, i.e. $e \le 0.16$. Using $(2, 2, 2, 1, 1)$ and proceeding as in Case 2 of Claim (v), we just need to check that (5.5.9) is not true at the end point $e = 0.16$. Let $\phi(0.16) = \psi(h)$. As $\psi''(h) > 0$ we have $\psi(h) \le \max\{\psi(0), \psi(a)\}$, which is negative for $0 < c < 0.155$ and $1.3 < a < 1.4509$.

CLAIM (viii). $F > 0.595$, $B^4FGHA \le 2$ *and* $B < 1.2475$.

From Claim (iii) we get $EF \ge DEF \ge \frac{1}{2}$. Therefore by Claim (vii), we obtain $F > \frac{1}{2E} > 0.595$.

Now suppose $B^4FGHA > 2$. Then $(1, 4, 2, 1)$ holds. Using the AM-GM inequality we have

$$A + 4B + 4F - 2B^{5/2}F^{3/2}H^{1/2}A^{1/2} + H > 8.$$

The left side is a decreasing function of F for $A > 1.3$, $B > 1.185$ and $F > 0.595$. Therefore we can replace F by 0.595 to get $\phi(H) = A + 4B + 4 \times 0.595 - 2B^{5/2}(0.595)^{3/2}H^{1/2}A^{1/2} + H > 8$. Now $\phi''(H) > 0$ and $1 < H \le A$, therefore $\phi(H) \le \max\{\phi(1), \phi(A)\}$, which is less than 8 for $1.3 < A < 1.4509$ and $1 < B < 1.322$. This gives a contradiction.

Now $2 \geq B^4FGHA \geq \frac{B^3}{CE} > \frac{B^3}{1.155\times 0.84}$ gives $B < 1.2475$.

Claim (ix). $A < 1.4$.

Suppose $A \geq 1.4$.

If $B \leq 1.226$, we use inequality $(2, 5^*, 1)$ and work as in Claim (vi) to get a contradiction to (5.5.10).

If $B > 1.226$, we prove that $B^4FGHA > 2$, which will give a contradiction to Claim (viii). From (5.5.4) and (5.5.1), we have $f < b + \frac{h}{2} - \frac{g}{2}$ and $g < b + \frac{h}{2}$. Therefore $B^4FGHA > B^4(1-b-\frac{h}{2}+\frac{g}{2})(1-g)AH = \phi(g) \geq \min\{\phi(0), \phi(b+\frac{h}{2})\}$. Now $\phi(0) = B^4(1-b-\frac{h}{2})A(1+h) = \psi(h) \geq \min\{\psi(0), \psi(a)\} > 2$ for $b > 0.226$ and $a > 0.4$. Similarly $\phi(b+\frac{h}{2}) = B^4(1-\frac{b}{2}-\frac{h}{4})(1-b-\frac{h}{2})A(1+h) = \vartheta(h) \geq \min\{\vartheta(0), \vartheta(a)\} > 2$ for $b > 0.226$ and $a > 0.4$.

Claim (x). $g < b + 0.15h$.

Suppose $g \geq b + 0.15h$. From inequality (5.5.4), we get $d + f < 0.5b + 0.425h$. Using inequality $(1, 2, 2, 2, 1)$ and applying the AM-GM inequality we get

$$6 + a + 4b - 4(d + f) + h - 6B(1 - d - f)A^{1/3}H^{1/3} > 0. \tag{5.5.11}$$

Replacing $d + f$ by $0.5b + 0.425h$ we have $\phi(h) = 6 + a + 2b - 0.7h - 6B(1 - 0.5b - 0.425h)A^{1/3}H^{1/3} > 0$. As $\phi''(h) > 0$, $\phi(h) \leq \max\{\phi(0), \phi(a)\}$, which is negative for $0.3 < a < 0.4$ and $0.185 < b < 0.2475$. This gives a contradiction.

Claim (xi). $B < 1.2214$.

If $B \geq 1.2214$, working as in Claim (ix) and using $g < b + 0.15h$ in place of $g < b + \frac{h}{2}$, we find that $B^4FGHA > 2$ for $a > 0.3$, which is a contradiction to Claim (viii).

Claim (xii). $g < 0.6b + 0.34h$.

Suppose $g \geq 0.6b + 0.34h$. From (5.4.5), we get $d + f < 0.7b + 0.33h$. Using inequality $(1, 2, 2, 2, 1)$ and working as in Claim (x), we arrive at a contradiction to (5.5.11) for $0.3 < a < 0.4$ and $0.185 < b < 0.2214$.

Claim (xiii). $B < 1.2$.

If $B \geq 1.2$, working as in Claim (ix) and using $g < 0.6b + 0.34h$ in place of $g < b + \frac{h}{2}$, we find that $B^4FGHA > 2$ for $a > 0.3$, which is a contradiction to Claim (viii).

Claim (xiv). $H > B$.

Suppose $H \leq B$. Using inequality $(2, 5^*, 1)$ and proceeding as in Claim (vi), we have $4A - \frac{2A^2}{B} + \frac{5}{(AB^2)^{1/5}} + B > 8$, which is not possible for $1.185 \leq B \leq 1.2$ and $1.3 < A < 1.4$. This contradicts (5.5.10).

Final contradiction. We now have $b < h \leq a$, $g < 0.6b + 0.34h$, $b > 0.185$ and $a > 0.3$. Proceeding as in Claim (ix), we find that $B^4FGHA > 2$, which is a contradiction to Claim (viii).

5.6. Case (32)

PROPOSITION 22. *Case* (32), *i.e.* $A > 1$, $B > 1$, $C > 1$, $D \leq 1$, $E \leq 1$, $F \leq 1$, $G \leq 1$, $H \leq 1$, *does not arise.*

Proof. Here $a \leq 1$, $b \leq \frac{1}{2}$ and $c \leq \frac{1}{3}$ by Lemma 3. Also $2E > 1 \geq F$ and $2F > 1 \geq G$. Using inequalities $(2,2,1,2,1)$, $(2,2,2,1,1)$ and $(1,2,2,1,1,1)$ we get

$$2b - 2d - e - 2g - h > 0, \tag{5.6.1}$$

$$2b - 2d - 2f - g - h > 0, \tag{5.6.2}$$

$$a + 2c - 2e - f - g - h > 0. \tag{5.6.3}$$

Further from (5.6.1) we have $g < b \leq \frac{1}{2}$, which gives $2G > 1 > H$. Therefore inequality $(2,2,2,2)$ also holds, which gives

$$2b - 2d - 2f - 2h > 0. \tag{5.6.4}$$

CLAIM (i). $B > C$.

Suppose $C \geq B$. Now $C^2 > DE$, therefore $(1,1,3,1,1,1)$ holds, i.e. $A + B + 4C - C^4FGHAB + F + G + H > 8$. From (5.6.1) and (5.6.4), we have $g < b$ and $f + h < b$, hence $C^3FGHAB \geq B^5FGH \geq (1+b)^5(1-b)^2 > 1$ for $b < \frac{1}{2}$. We can successively replace C by B and A by B to get $6B + F + G + H - B^6FGH > 8$, which implies $1 + 6b - (f+h) - g - (1+b)^6(1-(f+h))(1-g) > 0$. As the coefficient of $f + h$ is positive, we can replace $f + h$ by b to get $1 + 5b - g - (1+b)^6(1-b)(1-g) > 0$, which is not true for $g < b$ and $b \leq c \leq \frac{1}{3}$.

CLAIM (ii). $b < 0.2866$.

Suppose $b \geq 0.2866$. From (5.6.1) and (5.6.2) we have $g < b - \frac{h}{2}$ and $f < b - \frac{g+h}{2}$. Also $h < b$. Therefore $B^4FGHA \geq (1+b)^5(1 - b + \frac{g+h}{2})(1-g) \times (1-h) = \phi(g) \geq \phi(b - \frac{h}{2}) = \psi(h) \geq \min\{\psi(0), \psi(b)\} > 2$ for $b \geq 0.2866$. Hence $(1,4,1,1,1)$ holds, i.e. $A + 4B - \frac{1}{2}B^5FGHA + F + G + H > 8$. As the coefficient of A is negative, we can replace A by B to get $5B - \frac{1}{2}B^6FGH + F + G + H > 8$. Now the coefficient of F is negative and $F > 1 - b + \frac{g+h}{2}$, therefore we get

$$\vartheta(g) = 4b - \frac{1}{2}(1+b)^6\left(1 - b + \frac{g+h}{2}\right)(1-g)(1-h) - \frac{g+h}{2} > 0. \tag{5.6.5}$$

As $\vartheta''(g) > 0$, we have $\vartheta(g) \leq \max\{\vartheta(0), \vartheta(b - \frac{h}{2})\} \leq 0$ for $0.2866 \leq b \leq \frac{1}{2}$ and $0 < h < b$. This can be seen to give a contradiction.

Claim (iii). $c < 0.19$.

Suppose $c \geq 0.19$. Now $C^4GHAB \geq C^4AB(1-b+\frac{h}{2})(1-h) \geq C^4B^2 \times (1-\frac{b}{2})(1-b) = \phi(b) \geq \min\{\phi(c), \phi(0.2866)\} > 2$, therefore $(1,1,4,1,1)$ holds. That is, $A+B+4C-\frac{1}{2}C^5GHAB+G+H > 8$. As the coefficient of G is negative, we can replace G by $1-b+\frac{h}{2}$ to get

$$\psi(h) = a + 4c - \frac{h}{2} - \frac{1}{2}(1+c)^5(1+a)(1+b)(1-h)\left(1-b+\frac{h}{2}\right) > 0.$$

As $\psi''(h) > 0$ and $0 < h < b$, it follows that $\psi(h) \leq \max\{\psi(0), \psi(b)\}$. Now $\psi(0) = a+4c-\frac{1}{2}(1+c)^5(1+a)(1-b^2)$ is a decreasing function of a and $a \geq c$, therefore we get $\psi(0) \leq 5c-\frac{1}{2}(1+c)^6(1-b^2)$, which is negative for $c \geq 0.19$ and $b < 0.2866$. Also $\psi(b) = a+4c-\frac{b}{2}-\frac{1}{2}(1+c)^5(1+a)(1+b)(1-b)(1-\frac{b}{2}) \leq \max\{\psi(c), \psi(0.2866)\}$, which is negative for $c \leq a \leq 1$ and $c \geq 0.19$, a contradiction.

Claim (iv). $f+g+h > 2c$.

Suppose $f+g+h \leq 2c$. Using inequality $(1,1,3,1,1,1)$, we have $A+B+4C-C^4FGHAB+F+G+H > 8$. We can replace B by C as the coefficient of B is negative and then A by C to get $6C-C^6FGH+F+G+H > 8$. This implies $1+6c-(f+g+h)-(1+c)^6(1-f-g-h) > 0$, which is not true for $f+g+h \leq 2c$ and $c < 0.19$.

Claim (v). $a < 0.453$.

Suppose $a \geq 0.453$. Now $AEFGH = \frac{1}{DBC} > \frac{1}{BC} \geq \frac{1}{1.2866\times 1.19} > 0.653$. Therefore $A^4EFGH > 2$ for $A \geq 1.453$. Hence $(4,1,1,1,1)$ holds, i.e. $4A-\frac{1}{2}A^5EFGH+E+F+G+H > 8$. Since $E+F+G+H < 3+EFGH$ we have $4A-\frac{1}{2}A^5EFGH+3+EFGH > 8$. Now the coefficient of $EFGH$ is negative and $EFGH > \frac{0.653}{A}$, therefore we can replace $EFGH$ by $\frac{0.653}{A}$ to get $\phi(A) = 4A-\frac{1}{2}A^4 \times 0.653 + \frac{0.653}{A} > 5$. As $\phi(A)$ is a decreasing function of A and $A \geq 1.453$, it follows that $\phi(A) \leq \phi(1.453) < 5$, a contradiction.

Claim (vi). $d+f > 0.65b$.

Suppose $d+f \leq 0.65b$. Using $(1,2,2,2,1)$ and applying the AM-GM inequality we have $6+a+4b-4(d+f)-h-6(1+b)(1-d-f)(1+a)^{1/3}(1-h)^{1/3} > 0$. As the left side is an increasing function of h and $h < b-d-f$, replacing h by $b-d-f$ we get

$$6+a+3b-3(d+f)-6(1+b)(1-d-f)(1+a)^{1/3}(1-b+d+f)^{1/3} > 0. \tag{5.6.6}$$

Again the left side is an increasing function of $d+f$ and $d+f < 0.65b$, therefore $\Phi(a) = 6+a+1.05b-6(1+b)(1-0.65b)(1+a)^{1/3}(1-0.35b)^{1/3} > 0$. As $\Phi(a)$ is a decreasing function of a for $a < 0.453$, we get $\Phi(a) \leq \Phi(b) =$

$6+2.05b-6(1-0.65b)(1+b)^{4/3}(1-0.35b)^{1/3}$ which is negative for $b < 0.2866$, a contradiction.

CLAIM (vii). $b < 0.2282$.

Suppose $b \geq 0.2282$. From Claim (vi) and inequalities (5.6.4) and (5.6.2) we have $h < 0.35b$ and $g + h < 0.7b$. Using these better bounds on g and h, i.e. $g < 0.7b - h$ and $h < 0.35b$, and working as in Claim (ii) we find that $B^4AFGH > 2$ for $B \geq 1.2282$. Then $(1, 4, 1, 1, 1)$ holds. But then working as in Claim (ii), we find that (5.6.5) is not true, which gives a contradiction.

CLAIM (viii). $d + f > 0.75b$.

Suppose $d + f \leq 0.75b$. Using $(1, 2, 2, 2, 1)$ and proceeding as in Claim (vi) we find that (5.6.6) is not true for $d + f < 0.75b$, $a \geq b$ and $b \leq 0.2282$.

Final contradiction. Using Claim (viii) and Claim (iv), and inequalities (5.6.2)–(5.6.4) we have $g + h < 0.5b$, $e < \frac{a}{2}$ and $d + f + h < b$. As inequality $(1, 2, 1, 1, 1, 1, 1)$ holds we have $A + 4B - 2B^3DEFGHA + D + E + F + G + H > 8$. As the coefficient of E is negative and $E > 1 - \frac{a}{2}$, we can replace E by $1 - \frac{a}{2}$ to get $2 + 0.5a + 4b - (d + f + h) - g - 2(1 + b)^3(1 + a)(1 - (d + f + h))(1 - g)(1 - \frac{a}{2}) > 0$. We can successively replace $d + f + h$ by b and g by $\frac{b}{2}$ to get $2 + 0.5a + 2.5b - 2(1 + b)^3(1 + a)(1 - b)(1 - \frac{b}{2})(1 - \frac{a}{2}) > 0$, which is not possible for $b \leq a < 0.453$ and $b < 0.2282$, giving thereby a contradiction.

5.7. Case (57)

PROPOSITION 23. *Case* (57), *i.e.* $A > 1$, $B > 1$, $C \leq 1$, $D \leq 1$, $E \leq 1$, $F > 1$, $G > 1$, $H > 1$, *does not arise.*

Proof. Here $a \leq \frac{1}{2}$, $b \leq \frac{1}{3}$ by Lemma 3. Using inequalities $(1, 2, 2, 1, 1, 1)$, $(2, 1, 2, 1, 1, 1)$ and $(2, 2, 1, 1, 1, 1)$ we get

$$a - 2c - 2e + f + g + h > 0, \tag{5.7.1}$$

$$2b - c - 2e + f + g + h > 0, \tag{5.7.2}$$

$$2b - 2d - e + f + g + h > 0. \tag{5.7.3}$$

CLAIM (i). $B^4FGHA \leq 2$ and $B < 1.149$.

Suppose $B^4FGHA > 2$. Then $(1, 4, 1, 1, 1)$ holds, i.e. $A + 4B - \frac{1}{2}B^5FGHA + F + G + H > 8$. This is not true, by Lemma 7(ii) with $X_2 = B$, $X_3 = F$, $X_4 = G$, $X_5 = H$. Now $B^4FGHA \leq 2$ implies $B^5 \leq 2$, i.e. $B < 1.149$.

CLAIM (ii). $e > 2b$ *and* $f + g + h > 2b$.

Assume $e \leq 2b$. Using inequality $(1, 3, 1, 1, 1, 1)$ and applying Lemma 6(v) with $X_1 = B$, $X_2 = A$, $X_3 = E$, $X_4 = F$, $X_5 = G$, $X_6 = H$, $\gamma = e$ and $\delta = a + f + g + h > 2e = 2\gamma$ (from (5.7.1)) we get a contradiction as $\gamma \leq 2b = 2x_1$, $x_1 < 0.149$. So we must have $e > 2b$. Now (5.7.2) gives $f + g + h > 2b$.

Claim (iii). $a > 0.1433$.

Assume $a \le 0.1433$. Let $k = f + g + h$. From (5.7.3) and Claim (ii), $d < \frac{k}{2}$. Also from (5.7.1), $c + e < \frac{a+k}{2}$. As $(2,1,\dots,1)$ holds, we have $4A - 2A^3CDEFGH + C + D + E + F + G + H > 8$. This gives $4A - 2A^3(1 - (c+e))DFGH + D + (1-(c+e)) + k > 4$. Now $DFGH > D(1+k) > (1-\frac{k}{2})(1+k) > 1$ as $k \le 3a < 1$. Therefore the coefficient of $c+e$ is positive. So we have $4A - 2A^3(1 - \frac{a+k}{2})DFGH + D + (1 - \frac{a+k}{2}) + k > 4$. Again the coefficient of D is negative and $D > 1 - \frac{k}{2}$, therefore we get

$$\phi(f) = 2 + \frac{7a}{2} - 2A^3\left(1 - \frac{a+f+g+h}{2}\right)\left(1 - \frac{f+g+h}{2}\right)(1+f)(1+g)(1+h) > 0.$$

As $\phi(f)$ is an increasing function of f and $f \le a$, we get $2 + \frac{7a}{2} - 2(1+a)^4 \times (1 - \frac{2a+g+h}{2})(1 - \frac{a+g+h}{2})(1+g)(1+h) > 0$. Using similar arguments we can replace g and h successively by a to get $2 + \frac{7}{2}a - 2(1+a)^6(1-2a)(1-\frac{3a}{2}) > 0$, which is not true for $a \le 0.1433$.

Claim (iv). $b < 0.1$.

Assume $b \ge 0.1$. Using Claim (iii) and $k = f + g + h > 2b$ we have $B^4FGHA > 1.1433B^4(1+k) > 1.1433B^4(1+2b) > 2$ for $b \ge 0.1$. This contradicts Claim (i).

Claim (v). $a < 0.4$.

Suppose $a \ge 0.4$. Then $A^5 > \frac{16}{3}$ and therefore $A^5FGH > \frac{16}{3}$. So $(5,1,1,1)$ holds, i.e. $4A - \frac{3}{16}A^6FGH + F + G + H > 8$. This gives $4A - \frac{3}{16}A^6FGH + FGH > 6$. As the coefficient of FGH, namely $1 - \frac{3}{16}A^6$, is negative and $FGH > 1$, replacing FGH by 1 we get $4A - \frac{3}{16}A^6 > 5$, which is clearly not true for $a \ge 0.4$.

Claim (vi). $a < 0.202$.

Assume $a \ge 0.202$. Using (5.7.1) and $E \ge \frac{2C}{3}$ we have $c + e < \frac{a+k}{2}$ and $1 - c < \frac{3}{2}(1-e)$. This gives $e < \frac{1+a+k}{5}$. Also from inequality (5.7.2) we have $e < b + \frac{k}{2}$.

Now $A^4EFGH > (1+a)^4(1 - b - \frac{k}{2})(1+k) = \psi(k)$, say. As $\psi''(k) < 0$ and $2b < k \le 3a$, we have $A^4EFGH \ge \min\{\psi(2b), \psi(3a)\} > 2$ for $b < 0.1$ and $0.202 \le a \le 0.4$. Thus $A^4EFGH > 2$ for $a > 0.202$ and so $(4,1,1,1,1)$ holds. That is, $4A - \frac{1}{2}A^5EFGH + E + F + G + H > 8$. This gives $4A - \frac{1}{2}A^5(1-e)(1+k) - e + k > 4$. As the coefficient of e is positive and $e < \frac{1+a+k}{5}$ we can replace e by $\frac{1+a+k}{5}$ to get

$$\varphi(k) = -\frac{1}{5} + \frac{19a}{5} + \frac{4k}{5} - \frac{1}{2}(1+a)^5\left(\frac{4}{5} - \frac{a+k}{5}\right)(1+k) > 0.$$

As $\varphi''(k) > 0$ and $0 \le k \le 3a$, we have $\varphi(k) \le \max\{\varphi(0), \varphi(3a)\}$, which is negative for $a < 0.4$, giving thereby a contradiction.

Claim (vii). $e > 2b + 0.21k$.

Assume $e \le 2b+0.21k$. Using $(1,3,1,1,1,1)$ we have $A+4B-B^4EFGH+E+F+G+H > 8$. As the coefficient of E is negative and $E > 1-2b-0.21k$, we have $A+4B-B^4(1-2b-0.21k)FGH+1-2b-0.21k+F+G+H > 8$. This gives $\theta(k) = 1+a+2b+0.79k-(1+b)^4(1-2b-0.21k)(1+a)(1+k) > 0$. As $\theta''(k) > 0$ and $2b < k \le 3a$, we get $\theta(k) \le \max\{\theta(2b), \theta(3a)\} < 0$ for $0.202 > a > 0.1433$ and $b < 0.1$. This gives a contradiction.

Claim (viii). $a > 0.19$.

Assume $a \le 0.19$. Using Claim (vii) and inequalities (5.7.2) and (5.7.3) we have $f+g+h > \frac{2b}{0.58}$ and $d < 0.395k$. Proceeding as in Claim (iii) and using $(2,1,\ldots,1)$ we get $2+3.815a-2(1+a)^6(1-2a)(1-1.185a) > 0$, which is not true for $a < 0.19$.

Claim (ix). $b < 0.075$.

Assume $b \ge 0.075$. Using $k = f+g+h > \frac{2b}{0.58}$ we have $B^4FGHA > 1.19B^4(1+k) > 1.19B^4(1+\frac{2b}{0.58}) > 2$ for $b \ge 0.075$. This contradicts Claim (ii).

Final contradiction. Proceeding as in Claim (vi) we have $A^4EFGH > (1+a)^4(1-b-\frac{k}{2})(1+k) = \psi(k)$, say. As $\psi''(k) < 0$ and $\frac{2b}{0.58} < k \le 3a$, it follows that $A^4EFGH \ge \min\{\psi(\frac{2b}{0.58}), \psi(3a)\} > 2$ for $b < 0.075$ and $a > 0.19$. Therefore $(4,1,1,1,1)$ holds and proceeding as in Claim (vi) we get a contradiction.

5.8. Case (61)

Proposition 24. *Case* (61), *i.e.* $A > 1$, $B > 1$, $C \le 1$, $D \le 1$, $E \le 1$, $F \le 1$, $G > 1$, $H > 1$, *does not arise.*

Proof. Here by Lemma 3, $a \le \frac{1}{2}$, $b \le \frac{1}{3}$, $c \le \frac{1}{4}$, $d \le \frac{1}{3}$ and $f \le \frac{5}{9}$. Also $2E \ge B > 1 \ge F$. Using inequalities $(1,2,2,1,1,1)$, $(1,2,1,2,1,1)$, $(2,2,2,1,1)$, $(2,2,1,1,1,1)$ and $(2,1,2,1,1,1)$ we get

$$a-2c-2e-f+g+h > 0, \tag{5.8.1}$$

$$a-2c-d-2f+g+h > 0, \tag{5.8.2}$$

$$2b-2d-2f+g+h > 0, \tag{5.8.3}$$

$$2b-2d-e-f+g+h > 0, \tag{5.8.4}$$

$$2b-c-2e-f+g+h > 0. \tag{5.8.5}$$

In the forthcoming discussion, all the expressions considered as functions of variables g and h can be shown to have their second derivatives with respect to g as well as with respect to h either always positive or always negative throughout the ranges of the variables. Hence their maximum value (or minimum value as the case may be) can occur at the end points of g and h only, i.e. at $(g,h)=(0,0),(a,0),(0,a)$ or (a,a). These functions are symmetric in g and h, so we just need to consider their values at $(0,0),(a,0)$ and at (a,a).

Claim (i). $a<0.46$ *or* $g+h<\frac{3a}{2}$.

Suppose that $a\ge 0.46$ and $g+h\ge\frac{3a}{2}$. Then $A^6GH\ge A^6(1+\frac{3a}{2})>16$. Therefore $(6,1,1)$ holds, i.e. $4A-\frac{1}{16}A^7GH+G+H>8$. This implies that $4A-\frac{1}{16}A^7(1+g+h)+g+h>6$, which is not possible for $\frac{3a}{2}<g+h\le 2a$ and $0.46\le a\le\frac{1}{2}$.

Claim (ii). $a<0.274$.

Assume $a\ge 0.274$. Firstly we show that $A^4EFGH>2$.

If $A>1.4$, then $A^4EFGH>\frac{A^3}{B}>\frac{3}{4}A^3>2$ as $CD<1$.

Now suppose that $A\le 1.4$. From (5.8.1) and (5.8.2), we have $e<\frac{a+g+h}{2}-\frac{f}{2}$ and $0<f<\frac{a+g+h}{2}$. Therefore $A^4EFGH>A^4(1-\frac{a+g+h}{2}+\frac{f}{2})(1-f)\times(1+g)(1+h)=\phi(f)\ge\min\{\phi(0),\phi(\frac{a+g+h}{2})\}$, as $\phi''(f)<0$. Now $\phi(0)=\psi_1(g,h)=A^4(1-\frac{a+g+h}{2})(1+g)(1+h)\ge\min\{\psi_1(0,0),\psi_1(a,0),\psi_1(a,a)\}$ which can be shown to be greater than 2 for $a\ge 0.274$. Similarly $\phi(\frac{a+g+h}{2})=\psi_2(g,h)=A^4(1-\frac{a+g+h}{4})(1-\frac{a+g+h}{2})(1+g)(1+h)\ge\min\{\psi_2(0,0),\psi_2(a,0),\psi_2(a,a)\}>2$ for $0.274\le a\le 0.4$.

Therefore $A^4EFGH>2$ in both the cases. Hence $(4,1,1,1,1)$ holds, i.e. $4A-\frac{1}{2}A^5EFGH+E+F+G+H>8$. As the coefficient of E on the left side is negative and $E>1-\frac{a+g+h}{2}+\frac{f}{2}$, replacing E by $1-\frac{a+g+h}{2}+\frac{f}{2}$ we get

$$\text{(5.8.6)}\quad \varphi(f)=\frac{7a}{2}-\frac{f}{2}+\frac{g+h}{2}-\frac{1}{2}A^5\left(1-\frac{a+g+h}{2}+\frac{f}{2}\right)(1-f)(1+g)(1+h)>0.$$

As $\varphi''(f)>0$ and $0<f\le\frac{5}{9}$, we have $\varphi(f)\le\max\{\varphi(0),\varphi(\frac{5}{9})\}$. Let now

$$\begin{aligned}\varphi\left(\frac{5}{9}\right)&=\frac{7a}{2}-\frac{5}{18}+\frac{g+h}{2}-\frac{2}{9}A^5\left(1-\frac{a+g+h}{2}+\frac{5}{18}\right)(1+g)(1+h)\\&=\psi_3(g,h),\\ \varphi(0)&=\frac{7a}{2}+\frac{g+h}{2}-\frac{1}{2}A^5\left(1-\frac{a+g+h}{2}\right)(1+g)(1+h)=\psi_4(g,h).\end{aligned}$$

At each of the end points $(g,h)=(0,0),(a,0)$ or (a,a) one can verify that $\psi_3(g,h)<0$ for $0.274\le a\le 0.5$. Also $\psi_4(g,h)<0$ at the end points $(g,h)=(0,0)$ and $(a,0)$ for $0.274\le a\le 0.5$. But $\psi_4(a,a)$ is non-negative in the full range of a. It is certainly so for $0.274 < a \le 0.46$. If $a > 0.46$ we already have from Claim (i), $g+h \le \frac{3a}{2}$. In this case

$$\psi_4(g,h) < \frac{7a}{2} + \frac{k}{2} - \frac{1}{2}A^5\left(1 - \frac{a+k}{2}\right)(1+k) = \chi(k),$$

say where $k = g+h$. It is easy to see that $\chi(k) < 0$ and hence $\psi_4(g,h) < 0$ for $0 < k \le \frac{3a}{2}$ and $0.46 \le a \le \frac{1}{2}$. Therefore $\varphi(f) < 0$ for $a \ge 0.274$, giving a contradiction to (5.8.6).

Claim (iii). $b < 0.17$.

Suppose that $b \ge 0.17$. From (5.8.2) we have $f < \frac{a+g+h}{2}$ and so $B^4AFGH > B^4(1+a)(1-\frac{a+g+h}{2})(1+g)(1+h) = \psi_5(g,h)$, say. One verifies that $\psi_5(g,h) \ge \min\{\psi_5(0,0), \psi_5(a,0), \psi_5(a,a)\} > 2$ for $a \ge b \ge 0.17$. Therefore inequality $(1,4,1,1,1)$ holds, i.e. $A + 4B - \frac{1}{2}B^5AFGH + F + G + H > 8$. As the coefficient of F on the left side is negative and $F > 1 - \frac{a+g+h}{2}$, replacing F by $1 - \frac{a+g+h}{2}$ we get

$$\psi_6(g,h) = \frac{a}{2} + 4b + \frac{g+h}{2} - \frac{1}{2}B^5A\left(1 - \frac{a+g+h}{2}\right)(1+g)(1+h) > 0.$$

But $\psi_6(g,h) \le \max\{\psi_6(0,0), \psi_6(a,0), \psi_6(a,a)\}$, which is non-positive for $0.17 \le b \le a \le 0.274$. This gives a contradiction.

Claim (iv). *$e+f > 2b$ and $d < \frac{g+h}{2}$.*

Suppose $e+f \le 2b$. From (5.8.1) and (5.8.2) we have $2e+f < a+g+h$ and $f < \frac{a+g+h}{2}$. This gives $e+f < \frac{3}{4}(a+g+h)$. Using $(1,3,1,1,1,1)$ and applying Lemma 6(vi) with $\gamma = e+f$, $\delta = a+g+h$, $\gamma < \frac{3\delta}{4}$, $\gamma < 2b$ and $b < 0.17$ we get a contradiction. Hence $e+f > 2b$.

Now (5.8.4) gives $d < \frac{g+h}{2}$.

Claim (v). *$a > 0.2$ and $g+h > a$.*

Suppose first that $a \le 0.2$. Using $(1,2,1,1,1,1,1)$ we have $A + 4B - 2B^3ADEFGH + D + E + F + G + H > 8$. The coefficient of E on the left side is negative because $ABDFGH > 1$. Also from inequality (5.8.5) we have $E > 1 - b - \frac{g+h}{2} + \frac{f}{2}$, therefore replacing E by $1 - b - \frac{g+h}{2} + \frac{f}{2}$ and simplifying we get

$$\begin{aligned}(5.8.7)\qquad \phi(f) = 2 + a + 3b - d - \frac{f}{2} + \frac{g+h}{2} - 2B^3A(1-d)\\ \times\left(1 - b - \frac{g+h}{2} + \frac{f}{2}\right)(1-f)(1+g)(1+h) > 0.\end{aligned}$$

We shall prove that (5.8.7) is not true for variables f, d, g, h, a and b lying in the given intervals. As $\phi''(f) > 0$ and from (5.8.3), $0 < f < b - d + \frac{g+h}{2}$, therefore $\phi(f) \le \max\{\phi(0), \phi(b - d + \frac{g+h}{2})\}$. The coefficient of d in $\phi(0)$ is positive since $B^2(1-b-\frac{g+h}{2})(1+g)(1+h) > (1+2b+g+h)(1-\frac{2b+g+h}{2}) > 1$ for $2b+g+h < 4a < 1$. Also $d < \frac{g+h}{2}$. So

$$\phi(0) < 2+a+3b-2B^3A\left(1-\frac{g+h}{2}\right)\left(1-b-\frac{g+h}{2}\right)(1+g)(1+h) = \psi_7(g,h).$$

Further $\phi(b-d+\frac{g+h}{2}) = 2+a+\frac{5b}{2}-\frac{d}{2}+\frac{g+h}{4}-2B^3(1-d)A(1-\frac{b}{2}-\frac{g+h}{4}-\frac{d}{2}) \times (1-b-\frac{g+h}{2}+d)(1+g)(1+h) = \theta(d)$, say. For $d \le \frac{1}{3}$ one finds that $\theta''(d) > 0$. Therefore $\theta(d) \le \max\{\theta(0), \theta(\frac{g+h}{2})\}$. Let $\theta(0) = \psi_8(g,h)$ and $\theta(\frac{g+h}{2}) = \psi_9(g,h)$, where

$$\psi_8(g,h) = 2+a+\frac{5b}{2}+\frac{g+h}{4}-2B^3A\left(1-\frac{b}{2}-\frac{g+h}{4}\right)\left(1-b-\frac{g+h}{2}\right)(1+g)(1+h),$$

and

$$\psi_9(g,h) = 2+a+\frac{5b}{2}-2B^3A\left(1-\frac{b}{2}-\frac{g+h}{2}\right)\left(1-\frac{g+h}{2}\right)(1-b)(1+g)(1+h).$$

At each of the end points $(g,h) = (0,0), (a,0)$ or (a,a) one can verify that each of $\psi_7(g,h), \psi_8(g,h)$ and $\psi_9(g,h)$ is non-positive for $0 < b \le 0.17$ and $0 < a \le 0.2$. This implies that $\phi(f) < 0$, which is a contradiction to (5.8.7) for $a \le 0.2$.

Suppose now $g+h = k \le a$. In the above discussion we notice that $\psi_7(g,h), \psi_8(g,h)$ and $\psi_9(g,h)$ are negative at the end points $(g,h) = (0,0)$, $(a,0)$ for the full range of a, namely $a \le 0.274$. Using $GH > 1+k$, inequality (5.8.7) reduces to

$$\phi(f) = 2+a+3b-d-\frac{f}{2}+\frac{k}{2}-2B^3A(1-d) \times \left(1-b-\frac{k}{2}+\frac{f}{2}\right)(1-f)(1+k) > 0. \tag{5.8.8}$$

If $k \le a$, proceeding as above we find that (5.8.8) is not true for $0 < b \le 0.17$ and $0 < a \le 0.274$. Therefore we must have $g+h > a$.

Claim (vi). $b < 0.148$.

Assume $b \ge 0.148$. We proceed as in Claim (iii). We get $B^4AFGH > 2$ for $g+h > a$, $a > 0.2$ and $b \ge 0.148$. Then we use $(1,4,1,1,1)$ to get a contradiction.

Claim (vii). $e+f \ge 1.5b+0.4(g+h)$ *and* $d < \frac{b}{4}+\frac{3(g+h)}{10}$.

Suppose $e+f < 1.5b+0.4(g+h)$. Using $(1,3,1,1,1,1)$ we have $A+4B-B^4EFGHA+E+F+G+H > 8$, which implies that $A+4B-B^4(1-(e+f))GHA-(e+f)+G+H > 6$. As the coefficient of $e+f$ on the left side is positive, replacing $e + f$ by $1.5b + 0.4(g + h)$ we get

(5.8.9) $\varphi(g) = 1+a+2.5b+0.6(g+h)-B^4(1-1.5b-0.4(g+h))GHA > 0.$

As $\varphi''(g) > 0$ and from Claim (v) we have $a - h < g \le a$ therefore $\varphi(g) \le \max\{\varphi(a-h), \varphi(a)\}$. Now

$$\varphi(a-h) = 1+a+2.5b+0.6a-B^4(1-1.5b-0.4a)(1+a-h)(1+h)A = \omega(h),$$

say. As $\omega''(h) > 0$ and $0 < h \le a$ therefore $\omega(h) \le \max\{\omega(0), \omega(a)\} < 0$ for $0.2 \le a \le 0.274$ and $0 < b \le 0.148$. Similarly

$$\varphi(a) = 1 + a + 2.5b + 0.6(a + h) - B^4(1 - 1.5b - 0.4(a + h))HA^2 = \nu(h),$$

say. As $\nu''(h) > 0$ and $0 < h \le a$, we have $\nu(h) \le \max\{\nu(0), \nu(a)\}$. It is easy to see that $\nu(0) < 0$ as well as $\nu(a) < 0$ and hence $\nu(h) < 0$ for $0.2 \le a \le 0.274$ and $0 < b \le 0.148$. This gives $\varphi(g) < 0$, which is a contradiction to (5.8.9).

Therefore we must have $e + f \ge 1.5b + 0.4(g + h)$.

Now (5.8.4) gives $d < \frac{b}{4} + \frac{3(g+h)}{10}$.

Final contradiction. We use inequality $(1, 2, 1, 1, 1, 1, 1)$ and proceed as in Claim (iv). Here we have $d < \frac{b}{4} + \frac{3(g+h)}{10}$ in place of $d < \frac{g+h}{2}$. Using this upper bound on d we get $\psi_{10}(g, h)$ in place of $\psi_7(g, h)$ and $\psi_{11}(g, h)$ in place of $\psi_9(g, h)$ (for the end point $d = 0$, $\psi_8(g, h)$ remains unchanged), where

$$\psi_{10}(g,h) = 2 + a + \frac{11b}{4} + \frac{g+h}{5} - 2B^3A\left(1 - \frac{b}{4} - \frac{3(g+h)}{10}\right)\left(1 - b - \frac{g+h}{2}\right)GH,$$

$$\psi_{11}(g,h) = 2 + a + \frac{19b}{8} + \frac{g+h}{10} - 2B^3A\left(1 - \frac{5b}{8} - \frac{4(g+h)}{10}\right)\left(1 - \frac{3b}{4} - \frac{g+h}{5}\right)GH.$$

The second derivative of the function $\psi_{10}(g, h)$ with respect to g turns out to be positive. As $a - h < g \le a$, we have $\psi_{10}(g, h) \le \max\{\psi_{10}(a - h, h), \psi_{10}(a, h)\}$. Considering $\psi_{10}(a - h, h)$ and $\psi_{10}(a, h)$ as functions of h, their second derivatives are positive and $0 < h \le a$; so $\psi_{10}(g, h) \le \max\{\psi_{10}(a, 0), \psi_{10}(a, a)\}$. Similarly $\psi_{11}(g, h) \le \max\{\psi_{11}(a, 0), \psi_{11}(a, a)\}$. One can easily verify that $\psi_8(g, h), \psi_{10}(g, h)$ and $\psi_{11}(g, h)$ are negative at the end points $(g, h) = (a, 0), (a, a)$ for $0.2 \le a \le 0.274$ and $0 < b \le 0.148$. This contradicts (5.8.7).

5.9. Case (63)

PROPOSITION 25. *Case* (63), *i.e.* $A > 1$, $B > 1$, $C \le 1$, $D \le 1$, $E \le 1$, $F \le 1$, $G \le 1$, $H > 1$, *does not arise.*

Proof. Here $a \le \frac{1}{2}$, $b \le \frac{1}{3}$, $2E \ge B > 1 \ge F$ by Lemma 3. Also $2F > 1 > G$, for if $F \le \frac{1}{2}$ then $E < \frac{4F}{3} < \frac{2}{3}$ so that $EF < \frac{1}{3}$, which implies $ABCDGH > 3$. But $ABCDGH \le ABH \le A^2B \le \frac{9}{4}\cdot\frac{4}{3} = 3$, a contradiction. Using inequalities $(1,2,2,1,1,1), (1,2,1,2,1,1)$, $(1,2,2,2,1)$ and $(2,2,1,1,1,1)$ we get

$$a - 2c - 2e - f - g + h > 0, \tag{5.9.1}$$

$$a - 2c - d - 2f - g + h > 0, \tag{5.9.2}$$

$$a - 2c - 2e - 2g + h > 0, \tag{5.9.3}$$

$$2b - 2d - e - f - g + h > 0. \tag{5.9.4}$$

Claim (i). $a < 0.287$.

Assume $a \ge 0.287$. From (5.9.1)–(5.9.3) we get $e < \frac{a+h}{2} - \frac{f+g}{2}$, $f < \frac{a+h}{2} - \frac{g}{2}$ and $g < \frac{a+h}{2}$. We will first prove that $A^4EFGH > 2$ for $a \ge 0.287$. Now $A^4EFGH > A^4(1 - \frac{a+h}{2} + \frac{f+g}{2})(1-f)(1-g)(1+h) = \theta(f)$, say. One finds that $\theta(f)$ is a decreasing function of f, so $\theta(f) \ge \theta(\frac{a+h}{2} - \frac{g}{2}) = A^4(1 - \frac{a+h}{4} + \frac{g}{4})(1 - \frac{a+h}{2} + \frac{g}{2})(1-g)(1+h) = \phi(g)$, say. As $\phi''(g) \le 0$, we get $\phi(g) \ge \min\{\phi(0), \phi(\frac{a+h}{2})\}$. Now

$$\begin{aligned}
\phi(0) &= A^4\left(1 - \frac{a+h}{4}\right)\left(1 - \frac{a+h}{2}\right)(1+h) \\
&\ge \min\left\{(1+a)^4\left(1-\frac{a}{4}\right)\left(1-\frac{a}{2}\right), (1+a)^5\left(1-\frac{a}{2}\right)(1-a)\right\} \\
&= (1+a)^5\left(1-\frac{a}{2}\right)(1-a) > 2
\end{aligned}$$

for $a \ge 0.287$. Similarly

$$\begin{aligned}
\phi\left(\frac{a+h}{2}\right) &= A^4\left(1 - \frac{a+h}{8}\right)\left(1 - \frac{a+h}{4}\right)\left(1 - \frac{a+h}{2}\right)(1+h) \\
&\ge \min\left\{(1+a)^4\left(1-\frac{a}{8}\right)\left(1-\frac{a}{4}\right)\left(1-\frac{a}{2}\right),\right. \\
&\qquad\qquad \left.(1+a)^5\left(1-\frac{a}{4}\right)\left(1-\frac{a}{2}\right)(1-a)\right\} \\
&= (1+a)^5\left(1-\frac{a}{4}\right)\left(1-\frac{a}{2}\right)(1-a) > 2
\end{aligned}$$

for $a \ge 0.287$. Hence $A^4EFGH > 2$ for $a \ge 0.287$. Therefore $(4,1,1,1,1)$ holds, i.e. $4A - \frac{1}{2}A^5EFGH + E + F + G + H > 8$. As the coefficient of E on the left hand side is negative, replacing E by $1 - \frac{a+h}{2} + \frac{f+g}{2}$ and simplifying we get

$$(5.9.5)\quad \varphi(f) = \frac{7a}{2} - \frac{f+g}{2} + \frac{h}{2} - \frac{1}{2}A^5\left(1 - \frac{a+h}{2} + \frac{f+g}{2}\right)(1-f)(1-g)(1+h) > 0.$$

As $\varphi''(f) > 0$, we have $\varphi(f) \le \max\{\varphi(0), \varphi(\frac{a+h}{2} - \frac{g}{2})\}$. Now

$$\varphi\left(\frac{a+h}{2} - \frac{g}{2}\right) = \frac{13a}{4} - \frac{g}{4} + \frac{h}{4} - \frac{1}{2}A^5\left(1 - \frac{a+h}{2} + \frac{g}{2}\right)\left(1 - \frac{a+h}{4} + \frac{g}{4}\right)(1-g)(1+h) = \psi(g),$$

say. As $\psi''(g) > 0$ so $\psi(g) \le \max\{\psi(0), \psi(\frac{a+h}{2})\}$. It is easy to see that $\psi(0)$ and $\psi(\frac{a+h}{2})$ are negative for $0 < h \le a$ and $0.287 \le a \le \frac{1}{2}$, which shows that $\varphi(\frac{a+h}{2} - \frac{g}{2}) < 0$. Using a similar argument we can show that $\varphi(0) < 0$ for $0 < h \le a$ and $0.287 \le a \le \frac{1}{2}$. It now follows that $\varphi(f) < 0$, which contradicts (5.9.5).

Claim (ii). $c + e > 0.32(a + h)$.

Assume that $x = c + e \le 0.32(a + h)$. After applying the AM-GM inequality to $(2, 2, 2, 1, 1)$ we have $4A + 4C + 4E - 6ACEG^{1/3}H^{1/3} + G + H > 8$, which implies $6+4a-4x-g+h-6(1+a)(1-x)(1-g)^{1/3}(1+h)^{1/3} > 0$. Since $(1+a)(1-x) > 1$ for $x \le 0.64a$ and $a \le \frac{1}{2}$, the left side is an increasing function of g, therefore replacing g by $\frac{a+h}{2} - c - e$ we get

$$(5.9.6)\quad \varphi(x) = 6 + \frac{7a}{2} - 3x + \frac{h}{2} - 6(1+a)(1-x)\left(1 - \frac{a+h}{2} + x\right)^{1/3}(1+h)^{1/3} > 0.$$

Since $\varphi''(x) > 0$ and $0 \le x \le 0.32(a+h)$, it follows that $\varphi(x) \le \max\{\varphi(0), \varphi(0.32(a+h))\}$, which can be easily seen to be negative for $0 < h \le a$ and $0 < a < 0.287$, a contradiction to (5.9.6).

Claim (iii). $a < 0.25$.

Assume that $a \ge 0.25$. Using Claim (ii) and inequalities (5.9.1) and (5.9.3) we get $f < 0.36(a + h) - g$ and $g < 0.18(a + h)$. Now proceeding as in Claim (i) and using $f < 0.36(a + h) - g$ in place of $f < \frac{a+h}{2} - \frac{g}{2}$ and $g < 0.18(a+h)$ in place of $g < \frac{a+h}{2}$ we find that $A^4EFGH \ge 2$ for $a \ge 0.25$. Therefore $(4, 1, 1, 1, 1)$ holds. Proceeding as in Claim (i) we find that (5.9.5) is false for $f < 0.36(a + h) - g$, $g < 0.18(a + h)$, $0 < h \le a$ and $a \ge 0.25$.

Claim (iv). $c + e > 0.35(a + h)$.

Assume that $c + e \le 0.35(a + h)$. Using $(2, 2, 2, 1, 1)$ and proceeding as in Claim (ii) we find that (5.9.6) is not true for $x \le 0.35(a + h)$, $0 < h \le a$ and $0 < a < 0.25$.

Claim (v). $b < 0.17$.

Suppose that $b \geq 0.17$. Using Claim (iv) and inequalities (5.9.1) and (5.9.3) we get $f + g < 0.3(a + h)$ and $g < 0.15(a + h)$. Let $y = f + g$. Then $AFGH > (1+a+h)(1-y) > (1+a+h)(1-0.3(a+h)) > (1+a)(1-0.3a) > (1+b)(1-0.3b)$. This gives $B^4AFGH > (1+b)^5(1-0.3b) > 2$ for $b \geq 0.17$. Therefore $(1, 4, 1, 1, 1)$ holds. That is, $A+4B-\frac{1}{2}B^5AFGH+F+G+H > 8$, which implies that

$$\phi(y) = a + 4b - y + h - \tfrac{1}{2}(1 + b)^5(1 + a)(1 - y)(1 + h) > 0. \tag{5.9.7}$$

As the coefficient of y on the left side is positive and $y < 0.3(a+h)$, we have $\phi(y) < \phi(0.3(a+h)) = 0.7a+4b+0.7h-\frac{1}{2}(1+b)^5(1+a)(1-0.3(a+h))(1+h) = \psi(h)$, say. As $\psi''(h) > 0$, we have $\psi(h) \leq \max\{\psi(0), \psi(a)\}$, which is easily seen to be negative for $0.17 \leq b \leq a < 0.25$. This gives a contradiction to (5.9.7).

Claim (vi). $e + f + g > 2b$.

Assume $e + f + g \leq 2b$. From (5.9.1) we have $2e + f + g < a + h$. Also $f + g < 0.3(a + h)$. Adding these two we get $e + f + g < 0.65(a + h)$. Using inequality $(1, 3, 1, 1, 1, 1)$ we have $A+4B-B^4EFGHA+E+F+G+H > 8$. Now we get a contradiction from Lemma 6(vi) with $\gamma = e + f + g < \frac{3}{4}\delta$ where $\delta = a + h$ and $\gamma < 2b$, $b = x_1 < 0.17$.

Final contradiction. Using inequality (5.9.4) and Claim (vi) we get $d < \frac{h}{2}$. Using $(3, 1, 1, 1, 1, 1)$ we have $4A - A^4DEFGH + D + E + F + G + H > 8$. This gives

$$1 + 4a - d - (e + f + g) + h - (1 + a)^4(1 - d)(1 - (e + f + g))(1 + h) > 0.$$

As the coefficient of $e+f+g$ is positive and $e+f+g < 0.65(a+h)$ (proved in the last Claim), so we can replace $e+f+g$ by $0.65(a+h)$ and then d by $\frac{h}{2}$ to get $1+4a-\frac{h}{2}-0.65(a+h)+h-(1+a)^4(1-\frac{h}{2})(1-0.65(a+h))(1+h) > 0$, which is not possible for $0 < h \leq a < 0.25$. This gives a contradiction.

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