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On Hall's conjecture

by

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Hall's conjecture asserts that for any $\varepsilon > 0$, there exists a constant $c(\varepsilon) > 0$ such that if x and y are positive integers satisfying $x^3 - y^2 \neq 0$, then $|x^3 - y^2| > c(\varepsilon)x^{1/2-\varepsilon}$. It is known that Hall's conjecture follows from the *abc*-conjecture. For a stronger version of Hall's conjecture which is equivalent to the *abc*-conjecture see [3, Ch. 12.5]. Originally, Hall [8] conjectured that there is $C > 0$ such that $|x^3 - y^2| \geq C\sqrt{x}$ for positive integers x, y with $x^3 - y^2 \neq 0$, but this formulation is unlikely to be true. Danilov [4] proved that $0 < |x^3 - y^2| < 0.97\sqrt{x}$ has infinitely many solutions in positive integers x, y; here 0.97 comes from $54\sqrt{5}/125$. For examples with "very small" quotients $|x^3 - y^2|/\sqrt{x}$, up to 0.021, see [7] and [9].

It is well known that for nonconstant complex polynomials x and y, such that $x^3 \neq y^2$, we have $\deg(x^3 - y^2)/\deg(x) > 1/2$. More precisely, Davenport [6] proved that for such polynomials the inequality

$$\deg(x^3 - y^2) \geq \tfrac{1}{2}\deg(x) + 1 \tag{1}$$

holds. This statement also follows from Stothers–Mason's *abc* theorem for polynomials (see, e.g., [10, Ch. 4.7]). Zannier [12] proved that for any positive integer δ there exist complex polynomials x and y such that $\deg(x) = 2\delta$, $\deg(y) = 3\delta$ and x, y satisfy the equality in Davenport's bound (1). In his previous paper [11], he related the existence of such examples to coverings of the Riemann sphere, unramified except above 0, 1 and ∞.

It is natural to ask whether examples with equality in (1) exist for polynomials with integer (rational) coefficients. Such examples are known only for $\delta = 1, 2, 3, 4, 5$ (see [1, 7]). The first example for $\delta = 5$ was found by Birch, Chowla, Hall and Schinzel [2]. It is given by

$$x = \frac{t}{9}(t^9 + 6t^6 + 15t^3 + 12), \qquad y = \frac{1}{54}(2t^{15} + 18t^{12} + 72t^9 + 144t^6 + 135t^3 + 27),$$

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while then

$$x^3 - y^2 = -\frac{1}{108}(3t^6 + 14t^3 + 27)$$

(note that x, y are integers for $t \equiv 3 \pmod 6$). One more example for $\delta = 5$ has been found by Elkies [7]:

$$\begin{aligned} x &= t^{10} - 2t^9 + 33t^8 - 12t^7 + 378t^6 + 336t^5 + 2862t^4 + 2652t^3 + 14397t^2 \\ &\quad + 9922t + 18553, \\ y &= t^{15} - 3t^{14} + 51t^{13} - 67t^{12} + 969t^{11} + 33t^{10} + 10963t^9 + 9729t^8 \\ &\quad + 96507t^7 + 108631t^6 + 580785t^5 + 700503t^4 + 2102099t^3 + 1877667t^2 \\ &\quad + 3904161t + 1164691, \\ x^3 - y^2 &= 4591650240t^6 - 5509980288t^5 + 101934635328t^4 \\ &\quad + 58773123072t^3 + 730072388160t^2 + 1151585880192t \\ &\quad + 5029693672896. \end{aligned}$$

In these examples we have

$$\deg(x^3 - y^2)/\deg(x) = 0.6,$$

and it seems that no examples of polynomials with integer coefficients, satisfying $x^3 - y^2 \neq 0$ and $\deg(x^3 - y^2)/\deg(x) < 0.6$, have been published before.

In this note we will show the following result.

Theorem 1. *For any $\varepsilon > 0$ there exist polynomials x and y with integer coefficients such that $x^3 \neq y^2$ and $\deg(x^3 - y^2)/\deg(x) < 1/2 + \varepsilon$. More precisely, for any even positive integer δ there exist polynomials x and y with integer coefficients such that $\deg(x) = 2\delta$, $\deg(y) = 3\delta$ and $\deg(x^3 - y^2) = \delta + 5$.*

As an immediate corollary we obtain a nontrivial lower bound for the number of integer solutions to the inequality $|x^3 - y^2| < x^{1/2+\varepsilon}$ with $1 \leq x \leq N$ (heuristically, it is expected that this number is around N^ε).

Corollary 1. *For any $\varepsilon > 0$ and positive integer N, denote by $\mathcal{S}(\varepsilon, N)$ the number of integers x, $1 \leq x \leq N$, for which there exists an integer y such that $0 < |x^3 - y^2| < x^{1/2+\varepsilon}$. Then*

$$\mathcal{S}(\varepsilon, N) \gg N^{\varepsilon/(5+4\varepsilon)}.$$

Indeed, take δ to be the smallest even integer greater than $5/(2\varepsilon)$, so that $5/(2\varepsilon) < \delta < 5/(2\varepsilon) + 2$, and take $x = x(t)$, $y = y(t)$ as in Theorem 1. Then for sufficiently large t we have $x = O(t^{2\delta})$ and $|x^3 - y^2| = O(t^{\delta+5}) = O(x^{1/2+5/(2\delta)}) < x^{1/2+\varepsilon}$. Therefore,

$$\mathcal{S}(\varepsilon, N) \gg N^{1/(2\delta)} \gg N^{\varepsilon/(5+4\varepsilon)}.$$

Here is an explicit example which improves the quotient $\deg(x^3 - y^2)/\deg(x) = 0.6$ from the above mentioned examples by Birch, Chowla, Hall, Schinzel and Elkies, as $\deg(x^3 - y^2)/\deg(x) = 31/52 = 0.5961\ldots$:

$$\begin{aligned}
x ={}& 281474976710656t^{52} + 3799912185593856t^{50} + 24189255811072000t^{48} + 96537120918732800t^{46} \\
&+ 270892177293312000t^{44} + 568175382432317440t^{42} + 924393098014883840t^{40} \\
&+ 1194971570896896000t^{38} + 1247222961904025600t^{36} + 1062249296822272000t^{34} \\
&+ 743181990714408960t^{32} + 428630517911388160t^{30} + 203971125837824000t^{28} + 100663296t^{27} \\
&+ 79960271015116800t^{26} + 729808896t^{25} + 25720746147840000t^{24} + 2359296000t^{23} \\
&+ 6745085391667200t^{22} + 4482662400t^{21} + 1428736897843200t^{20} + 5554176000t^{19} \\
&+ 241375027200000t^{18} + 4706795520t^{17} + 31982191104000t^{16} + 2782494720t^{15} + 3250264320000t^{14} \\
&+ 1148928000t^{13} + 245895686400t^{12} + 326476800t^{11} + 13292822400t^{10} + 61776000t^{9} \\
&+ 484380000t^{8} + 7344480t^{7} + 10894000t^{6} + 496080t^{5} + 130625t^{4} + 15750t^{3} + 629t^{2} + 150t + 4,
\end{aligned}$$

$$\begin{aligned}
y ={}& 4722366482869645213696t^{78} + 95627921278110315577344t^{76} + 931486788746037518401536t^{74} \\
&+ 5812273909720700361375744t^{72} + 26102714713365300532740096t^{70} \\
&+ 89873242715073754863501312t^{68} + 246761827996223603178733568t^{66} \\
&+ 554869751478978106456276992t^{64} + 1041377162422256031202541568t^{62} \\
&+ 1654256777803799676753805312t^{60} + 2247766244734980591395536896t^{58} \\
&+ 2633529391786763986554322944t^{56} + 2676840149412734907329806336t^{54} \\
&+ 2533274790395904t^{53} + 2371433108159248512627769344t^{52} + 35465847065542656t^{51} \\
&+ 1837294956807449113993936896t^{50} + 234486247786020864t^{49} \\
&+ 1247823926411289395000770560t^{48} + 973569167884025856t^{47} + 743994544482135039635619840t^{46} \\
&+ 2847272221544546304t^{45} + 389682593956278112836648960t^{44} + 6236328797675716608t^{43} \\
&+ 179279686440609529032867840t^{42} + 10618254681610125312t^{41} + 72388134028773255869890560t^{40} \\
&+ 14399046085119049728t^{39} + 25611943886548098204303360t^{38} + 15806610071787405312t^{37} \\
&+ 7922395450159324505047040t^{36} + 14200560742834372608t^{35} + 2135839807968003238133760t^{34} \\
&+ 10514148446410113024t^{33} + 499883693495498613719040t^{32} + 6441026076788391936t^{31} \\
&+ 101073262762096181903360t^{30} + 3269189665642512384t^{29} + 17550157782838363029504t^{28} \\
&+ 1373442845007937536t^{27} + 2598168579136061177856t^{26} + 476068223096193024t^{25} \\
&+ 325093317533140516864t^{24} + 135395930768670720t^{23} + 34019036843474681856t^{22} \\
&+ 31339645700014080t^{21} + 2939255644452962304t^{20} + 5838612910571520t^{19} \\
&+ 206402445920944128t^{18} + 862650209710080t^{17} + 11551766627438592t^{16} + 99129281310720t^{15} \\
&+ 502656091170048t^{14} + 8633278321920t^{13} + 16468534726592t^{12} + 550276346880t^{11} \\
&+ 389483950128t^{10} + 24450210720t^{9} + 6312333144t^{8} + 705350880t^{7} + 68685241t^{6} + 11812545t^{5} \\
&+ 642429t^{4} + 94050t^{3} + 6591t^{2} + 225t + 19,
\end{aligned}$$

$$\begin{aligned}
x^3 - y^2 ={}& -905969664t^{31} - 8380219392t^{29} - 35276193792t^{27} - 89379569664t^{25} \\
&- 151909171200t^{23} - 182680289280t^{21} - 159752355840t^{19} - 102786416640t^{17} - 48661447680t^{15} \\
&- 16772918400t^{13} - 4116359520t^{11} - 692649360t^{9} - 75171510t^{7} - 297t^{6} - 4749570t^{5} - 891t^{4} \\
&- 144450t^{3} - 891t^{2} - 1350t - 297.
\end{aligned}$$

Now we describe the general construction. Let us define the binary recursive sequence by

$$a_1 = 0, \quad a_2 = t^2 + 1, \quad a_m = 2ta_{m-1} + a_{m-2}.$$

Thus, for $m \geq 2$, a_m is a polynomial in variable t, of degree m. Put $u = a_{k-1}$ and $v = a_k$ for an odd positive integer $k \geq 3$. We search for examples with

$x = O(v^2)$, $y = O(v^3)$ and $x^3 - y^2 = O(v)$. Note that

$$v^2 - 2tuv - u^2 = -(a_2^2 - 2ta_1a_2 - a_1^2) = -(t^2+1)^2. \tag{2}$$

Therefore, we may take

$$x = av^2 + buv + cu + dv + e,$$
$$y = fv^3 + gv^2u + hv^2 + iuv + ju + mv + n,$$

with unknown coefficients $a, b, c, \dots, n$, which will be determined so that in the expression for $x^3 - y^2$ the coefficients with $v^6, uv^5, v^5, \dots, v^2, uv$ are equal to 0. We find the following (polynomial) solution:

$$x = v^2 - 2tuv + 6v - 6tu + (t^4 + 5t^2 + 4),$$
$$y = -2tv^3 + (4t^2+1)uv^2 - 9tv^2 + (18t^2+9)uv + (-2t^5 - 4t^3 - 2t)v + (t^4 + 20t^2 + 19)u + (-9t^5 - 18t^3 - 9t).$$

Using (2), it is easy to check that

$$x^3 - y^2 = -27(t^2+1)^2(2v - 2tu + 11t^2 + 11).$$

Therefore, $\deg(x) = 2k - 2$ and $\deg(x^3 - y^2) = k + 4$. Also,

$$\deg(x^3 - y^2)/\deg(x) = (k+4)/(2k-2),$$

which tends to $1/2$ when k tends to infinity. The above explicit example corresponds to $k = 27$.

Compared with Davenport's bound, our polynomial x and y satisfy

$$\deg(x^3 - y^2) = \tfrac{1}{2}\deg(x) + 5.$$

Thus, although our examples (x, y) do not give equality in Davenport's bound (1), they are very close to the best possible result for $\deg(x^3 - y^2)$, and it seems that this is the first known result where $\deg(x^3 - y^2) - \frac{1}{2}\deg(x)$ is bounded by an absolute constant, for polynomials x, y with integer coefficients and arbitrarily large degrees.

Since $t^2 + 1$ divides a_m for all m, it divides x and $(t^2+1)^2$ divides y. Hence, with $x = (t^2+1)X$ and $y = (t^2+1)^2Y$, we have

$$\deg(X^3 - (t^2+1)Y^2) = \tfrac{1}{2}\deg(X).$$

This shows that the only branch points of the rational function x^3/y^2 are 0, 1 and ∞, which is in agreement with the results of Zannier [11, 12].

Let us give an interpretation of our result in terms of polynomial Pell's equations. Following a suggestion by N. Elkies, we put $v - tu = (t^2+1)z$. Then the expressions of x and $x^3 - y^2$ simplify considerably, and we get $x = (t^2+1)(z^2+6z+4)$, $x^3 - y^2 = -27(t^2+1)^3(2z+11)$, which gives $y^2 = (t^2+1)^3(z^2+1)(z^2+9z+19)^2$. Thus, we need that $z^2+1 = (t^2+1)w^2$, i.e.

$$z^2 - (t^2+1)w^2 = -1. \tag{3}$$

The fundamental solution of Pell's equation (3) is $(z, w) = (t, 1)$. Taking $t = z$, we obtain the identity

$$(z^2 + 6z + 4)^3 - (z^2 + 1)(z^2 + 9z + 19)^2 = -27(2z + 11),$$

which is equivalent to Danilov's example [4] (and by taking $z^2 + 1 = 5w^2$ and $2z + 11 \equiv 0 \pmod{125}$, we get a well-known sequence of numerical examples with $|x^3 - y^2| < \sqrt{x}$).

However, if we consider (3) as a polynomial Pell's equation (in variable t), we obtain the sequence of solutions

$$z_1 = t, \qquad z_2 = 4t^3 + 3t, \qquad z_k = (4t^2 + 2)z_{k-1} - z_{k-2}.$$

This gives exactly the sequences of polynomials x and y, as given above.

Remark 1. In [5], Danilov consireded small values of $|x^4 - Ay^2|$ for integers A satisfying certain conditions. Using the formula

$$(27z + 7)^4 - (81z + 20)^2 \cdot \frac{(81z + 22)^2 + 2}{81} = 4z + 1, \tag{4}$$

he proved that if the Pellian equation $u^2 - 81Av^2 = -2$ has a solution, then the inequality $|x^4 - Ay^2| < \frac{4}{27}|x|$ has infinitely many integer solutions x, y. By applying a similar construction, as above, to Danilov's formula (4), we obtain the sequences x_k and y_k of polynomials in variable t with $\deg(x_k) = 2k + 1$, $\deg(y_k) = 4k$ and $\deg(x^4 - (t^2 + 2)y^2) = \deg(x) = 2k + 1$. For example, for $k = 3$ we have

$$x = 8t^7 + 28t^5 + 28t^3 + 7t - 1,$$
$$y = 64t^{13} + 384t^{11} + 880t^9 + 960t^7 - 16t^6 + 504t^5 - 40t^4 + 112t^3 - 24t^2 + 7t - 2,$$

and then

$$x^4 - (t^2 + 2)y^2 = 32t^7 + 112t^5 + 112t^3 + 28t - 7.$$

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References

[1] F. Beukers and C. L. Stewart, *Neighboring powers*, J. Number Theory 130 (2010), 660–679.

[2] B. J. Birch, S. Chowla, M. Hall, Jr., and A. Schinzel, *On the difference* $x^3 - y^2$, Norske Vid. Selsk. Forh. (Trondheim) 38 (1965), 65–69.

[3] E. Bombieri and W. Gubler, *Heights in Diophantine Geometry*, Cambridge Univ. Press, Cambridge, 2006.

[4] L. V. Danilov, *The Diophantine equation $x^3 - y^2 = k$ and Hall's conjecture*, Math. Notes 32 (1982), 617–618; see also ibid. 36 (1984), 726.
[5] —, *The Diophantine equations $x^m - Ay^n = k$*, ibid. 46 (1989), 914–919.
[6] H. Davenport, *On $f^3(t) - g^2(t)$*, Norske Vid. Selsk. Forh. (Trondheim) 38 (1965), 86–87.
[7] N. Elkies, *Rational points near curves and small nonzero $|x^3 - y^2|$ via lattice reduction*, in: Algorithmic Number Theory (Leiden, 2000), W. Bosma (ed.), Lecture Notes in Comput. Sci. 1838, Springer, 2000, 33–63.
[8] M. Hall, Jr., *The Diophantine equation $x^3 - y^2 = k$*, in: Computers in Number Theory (Oxford, 1969), A. O. L. Atkin and B. J. Birch (eds.), Academic Press, 1971, 173–198.
[9] I. Jiménez Calvo, J. Herranz and G. Sáez, *A new algorithm to search for small nonzero $|x^3 - y^2|$ values*, Math. Comp. 78 (2009), 2435–2444.
[10] S. Lang, *Algebra*, 3rd ed., Addison-Wesley, Reading, 1993.
[11] U. Zannier, *Some remarks on the S-unit equation in function fields*, Acta Arith. 64 (1993), 87–98.
[12] —, *On Davenport's bound for the degree of $f^3 - g^2$ and Riemann's Existence Theorem*, ibid. 71 (1995), 107–137.

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