Algebraic relations between
special values of multiple sine functions

by

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1. Introduction. The algebraicity of a product of division values of
the double sine function $S_2(\alpha, (1, \tau))$ at $\alpha \in \mathbb{Q} + \mathbb{Q}\tau$ (see [KW], [S], [Tang])
is very interesting from the viewpoint of Kronecker's Jugendtraum for a
real quadratic field. We expect the similar situation for division values of
multiple sine functions. In this paper, we study algebraic relations between
special values of multiple sine functions. Recall that the multiple sine func-
tion $S_r(x, (\omega_1, \dots, \omega_r))$ is defined by

$$S_r(x, (\omega_1, \dots, \omega_r))$$
$$= \prod_{n_1, \dots, n_r \geq 0} (n_1\omega_1 + \dots + n_r\omega_r + x) \Big(\prod_{n_1, \dots, n_r \geq 1} (n_1\omega_1 + \dots + n_r\omega_r - x) \Big)^{(-1)^{r-1}},$$

where $\prod$ denotes the regularized product of Deninger [D]:

$$\prod_{\lambda \in \Lambda} \lambda = \exp\Big(-\frac{d}{ds} \sum_{\lambda \in \Lambda} \lambda^{-s} \Big|_{s=0} \Big).$$

We refer to [KoKu], [KuKo], [KW] for the details of the theory of multiple
sine functions. We write $S_r(x, (1, \dots, 1)) = S_r(x)$ for simplicity.

THEOREM 1. *Let m be a positive integer. Then*

$$\prod_{k=0}^{m-1} S_{2m-k}(1)^{(-1)^k \binom{m-1}{k}} = 1.$$

THEOREM 2. *Let*

$$\delta(r) := \begin{cases} 1 & \text{if } r \text{ is odd,} \\ 0 & \text{if } r \text{ is even.} \end{cases}$$

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Then

$$\prod_{k=0}^{[r/2]-1} S_{r-k}(1)^{(-1)^k \sum_{l=1+k}^{[r/2]} \binom{r}{2l}\binom{l-1}{k}}$$

$$\times \prod_{k=0}^{[r/2]+\delta(r)-1} S_{r-k}(1/2)^{(-1)^k \sum_{l=1+k}^{[r/2]+\delta(r)} \binom{r}{2l-1}\binom{l-1}{k}} = 2.$$

THEOREM 3.

(1) $\qquad S_3(1,(1,1,1/2))^{-8} S_3(1)^2 = 2,$

(2) $\qquad S_4(1,(1,1,1,1/2))^{-16} S_4(1)^{10} = 2,$

(3) $\qquad S_5(1,(1,1,1,1,1/2))^{-128} S_5(1)^8 S_4(1)^{82} = 32,$

(4) $\qquad S_6(1,(1,1,1,1,1,1/2))^{-256} S_6(1)^{-38} S_5(1)^{228} = 128.$

REMARK 1.1. Kurokawa and Koyama [KuKo, Lemma 3.2] calculated $S_r(1)$. For example, we have

$$S_3(1) = S_4(1) = \exp\left(\frac{\zeta(3)}{4\pi^2}\right),$$

$$S_5(1) = \exp\left(\frac{11}{48\pi^2}\zeta(3) - \frac{\zeta(5)}{16\pi^4}\right),$$

$$S_6(1) = \exp\left(\frac{5}{24\pi^2}\zeta(3) - \frac{\zeta(5)}{8\pi^4}\right).$$

Moreover, the value $S_r(\alpha)$ at $\alpha \in \mathbb{Q}-\mathbb{Z}$ was obtained in [Tana, Theorem 1.1].

Set $\underline{\omega}_k := (\overbrace{\omega,\ldots,\omega}^{k})$ for a positive integer k.

THEOREM 4. *Let m be a positive integer. Then*

$$S_{2m}(m,(\underline{1}_m, \underline{\tau}_m)) = S_{2m}(m,(\underline{1}_m, \underline{\tau^{-1}}_m))^{-1}.$$

REMARK 1.2. Put $m = 1$ in Theorem 4. Then

$$S_2(1,(1,\tau)) = S_2(1,(1,\tau^{-1}))^{-1}.$$

In fact, following Koyama, Kurokawa, Tangedal and Wakayama (see [KoKu, (4) or (5) of Theorem 1], [KW, Lemma 2.2], [Tang, Lemma 6] for details) we have

$$S_2(1,(1,\tau)) = \sqrt{\tau}.$$

THEOREM 5. *Let $r \geq 2$ be an integer and $\tau > 0$ be algebraic irrational.*

(1) *One of the numbers $S_r(t+1,(\underline{1}_{r-1},\tau))$, $t = 1,\ldots,r-1$, is transcendental.*

(2) *One of the numbers $S_r((t+1)\tau + 1/2,(\underline{1}_{r-1},\tau))$, $t = 1,\ldots,r-1$, is transcendental.*

(3) *One of the numbers $\prod_{k=1}^{N-1} S_r(t + k/N, (\underline{1}_{r-1}, \tau))$, $t = 1, \ldots, r-1$, is transcendental. In particular, one of the numbers $S_r((2t+1)/2, (\underline{1}_{r-1}, \tau))$, $t = 1, \ldots, r-1$, is transcendental.*
(4) *Let $\delta = 1, \tau$. Then one of the numbers $S_r((1+\tau)/2 + t\delta, (\underline{1}_{r-1}, \tau))$, $t = 1, \ldots, r-1$, is transcendental.*

REMARK 1.3. When $r = 2$, Theorem 5(1) is a result of Kurokawa and Wakayama [KW, Theorem 1.2].

THEOREM 6.

(1) *Let $n \geq 2$ be an integer. When $\tau > 0$ is algebraic irrational, the number $S_2(1/2 + n\tau, (1, \tau))$ is transcendental.*
(2) *Let $n \geq 1$ and $N \geq 2$ be integers. When $\tau > 0$ is algebraic irrational, the number $\prod_{k=1}^{N-1} S_2(k/N + n, (1, \tau))$ is transcendental. In particular, $S_2(1/2 + n, (1, \tau))$ is transcendental.*

THEOREM 7. *Let n_1, n_2 be non-negative integers, not both zero. When $\tau > 0$ is real quadratic irrational, $S_2((1+\tau)/2 + n_1 + n_2\tau, (1, \tau))$ is transcendental. When $n_1 = 0$ or $n_2 = 0$, this holds for any algebraic irrational $\tau > 0$.*

THEOREM 8. *When $\tau > 0$ is rational, the values in Theorems 6 and 7 are algebraic.*

Theorem 1.2 of [KW] is shown via the Gelfond–Schneider theorem (see [Bak, Theorem 2.1]). In the same way we prove Theorems 6 and 7.

2. Proofs of Theorems 1–4

Proof of Theorem 1. First we show that for any non-negative integer k,

$$(2.1) \qquad S_r(x) = \prod_{l=0}^{k} S_{r-l}(x - k)^{(-1)^l \binom{k}{l}}.$$

We recall the quasi-periodicity of multiple sine functions (see [KuKo] for details):

LEMMA 2.1 ([KuKo, Theorem 2.1(a)]). *For $j = 1, \ldots, r$,*

$$(2.2) \qquad S_r(x + \omega_j, (\omega_1, \ldots, \omega_r)) = \frac{S_r(x, (\omega_1, \ldots, \omega_r))}{S_{r-1}(x, (\omega_1, \ldots, \omega_{j-1}, \omega_{j+1}, \ldots, \omega_r))},$$

where we put $S_0(x, \cdot) = -1$.

When $k = 1$, from (2.2) we have

$$S_r(x) = \frac{S_r(x-1)}{S_{r-1}(x-1)}.$$

Inductively suppose (2.1) holds for k. Then using (2.2) we have

$$S_r(x) = \prod_{l=0}^{k}\{S_{r-l}(x-(k+1))S_{r-l-1}(x-(k+1))^{-1}\}^{(-1)^l\binom{k}{l}}$$

$$= S_r(x-(k+1))S_{r-(k+1)}(x-(k+1))^{(-1)^{k+1}}$$

$$\times \prod_{l=1}^{k}S_{r-l}(x-(k+1))^{(-1)^l\binom{k}{l}}$$

$$\times \prod_{l=0}^{k-1}S_{r-l-1}(x-(k+1))^{(-1)^{l+1}\binom{k}{l}}$$

$$= S_r(x-(k+1))S_{r-(k+1)}(x-(k+1))^{(-1)^{k+1}}$$

$$\times \prod_{l=1}^{k}S_{r-l}(x-(k+1))^{(-1)^l\{\binom{k}{l}+\binom{k}{l-1}\}}$$

$$= \prod_{l=0}^{k+1}S_{r-l}(x-(k+1))^{(-1)^l\binom{k+1}{l}}.$$

Hence we obtain (2.1). Put $r = 2m$, $x = m$ and $k = m-1$ in (2.1) to obtain

$$S_{2m}(m) = \prod_{l=0}^{m-1}S_{2m-l}(1)^{(-1)^l\binom{m-1}{l}}.$$

By the definition of the multiple sine function, we see that

$$S_{2m}(m) = \frac{\prod_{n_1,\ldots,n_{2m}\geq 0}(n_1+\cdots+n_{2m}+m)}{\prod_{n_1,\ldots,n_{2m}\geq 0}(n_1+\cdots+n_{2m}+m)} = 1.$$

Hence, Theorem 1 is proved. ∎

Proof of Theorem 2. First, we recall a lemma.

LEMMA 2.2 ([KuKo, Theorem 2.1(c)]). *Let $N \geq 2$ be an integer. Then*

(2.3) $$\prod_{\substack{k_1,\ldots,k_r=0 \\ (k_1,\ldots,k_r)\neq(0,\ldots,0)}}^{N-1} S_r\left(\frac{k_1\omega_1+\cdots+k_r\omega_r}{N},(\omega_1,\ldots,\omega_r)\right) = N.$$

Putting $N = 2$ and $\omega_1 = \cdots = \omega_r = 1$ in (2.3), we get

$$\prod_{k=1}^{r}S_r(k/2)^{\binom{r}{k}} = 2.$$

Therefore,

$$\prod_{l=1}^{[r/2]} S_r(l)^{\binom{r}{2l}} \prod_{l=1}^{[r/2]+\delta(r)} S_r(l-1/2)^{\binom{r}{2l-1}} = 2.$$

From (2.1) we have

$$S_r(l) = \prod_{k=0}^{l-1} S_{r-k}(1)^{(-1)^k \binom{l-1}{k}},$$

$$S_r(l-1/2) = \prod_{k=0}^{l-1} S_{r-k}(1/2)^{(-1)^k \binom{l-1}{k}}.$$

Hence,

$$\prod_{l=1}^{[r/2]} \prod_{k=0}^{l-1} S_{r-k}(1)^{(-1)^k \binom{r}{2l} \binom{l-1}{k}} \prod_{l=1}^{[r/2]+\delta(r)} \prod_{k=0}^{l-1} S_{r-k}(1/2)^{(-1)^k \binom{r}{2l-1} \binom{l-1}{k}} = 2.$$

This formula can be rewritten to give the formula of Theorem 2. ∎

EXAMPLE 2.1 (Application of Theorem 2).

(1) $\quad S_2(1/2)^2 = 2,$

(2) $\quad S_3(1)^3 S_3(1/2)^4 S_2(1/2)^{-1} = 2,$

(3) $\quad S_4(1)^6 S_4(1/2)^8 S_3(1/2)^{-4} = 2,$

(4) $\quad S_5(1)^{15} S_4(1)^{-5} S_5(1/2)^{16} S_4(1/2)^{-12} S_3(1/2) = 2,$

(5) $\quad S_6(1)^{30} S_5(1)^{-15} S_6(1/2)^{32} S_5(1/2)^{-32} S_4(1/2)^6 = 2.$

Proof. Putting $r = 3, 5$ in Theorem 2, we obtain (2) and (4) immediately. Putting $r = 2$ in Theorem 2 yields

$$S_2(1) S_2(1/2)^2 = 2,$$

and putting $m = 1$ in Theorem 1 gives

$$S_2(1) = 1.$$

Hence, (1) is proved. Similarly, put $r = 4$ and $r = 6$ in Theorem 2 to get,

$$S_4(1)^7 S_3(1)^{-1} S_4(1/2)^8 S_3(1/2)^{-4} = 2,$$
$$S_6(1)^{31} S_5(1)^{-17} S_4(1) S_6(1/2)^{32} S_5(1/2)^{-32} S_4(1/2)^6 = 2.$$

Putting $m = 2$ and $m = 3$ in Theorem 1, we see that

(2.4) $\quad S_4(1) S_3(1)^{-1} = 1,$

(2.5) $\quad S_6(1) S_5(1)^{-2} S_4(1) = 1.$

This yields (3) and (5). ∎

Proof of Theorem 3. We prepare a lemma.

LEMMA 2.3. *Let l, N be positive integers. Then*

$$S_r(x, (\underline{1}_{r-1}, l/N)) = \prod_{k=0}^{N-1} S_r(x + lk/N, (\underline{1}_{r-1}, l)).$$

Proof of Lemma 2.3. Using

$$\zeta_r(s, x, (\underline{1}_{r-1}, l/N)) := \sum_{n_1,\ldots,n_r \geq 0} (x + n_1 + \cdots + n_{r-1} + ln_r/N)^{-s}$$

$$= \sum_{k=0}^{N-1} \sum_{n_1,\ldots,n_r \geq 0} (x + lk/N + n_1 + \cdots + n_{r-1} + ln_r)^{-s},$$

we have

$$\prod_{n_1,\ldots,n_r \geq 0} (n_1 + \cdots + n_{r-1} + ln_r/N + x)^{-1}$$

$$= \prod_{k=0}^{N-1} \prod_{n_1,\ldots,n_r \geq 0} (n_1 + \cdots + n_{r-1} + ln_r + x + lk/N)^{-1}.$$

From

$$\prod_{n_1,\ldots,n_r \geq 1} (n_1 + \cdots + n_{r-1} + ln_r/N - x)^{-1}$$

$$= \prod_{n_1,\ldots,n_r \geq 0} (n_1 + \cdots + n_{r-1} + ln_r/N + r - 1 - x + l/N)^{-1}$$

$$= \prod_{k=0}^{N-1} \prod_{n_1,\ldots,n_r \geq 0} (n_1 + \cdots + n_{r-1} + ln_r + r - 1 - x + l(k+1)/N)^{-1}$$

$$= \prod_{k=0}^{N-1} \prod_{n_1,\ldots,n_r \geq 0} (n_1 + \cdots + n_{r-1} + ln_r + r - 1 - x + l(N-k)/N)^{-1}$$

$$= \prod_{k=0}^{N-1} \prod_{n_1,\ldots,n_r \geq 1} (n_1 + \cdots + n_{r-1} + ln_r - x - lk/N)^{-1}$$

we obtain Lemma 2.3. ∎

Put $x = 1$, $l = 1$ and $N = 2$ in Lemma 2.3 to obtain

$$(2.6) \qquad S_r(1, (\underline{1}_{r-1}, 1/2)) = S_r(1)S_r(3/2) = S_r(1)S_r(1/2)S_{r-1}(1/2)^{-1},$$

where we used the quasi-periodicity of multiple sine functions:

$$(2.7) \qquad\qquad S_r(3/2) = S_r(1/2)S_{r-1}(1/2)^{-1}.$$

Now we show (1) of Theorem 3. When $r = 3$, from (2.6) and Example 2.1(2) we see that

$$S_3(1, (1, 1, 1/2)) = S_3(1)S_2(1/2)^{-1}\{2S_3(1)^{-3}S_2(1/2)\}^{1/4}.$$

Since (see Example 2.1(1))

$$S_2(1/2) = \sqrt{2}$$

we obtain (1) of Theorem 3.

Next we show (2) of Theorem 3. When $r = 4$, from (2.6) and Example 2.1(3) we see that

$$S_4(1, (1, 1, 1, 1/2)) = S_4(1)S_3(1/2)^{-1}\{2S_4(1)^{-6}S_3(1/2)^4\}^{1/8}.$$

Hence, using Example 2.1(2) we have

$$S_4(1, (1, 1, 1, 1/2))^8 = 2S_4(1)^2 S_3(1/2)^{-4} = 2S_4(1)^2\{2S_3(1)^{-3}S_2(1/2)\}^{-1}.$$

Applying (2.4) we obtain (2) of Theorem 3.

Next we show (3) of Theorem 3. From (2.6) for $r = 5$ and Example 2.1(4) we see that

$$S_5(1, (1, 1, 1, 1, 1/2))$$
$$= S_5(1)S_4(1/2)^{-1}\{2S_5(1)^{-15}S_4(1)^5 S_4(1/2)^{12}S_3(1/2)^{-1}\}^{1/16}.$$

Hence, using Example 2.1(2) & (3) we have

$$S_5(1, (1, 1, 1, 1, 1/2))^{16} = 2S_5(1)S_4(1)^5 S_4(1/2)^{-4}S_3(1/2)^{-1}$$
$$= 2S_5(1)S_4(1)^5 S_3(1/2)^{-1}\{2S_4(1)^{-6}S_3(1/2)^4\}^{-1/2}$$
$$= 2^{1/2}S_5(1)S_4(1)^8 S_3(1/2)^{-3}$$
$$= 2^{1/2}S_5(1)S_4(1)^8\{2S_3(1)^{-3}S_2(1/2)\}^{-3/4}$$
$$= 2^{-5/8}S_5(1)S_4(1)^8 S_3(1)^{9/4}.$$

Applying (2.4) we obtain (3) of Theorem 3.

Finally, we show (4) of Theorem 3. From (2.6) for $r = 6$ and Example 2.1(5) we see that

$$S_6(1, (1, 1, 1, 1, 1, 1/2))$$
$$= S_6(1)S_5(1/2)^{-1}\{2S_6(1)^{-30}S_5(1)^{15}S_5(1/2)^{32}S_4(1/2)^{-6}\}^{1/32}.$$

Hence, using Example 2.1(2) & (3) we have

$$S_6(1, (1, 1, 1, 1, 1, 1/2))^{32} = 2S_6(1)^2 S_5(1)^{15}S_4(1/2)^{-6}$$
$$= 2S_6(1)^2 S_5(1)^{15}\{2S_4(1)^{-6}S_3(1/2)^4\}^{-3/4}$$
$$= 2^{1/4}S_6(1)^2 S_5(1)^{15}S_4(1)^{9/2}S_3(1/2)^{-3}$$
$$= 2^{1/4}S_6(1)^2 S_5(1)^{15}S_4(1)^{9/2}\{2S_3(1)^{-3}S_2(1/2)\}^{-3/4}$$
$$= 2^{-7/8}S_6(1)^2 S_5(1)^{15}S_4(1)^{27/4}.$$

Applying (2.5) we obtain (4) of Theorem 3. ∎

Proof of Theorem 4. Using the homogeneity (see [KuKo] for details)

$$S_r(cx, (c\omega_1, \ldots, c\omega_r)) = S_r(x, (\omega_1, \ldots, \omega_r))$$

and the construction of S_{2m} we obtain

$$S_{2m}(m, (\underline{1}_m, \underline{\tau}_m)) = S_{2m}(m/\tau, (\underline{1}_m, \underline{\tau^{-1}}_m)) = S_{2m}(m, (\underline{1}_m, \underline{\tau^{-1}}_m))^{-1}. \quad \blacksquare$$

3. Proofs of Theorems 5–8. First, we show Theorems 6–8. We express certain special values of double sine functions via products of usual sine or cosine functions.

LEMMA 3.1.

(1) *Let n be a positive integer. Then*

$$S_2\left(\frac{1}{2} + n\tau, (1, \tau)\right) = \sqrt{2} \prod_{l=0}^{n} (2\cos(l\pi\tau))^{-1},$$

$$S_2\left(\frac{1+\tau}{2} + n\tau, (1, \tau)\right) = \prod_{\substack{l=1 \\ l:\,odd}}^{2n-1} \left(2\cos\left(\frac{l\pi}{2}\tau\right)\right)^{-1}$$

and

$$S_2\left(\frac{1+\tau}{2} + n, (1, \tau)\right) = \prod_{\substack{l=1 \\ l:\,odd}}^{2n-1} \left(2\cos\left(\frac{l\pi}{2\tau}\right)\right)^{-1}.$$

(2) *Let $n \geq 1$ and $N \geq 2$ be integers. Then*

$$\prod_{k=1}^{N-1} S_2\left(\frac{k}{N} + n, (1, \tau)\right) = \sqrt{N} \prod_{k=1}^{N-1} \prod_{l=0}^{n-1} \left(2\sin\left(\frac{\pi}{\tau}\left(\frac{k}{N} + l\right)\right)\right)^{-1}.$$

(3) *Let n_1, n_2 be positive integers. Then*

$$S_2\left(\frac{1+\tau}{2} + n_1 + n_2\tau, (1, \tau)\right) = (-1)^{n_1 n_2} \prod_{\substack{l_1=1 \\ l_1:\,odd}}^{2n_1-1} \left(2\cos\left(\frac{l_1\pi}{2}\frac{1}{\tau}\right)\right)^{-1}$$

$$\times \prod_{\substack{l_2=1 \\ l_2:\,odd}}^{2n_2-1} \left(2\cos\left(\frac{l_2\pi}{2}\tau\right)\right)^{-1}.$$

Proof. From the quasi-periodicity of multiple sine functions we have

$$S_2\left(\frac{1}{2} + n\tau, (1, \tau)\right) = S_2\left(\frac{1}{2}, (1, \tau)\right) \prod_{l=0}^{n-1} S_1\left(\frac{1}{2} + l\tau\right)^{-1},$$

$$S_2\left(\frac{1+\tau}{2}+n\tau,(1,\tau)\right)=\prod_{l=0}^{n-1}S_1\left(\frac{1+\tau}{2}+l\tau\right)^{-1},$$

$$S_2\left(\frac{k}{N}+n,(1,\tau)\right)=S_2\left(\frac{k}{N},(1,\tau)\right)\prod_{l=0}^{n-1}S_1\left(\frac{k}{N}+l,(\tau)\right)^{-1},$$

$$S_2\left(\frac{1+\tau}{2}+n_1+n_2\tau,(1,\tau)\right)=\prod_{l_1=0}^{n_1-1}S_1\left(\frac{1+\tau}{2}+l_1,(\tau)\right)^{-1}$$

$$\times\prod_{l_2=0}^{n_2-1}S_1\left(\frac{1+\tau}{2}+n_1+l_2\tau\right)^{-1},$$

where we used

$$S_2\left(\frac{1+\tau}{2},(1,\tau)\right)=1.$$

Since $S_1(x)=2\sin(\pi x)$, which is obtained from Lerch's formula [L], and by the homogeneity of multiple sine functions we obtain

$$S_1\left(\frac{1}{2}+l\tau\right)=2\cos(l\pi\tau),$$

$$S_1\left(\frac{1+\tau}{2}+l\tau\right)=2\cos\left(\frac{(2l+1)\pi}{2}\tau\right),$$

$$S_1\left(\frac{k}{N}+l,(\tau)\right)=2\sin\left(\frac{\pi}{\tau}\left(\frac{k}{N}+l\right)\right),$$

$$S_1\left(\frac{1+\tau}{2}+l_1,(\tau)\right)=2\cos\left(\frac{(2l_1+1)\pi}{2\tau}\right),$$

$$S_1\left(\frac{1+\tau}{2}+n_1+l_2\tau\right)=(-1)^{n_1}2\cos\left(\frac{(2l_2+1)\pi}{2}\tau\right).$$

Hence (3) of Lemma 3.1 is proved.

Now we show

$$\prod_{k=1}^{N-1}S_2\left(\frac{k\omega_1}{N},(\omega_1,\omega_2)\right)=\sqrt{N}.$$

In fact, by the definition of S_2 we have

$$S_2(x,(\omega_1,\omega_2))S_2(\omega_1+\omega_2-x,(\omega_1,\omega_2))=1.$$

Let $x=k\omega_1/N$. Then

$$\prod_{k=1}^{N-1}S_2\left(\frac{k\omega_1}{N},(\omega_1,\omega_2)\right)\prod_{k=1}^{N-1}S_2\left(\omega_1-\frac{k\omega_1}{N},(\omega_1,\omega_2)\right)$$

$$=\prod_{k=1}^{N-1}S_1\left(\omega_1-\frac{k\omega_1}{N},(\omega_1)\right).$$

This gives

$$\left(\prod_{k=1}^{N-1} S_2\left(\frac{k\omega_1}{N}, (\omega_1, \omega_2)\right)\right)^2 = \prod_{k=1}^{N-1} 2\sin\left(\frac{k}{N}\pi\right) = N.$$

Hence (2) of Lemma 3.1 is proved.

When $N = 2$, we have

$$S_2(\omega_1/2, (\omega_1, \omega_2)) = \sqrt{2},$$

which was obtained by Koyama and Kurokawa [KoKu, Theorem 1(2)]. Moreover, from the homogeneity of multiple sine functions and

$$S_2\left(\frac{1+\tau}{2} + n\tau, (1, \tau)\right) = \prod_{\substack{l=1 \\ l:\,\mathrm{odd}}}^{2n-1} \left(2\cos\left(\frac{l\pi}{2}\tau\right)\right)^{-1},$$

we have

$$S_2\left(\frac{1+\tau}{2} + n, (1, \tau)\right) = \prod_{\substack{l=1 \\ l:\,\mathrm{odd}}}^{2n-1} \left(2\cos\left(\frac{l\pi}{2\tau}\right)\right)^{-1}.$$

Thus, we obtain (1) of Lemma 3.1. ∎

LEMMA 3.2.

(1) *Let $n \geq 2$ be an integer and $\tau \in \overline{\mathbb{Q}} - \mathbb{Q}$. Then*

$$\prod_{l=0}^{n-1} \cos(l\pi\tau) \notin \overline{\mathbb{Q}}.$$

(2) *Let n be a positive integer and $\tau \in \overline{\mathbb{Q}} - \mathbb{Q}$. Then*

$$\prod_{\substack{l=1 \\ l:\,odd}}^{2n-1} \cos\left(\frac{l\pi}{2}\tau\right) \notin \overline{\mathbb{Q}} \quad and \quad \prod_{\substack{l=1 \\ l:\,odd}}^{2n-1} \cos\left(\frac{l\pi}{2\tau}\right) \notin \overline{\mathbb{Q}}.$$

(3) *Let $n \geq 1$ and $N \geq 2$ be integers and $\tau \in \overline{\mathbb{Q}} - \mathbb{Q}$. Then*

$$\prod_{k=1}^{N-1} \prod_{l=0}^{n-1} \sin\left(\frac{\pi}{\tau}\left(\frac{k}{N} + l\right)\right) \notin \overline{\mathbb{Q}}.$$

(4) *Let n_1, n_2 be positive integers and $\tau \neq 0$ be quadratic irrational. Then*

$$\prod_{\substack{l_1=1 \\ l_1:\,odd}}^{2n_1-1} \cos\left(\frac{l_1\pi}{2}\frac{1}{\tau}\right) \prod_{\substack{l_2=1 \\ l_2:\,odd}}^{2n_2-1} \cos\left(\frac{l_2\pi}{2}\tau\right) \notin \overline{\mathbb{Q}}.$$

Proof. First we prove (1). Suppose otherwise. Put $q = e^{\pi i \tau}$. Then

$$\prod_{l=0}^{n-1} (q^l + q^{-l}) \in \overline{\mathbb{Q}}.$$

Hence there is a $\overline{\mathbb{Q}}$-algebraic equation for q. However, the Gelfond–Schneider theorem (see [Bak, Theorem 2.1]) gives $q = (-1)^\tau \notin \overline{\mathbb{Q}}$. The contradiction proves (1).

Next we prove (2). Suppose otherwise. Put $q = e^{\frac{\pi i}{2} \tau}$ or $q = e^{\frac{\pi i}{2\tau}}$. Then

$$\prod_{\substack{l=1 \\ l\,:\,\text{odd}}}^{2n-1} (q^l + q^{-l}) \in \overline{\mathbb{Q}}.$$

Similarly, this gives a contradiction.

Next we prove (3). Suppose otherwise. Put $q = e^{\frac{\pi i}{N\tau}}$. Then

$$\prod_{k=1}^{N-1} \prod_{l=0}^{n-1} (q^{k+lN} - q^{-(k+lN)}) \in \overline{\mathbb{Q}}.$$

Similarly, this gives a contradiction.

Finally, we prove (4). Since τ is a quadratic irrational number, there are integers N, M such that $\tau = \frac{N}{M} \frac{1}{\tau}$. Suppose otherwise. Put $q = e^{\frac{\pi i}{2M\tau}}$. Then

$$\prod_{\substack{l_1=1 \\ l_1\,:\,\text{odd}}}^{2n_1-1} (q^{Ml_1} + q^{-Ml_1}) \prod_{\substack{l_2=1 \\ l_2\,:\,\text{odd}}}^{2n_2-1} (q^{Nl_2} + q^{-Nl_2}) \in \overline{\mathbb{Q}}.$$

Similarly, this gives a contradiction. ∎

Proofs of Theorems 6 and 7. The theorems follow from Lemmas 3.1 and 3.2. ∎

Proof of Theorem 8. When α is rational, $\sin(\alpha\pi), \cos(\alpha\pi) \in \overline{\mathbb{Q}}$. Hence Theorem 8 follows. ∎

Proof of Theorem 5. We prove (1) of Theorem 5 by induction on $r = 2, 3, \ldots$. When $r = 2$, from the result of Kurokawa and Wakayama [KW, Theorem 1.2] we see that $S_2(2, (1, \tau))$, is transcendental. Assume that one of the numbers $S_r(t + 1, (\underline{1}_{r-1}, \tau))$, $t = 1, \ldots, r - 1$ is transcendental. Then from the quasi-periodicity of multiple sine functions for $k = 3, \ldots, r + 1$ we have

$$S_{r+1}(k, (\underline{1}_r, \tau)) S_{r+1}(k - 1, (\underline{1}_r, \tau))^{-1} = S_r(k - 1, (\underline{1}_{r-1}, \tau))^{-1}.$$

So, one of the r numbers $S_{r+1}(t + 1, (\underline{1}_r, \tau))$, $t = 1, \ldots, r$, is transcendental. Similarly, from Theorem 6(1) & (2) and Theorem 7 we have (2), (3) and (4) of Theorem 5 respectively. ∎

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