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An irregular $D(4)$-quadruple cannot be extended to a quintuple

by

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1. Introduction

Definition 1. Let n be an integer. A set of m positive integers is called a *Diophantine m-tuple with the property* $D(n)$, or simply $D(n)$-*m-tuple*, if the product of any two of them increased by n is a perfect square.

Diophantus was the first to look for such sets in the case $n = 1$. He found a set of four positive rational numbers with the above property: $\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$. Fermat found a first $D(1)$-quadruple, the set $\{1, 3, 8, 120\}$. Euler was later able to add the fifth positive rational, $\frac{777480}{8288641}$, to Fermat's set (see [3], [4, pp. 103–104, 232]). Recently, Gibbs [15] found several examples of $D(n)$-sextuples, e.g. $\{99, 315, 9920, 32768, 44460, 19534284\}$ is a $D(2985984)$-sextuple. There is a folklore conjecture that there does not exist a $D(1)$-quintuple. The first result supporting this conjecture is due to Baker and Davenport [1], who proved that Fermat's set cannot be extended to a $D(1)$-quintuple. Dujella [7] proved that there does not exist a $D(1)$-sextuple and that there are only finitely many $D(1)$-quintuples. Considering congruences modulo 8, it is easy to prove that a $D(4)$-m-tuple can contain at most two odd numbers. So Dujella's result implies that there does not exist a $D(4)$-8-tuple and that there are only finitely many $D(4)$-septuples (see [9]). The author [11, 12] improved this result by proving that there does not exist a $D(4)$-sextuple. In the present paper we further improve this result.

For $n = 4$ it is conjectured that there does not exist a $D(4)$-quintuple. Actually, there is even a stronger version of that conjecture.

Conjecture 1 (cf. [9, Conjecture 1]). *There does not exist a $D(4)$-quintuple. Moreover, if $\{a, b, c, d\}$ is a $D(4)$-quadruple such that $a < b < c < d$, then*

$$d = a + b + c + \tfrac{1}{2}(abc + rst),$$

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where r, s, t are positive integers defined by

$$ab + 4 = r^2, \quad ac + 4 = s^2, \quad bc + 4 = t^2.$$

If $d = a + b + c + \frac{1}{2}(abc + rst)$, then $\{a, b, c, d\}$ is a $D(4)$-quadruple. We will write d_+ for this d. We also define $d_- = a + b + c + \frac{1}{2}(abc - rst)$. If $d_- \neq 0$, the set $\{a, b, c, d_-\}$ is also a $D(4)$-quadruple, but $d_- < c$.

Definition 2. A $D(4)$-quadruple $\{a, b, c, d\}$ such that $d > \max\{a, b, c\}$ is called *regular* if $d = d_+$.

We have checked in [11] that all $D(4)$-quadruples $\{a, b, c, d\}$ such that $\max\{a, b, c, d\} \leq 4 \cdot 10^7$ are regular; we use this at several places in this paper.

Mohanty and Ramasamy [17] were the first to study the nonextendibility of $D(4)$-m-tuples. They proved that the $D(4)$-quadruple $\{1, 5, 12, 96\}$ cannot be extended to a $D(4)$-quintuple. Kedlaya [16] later proved that if $\{1, 5, 12, d\}$ is a $D(4)$-quadruple, then $d = 96$.

There are some generalizations of this result that support Conjecture 1. One was given by Dujella and Ramasamy [9] who proved Conjecture 1 for a parametric family of $D(4)$-quadruples. They showed that if k and d are positive integers and

$$\{F_{2k}, 5F_{2k}, 4F_{2k+2}, d\}$$

is a $D(4)$-quadruple, then $d = 4L_{2k}F_{4k+2}$, where F_k and L_k are Fibonacci and Lucas numbers. Another generalization was given by Fujita [14], who proved that if $k \geq 3$ is an integer and $\{k-2, k+2, 4k, d\}$ is a $D(4)$-quadruple, then $d = 4k^3 - 4k$.

Our main result is the following theorem.

Theorem 1. *Any $D(4)$-quintuple contains a regular $D(4)$-quadruple.*

The theorem implies that an irregular $D(4)$-quadruple cannot be extended to a quintuple with a larger element. In the proof we use the methods and results from [7, 11, 12]. We transform the problem of extending a $D(4)$-triple $\{a, b, c\}$ to a quadruple to solving a system of simultaneous Pellian equations. This reduces to finding the intersection of binary recurrence sequences. Here we examine the elements of the sequences with small indices more precisely than in [12] and obtain much better gap principles for irregular $D(4)$-quadruples. Precisely, we are able to prove that if $\{a, b, c, d\}$ is an irregular $D(4)$-quadruple, then $d > \max\{7b^{11}, 10^{26}\}$ except for finitely many a, b and c. This result, together with the results we have already proved in [12], will imply the main theorem except in finitely many cases. In particular, we prove that an irregular $D(4)$-quadruple cannot be extended to a quintuple except for finitely many $\{a, b, c, d\}$. But for finitely many remaining $D(4)$-triples $\{a, b, c\}$ we will prove that they can be extended to a quadruple in a unique way, using Baker–Davenport reduction. And because this unique extension yields a regular quadruple, we get a contradiction if we suppose that $\{a,b,c,d\}$ is irregular.

Let us mention that recently Fujita [13] has proved the analogous result for $D(1)$-m-tuples. The main difference in our proof is that we consider the binary recurrence sequences more carefully, so we obtain significantly improved gap principles. Doing that we do not need to define standard triples as in the case $n=1$. Fujita's result implies that an irregular $D(4)$-quadruple with even elements cannot be extended with a larger fifth even element. Results and methods for $n=1$ and $n=4$ are analogous, but they cannot be transferred to $n=l^2$ in general. E.g. uniqueness of extension does not hold for $n=16$. We know that the $D(16)$-triple $\{1,20,33\}$ has exactly two extensions to a $D(16)$-quadruple: $\{1,20,33,105\}$ and $\{1,20,33,273\}$ (see [10]). And there are also $D(n)$-quintuples for some n, since the sets $\{1,33,105,320,18240\}$ and $\{5,21,64,285,6720\}$ are $D(256)$-quintuples (see [5]).

2. System of Pellian equations. Let $\{a,b,c\}$ be a $D(4)$-triple such that $a<b<c$. Furthermore, let r,s,t be positive integers defined by

$$ab+4=r^2,\qquad ac+4=s^2,\qquad bc+4=t^2. \tag{1}$$

Assume that we can extend $\{a,b,c\}$ to an irregular $D(4)$-quadruple $\{a,b,c,d\}$. Then there exist positive integers x,y,z satisfying

$$ad+4=x^2,\qquad bd+4=y^2,\qquad cd+4=z^2.$$

If we eliminate d we get the system of simultaneous Pellian equations

$$az^2-cx^2=4(a-c), \tag{2}$$

$$bz^2-cy^2=4(b-c). \tag{3}$$

We describe the sets of solutions of (2) and (3) in the following lemma.

Lemma 1 (cf. [9, Lemma 2], [11, Lemma 1]). *There exist positive integers* i_0, j_0 *and integers* $z_0^{(i)}, x_0^{(i)}, z_1^{(j)}, y_1^{(j)}$, $i=1,\dots,i_0$, $j=1,\dots,j_0$, *with the following properties*:

(i) $(z_0^{(i)}, x_0^{(i)})$ *and* $(z_1^{(j)}, y_1^{(j)})$ *are solutions of* (2) *and* (3).

(ii) $z_0^{(i)}, x_0^{(i)}, z_1^{(j)}, y_1^{(j)}$ *satisfy the following inequalities*

$$1\le x_0^{(i)}\le\sqrt{\frac{a(c-a)}{s-2}}<\sqrt{s+2}<1.236\sqrt[4]{ac}, \tag{4}$$

$$|z_0^{(i)}|\le\sqrt{\frac{(s-2)(c-a)}{a}}<\sqrt{\frac{c\sqrt{c}}{\sqrt{a}}}<0.468c, \tag{5}$$

$$1 \le y_1^{(j)} \le \sqrt{\frac{b(c-b)}{t-2}} < \sqrt{t+2} < 1.122\sqrt[4]{bc}, \tag{6}$$

$$|z_1^{(j)}| \le \sqrt{\frac{(t-2)(c-b)}{b}} < \sqrt{\frac{c\sqrt{c}}{\sqrt{b}}} < 0.360c. \tag{7}$$

(iii) *If* (z,x) *and* (z,y) *are integer solutions of* (2) *and* (3), *then there exist* $i \in \{1,\dots,i_0\}$, $j \in \{1,\dots,j_0\}$ *and integers* $m,n \ge 0$ *such that*

$$z\sqrt{a} + x\sqrt{c} = (z_0^{(i)}\sqrt{a} + x_0^{(i)}\sqrt{c})\left(\frac{s+\sqrt{ac}}{2}\right)^m, \tag{8}$$

$$z\sqrt{b} + y\sqrt{c} = (z_1^{(j)}\sqrt{b} + y_1^{(j)}\sqrt{c})\left(\frac{t+\sqrt{bc}}{2}\right)^n. \tag{9}$$

Let (x,y,z) be a solution of the system (2)–(3). Then from (8) we get $z = v_m^{(i)}$ for some i and $m \ge 0$, where

$$v_0^{(i)} = z_0^{(i)}, \quad v_1^{(i)} = \tfrac{1}{2}(sz_0^{(i)} + cx_0^{(i)}), \quad v_{m+2}^{(i)} = sv_{m+1}^{(i)} - v_m^{(i)}. \tag{10}$$

From (9) we conclude that $z = w_n^{(j)}$ for some j and $n \ge 0$, where

$$w_0^{(j)} = z_1^{(j)}, \quad w_1^{(j)} = \tfrac{1}{2}(tz_1^{(j)} + cy_1^{(j)}), \quad w_{n+2}^{(j)} = tw_{n+1}^{(j)} - w_n^{(j)}. \tag{11}$$

For simplicity, we will omit the indices i and j from now on. Because we are interested in $D(4)$-quadruples that are not regular, we can take $m,n > 2$. This follows from [11, Lemma 6].

3. Gap principles. In this section we will significantly improve the gap principles from [12]. First we need some lemmata proved in [11, 12].

LEMMA 2 (cf. [11, Lemma 9]).

(i) *If the equation* $v_{2m} = w_{2n}$ *has a solution, then* $z_0 = z_1$. *Moreover*, $|z_0| = 2$, *or* $|z_0| = \frac{1}{2}(cr - st)$, *or* $|z_0| < 1.608a^{-5/14}c^{9/14}$.

(ii) *If the equation* $v_{2m+1} = w_{2n}$ *has a solution, then* $|z_0| = t$, $|z_1| = \frac{1}{2}(cr - st)$, $z_0z_1 < 0$.

(iii) *If the equation* $v_{2m} = w_{2n+1}$ *has a solution, then* $|z_1| = s$, $|z_0| = \frac{1}{2}(cr - st)$, $z_0z_1 < 0$.

(iv) *If the equation* $v_{2m+1} = w_{2n+1}$ *has a solution, then* $|z_0| = t$, $|z_1| = s$, $z_0z_1 > 0$.

LEMMA 3. *If* $z = v_m = w_n$, *then* $5 \le m \le 2n+1$, *or* $m = n = 4$ *and* $|z_0| < 1.608a^{-5/14}c^{9/14}$.

Proof. See the proofs of [11, Lemma 5] and [12, Lemma 5]. ∎

We are now ready to prove the main lemma that gives the desired gap principle. Let us mention that here we come upon some equations $v_m = w_n$ that are not as trivially solvable as in [11, 12], but we succeed in solving them by considering congruence relations more carefully and checking for finitely many a, b and c if we get any new $D(4)$-triple for which we did not prove the uniqueness of extension to a quadruple. For that we also need one more useful lemma.

LEMMA 4 (cf. [6, Lemma 3]). *If $\{a,b,c\}$ is a $D(4)$-triple such that $a < b < c$, then there exist positive integers e, x', y', z' such that*

$$ae + 16 = (x')^2, \qquad be + 16 = (y')^2, \qquad ce + 16 = (z')^2$$

and

$$c = a + b + \tfrac{1}{4}e + \tfrac{1}{8}(abe + rx'y').$$

From this lemma we can conclude that $c = a + b + 2r$ or $c > \frac{1}{4}aeb$, for some positive integer e satisfying the equalities in the lemma.

LEMMA 5. *Let $z = v_m = w_n$. Then at least one of the following two statements is valid*:

(i) $n \geq 7$,
(ii) $n \geq 4$ *and* $c > 0.036b^{3.5}$.

Proof. Let us mention that we get (ii) when $|z_0| < 1.608a^{-5/14}c^{9/14}$, as in [12]. So we can consider only the cases from Lemma 2 when we know the exact values of fundamental solutions. Because $n \geq 3$, we have to consider the cases $n = 3, 4, 5, 6$, and we succeed in obtaining a contradiction in all of them. Also in [12] we have proved, using Baker–Davenport reduction, uniqueness of extension of $D(4)$-triples $\{a,b,c\}$ when $ab^2c < 10^7$. In doing that, we have used slightly worse constants than needed. So in the proof of the lemma we may assume $c \geq 80$, $ac \geq 96$, and $bc \geq 6325$.

CASE 1. Let first $w_{2n+1} = v_{2m+1}$. Then $|z_0| = t$, $|z_1| = s$ and $z_0z_1 > 0$.

If we take $z_0 = t$, $z_1 = s$, then $x_0 = y_1 = r$ and

$$\begin{aligned}
w_3 &= \tfrac{1}{2}bc^2r + \tfrac{1}{2}bcst + \tfrac{3}{2}cr + \tfrac{1}{2}st,\\
w_5 &= \tfrac{1}{2}b^2c^3r + \tfrac{1}{2}b^2c^2st + \tfrac{5}{2}bc^2r + \tfrac{3}{2}bcst + \tfrac{5}{2}cr + \tfrac{1}{2}st,\\
v_5 &= \tfrac{1}{2}a^2c^3r + \tfrac{1}{2}a^2c^2st + \tfrac{5}{2}ac^2r + \tfrac{3}{2}acst + \tfrac{5}{2}cr + \tfrac{1}{2}st,\\
v_7 &= \tfrac{1}{2}a^3c^4r + \tfrac{1}{2}a^3c^3st + \tfrac{7}{2}a^2c^3r + \tfrac{5}{2}a^2c^2st + 7ac^2r + 3acst + \tfrac{7}{2}cr + \tfrac{1}{2}st.
\end{aligned}$$

Now from $b < a^2c$, we get $w_3 < v_5 < v_7$. Furthermore, if $b^2 < a^3c$, then we deduce $v_5 < w_5 < v_7 < v_9 < v_{11}$. And if $b^2 > a^3c$, from the estimate for $z_0 = t$ we get

$$bc < t^2 < \frac{c\sqrt{c}}{\sqrt{a}},$$

which implies $a^2 < 1$, a contradiction.

If $z_0 = -t$, $z_1 = -s$, then also $x_0 = y_1 = r$ and

$$w_3 = \tfrac{1}{2}bc^2r - \tfrac{1}{2}bcst + \tfrac{3}{2}cr - \tfrac{1}{2}st,$$
$$w_5 = \tfrac{1}{2}b^2c^3r - \tfrac{1}{2}b^2c^2st + \tfrac{5}{2}bc^2r - \tfrac{3}{2}bcst + \tfrac{5}{2}cr - \tfrac{1}{2}st,$$
$$v_5 = \tfrac{1}{2}a^2c^3r - \tfrac{1}{2}a^2c^2st + \tfrac{5}{2}ac^2r - \tfrac{3}{2}acst + \tfrac{5}{2}cr - \tfrac{1}{2}st,$$
$$v_7 = \tfrac{1}{2}a^3c^4r - \tfrac{1}{2}a^3c^3st + \tfrac{7}{2}a^2c^3r - \tfrac{5}{2}a^2c^2st + 7ac^2r - 3acst + \tfrac{7}{2}cr - \tfrac{1}{2}st.$$

Now we get a contradiction in the same way as in the last case, we only have to take into account that $cr > st$.

Case 2. Assume now $w_{2n+1} = v_{2m}$. Then $|z_0| = \frac{1}{2}(cr - st)$, $|z_1| = s$ and $z_0 z_1 < 0$.

Let us first take $z_1 = s$, $z_0 = \frac{1}{2}(st - cr)$. Then $x_0 = \frac{1}{2}(rs - at)$, $y_1 = r$ and

$$w_3 = \tfrac{1}{2}bc^2r + \tfrac{1}{2}bcst + \tfrac{3}{2}cr + \tfrac{1}{2}st,$$
$$w_5 = \tfrac{1}{2}b^2c^3r + \tfrac{1}{2}b^2c^2st + \tfrac{5}{2}bc^2r + \tfrac{3}{2}bcst + \tfrac{5}{2}cr + \tfrac{1}{2}st,$$
$$v_6 = \tfrac{1}{2}a^2c^3r + \tfrac{1}{2}a^2c^2st + \tfrac{5}{2}ac^2r + \tfrac{3}{2}acst + \tfrac{5}{2}cr + \tfrac{1}{2}st,$$
$$v_8 = \tfrac{1}{2}a^3c^4r + \tfrac{1}{2}a^3c^3st + \tfrac{7}{2}a^2c^3r + \tfrac{5}{2}a^2c^2st + 7ac^2r + 3acst + \tfrac{7}{2}cr + \tfrac{1}{2}st,$$
$$v_{10} = \tfrac{1}{2}a^4c^5r + \tfrac{1}{2}a^4c^4st + \tfrac{9}{2}a^3c^4r + \tfrac{7}{2}a^3c^3st + \tfrac{27}{2}a^2c^3r + \tfrac{15}{2}a^2c^2st + 15ac^2r + 5acst + \tfrac{9}{2}cr + \tfrac{1}{2}st.$$

Now obviously $w_3 < v_6$. Moreover, $v_6 < w_5 < v_{10}$, so the only possibility here is $w_5 = v_8$ and only if $b^2 > a^3c$. But from $w_1 = v_2 = \frac{1}{2}(cr + st)$, we have the estimates

$$(t-1)^4 v_2 < w_5 = v_8 < v_2 s^6,$$

which implies $0.95b^2c^2 < 1.1303a^3c^3$, i.e. $b^2 < 1.19a^3c$. Furthermore, we get this case when $\{a, b, d_-, c\}$ is a proper regular quadruple (see the proof of [11, Lemma 9]). So we can deduce $0 < d_- < b$. Otherwise we would get $c > b^2$, a contradiction. Now, $w_5 = v_8$ implies

$$\tfrac{3}{2}bst - 3ast \equiv r \pmod{c}.$$

If we multiply both sides by st we get

$$24b - 48a \equiv rst \pmod{c}.$$

Now it can be checked that the left side of this congruence is less than c. We see that a problem can only occur for small a and d_-, because we know $c > b + ad_-b$. But it is easy to check by computer that triples which satisfy $c < 24b$ and $a^3c < b^2 < 1.19a^3c$ do not exist. Here we use the fact that $c > \frac{1}{4}aeb$, because otherwise $d_- = 0$, and see what is happening for $ae \leq 96$, where a and e satisfy the conditions from Lemma 4. From $0 < d_- < b$, we conclude

$$2a + 2c + abc < rst < 2a + 2b + 2c + abc.$$

But then $24b-48a=\alpha$ for some α with $2a<\alpha<2a+2b$. We also know that $b>a\sqrt{c}>8.94a$, which gives us a contradiction because then $24b-48a>\alpha$.

The case $z_1=-s$, $z_0=\frac{1}{2}(cr-st)$ can be proven in exactly the same way.

Case 3. Let now $w_{2n}=v_{2m+1}$. Then $|z_0|=t$, $|z_1|=\frac{1}{2}(cr-st)$ and $z_0z_1<0$. But this case is exactly the same as the previous one. We get similar congruences. This case also comes from the case when $\{a,b,d_-,c\}$ is a proper regular quadruple.

Case 4. Let us now assume $w_{2n}=v_{2m}$. The case $|z_0|=|z_1|=\frac{1}{2}(cr-st)$ is the same as the previous two cases, so we can take $|z_0|=|z_1|=2$.

Let first $z_0=z_1=2$. Then it is easy to see that $v_1<w_1<2v_1$. From $(t-1)^3v_1<w_4<2v_1t^3$, we conclude that the only possible equation for w_4 that might have a solution is $w_4=v_6$, because $v_8>(s-1)^7v_1>2t^3v_1>w_4$. In the case $w_4=v_6$, we have

$$(t-1)^3v_1<w_4<2v_1t^3, \qquad (s-1)^5v_1<v_6<s^5v_1,$$

which after short computations gives

$$0.3816a^{2.5}c<b^{1.5}<1.1503a^{2.5}c.$$

Furthermore, we have $4b+2t\equiv 9a+3s \pmod{c}$ (see [11, Lemma 12]). We can also see that $b>11a$. Now we get a contradiction if this congruence is an equality, because $4b+2t>9a+3s$. To prove that, it is enough to show $4b+2t<c$, i.e. $c>12b$. But if $c\neq a+b+2r$, we have $c>\frac{1}{4}aeb$, where e is the positive integer from Lemma 4. So we check if there exist triples $\{a,b,c\}$ such that $ae<48$, $b<c<12b$, and

$$0.3816a^{2.5}c<b^{1.5}<1.1503a^{2.5}c.$$

We get only one such triple, namely $\{1,12,96\}$, but for this triple we have already used Baker–Davenport reduction to prove uniqueness of extension.

If $c=a+b+2r$, we get $s=a+r$ and $t=b+r$. Then $v_1=2a+b+3r$ and $w_1=a+2b+3r$. Here we have somewhat better estimates $v_1<w_1<1.5v_1$. So $w_4=v_6$ implies

$$0.5088a^{2.5}c<b^{1.5}<1.1503a^{2.5}c$$

and $4b+2t\equiv 9a+3s \pmod{c}$. From $c>b$ we get $b>0.5088a^5$ and $b>14a$, if $a=1$. Obviously the left side of the congruence is larger, so we conclude

$$4b+2(r+b)=9a+3(r+a)+A(a+b+2r)$$

for some integer $0<A<6$. This always yields a contradiction, because the left hand side of the equality is larger. To see this, it is enough to show $17a+11r<b$, because then we would have

$$(12+A)a+(3+2A)r+Ab\leq 17a+11r+2r+Ab<(1+A)b+2r\leq 6b+2r.$$

If $a \geq 3$, we have $b > 41.21a$, i.e. $17a + 11r < b$. If $a = 1, 2$ we can repeat the procedure already used. Namely, we have exact values for a, an estimate for b if $b \leq 17a + 11r$, and $c = a + b + 2r$. Checking with a short computer program does not give us any new triple.

The case when $n = 6$ is completely analogous.

The case when $z_0 = z_1 = -2$ gives us a contradiction in the same way, with slightly different congruences. ∎

Proposition 1. *If $\{a, b, c, d\}$ is an irregular $D(4)$-quadruple such that $a < b < c < d$, then at least one of the following two statements is valid*:

(i) $d \geq 0.173c^{6.5}a^{5.5}$.

(ii) $d \geq 0.087c^{3.5}b^{2.5}$ *and* $c > 0.036b^{3.5}$.

Proof. From Lemma 5 we have two possibilities. First, if $n \geq 7$, then from the proof of [11, Lemma 5],

$$z \geq w_7 > (t-1)^6 \frac{c}{2.224\sqrt[4]{bc}} > 0.416b^{2.75}c^{3.75},$$

which implies

$$d = \frac{z^2 - 4}{c} > 0.173c^{6.5}b^{5.5}.$$

The second statement can be proven the same way (see [12, Proposition 1]). ∎

4. Any $D(4)$-quintuple contains a regular quadruple. We have prepared almost everything for the proof of the main theorem; we only need the following lemma.

Lemma 6 (cf. [12, Lemma 8]). *Let $\{a, b, c, d\}$ be a $D(4)$-quadruple such that $a < b < c < d$ and $c > \max\{7b^{11}, 10^{26}\}$. Then $d = d_+$.*

In the proof of the lemma we have used congruence relations together with Bennett's theorem [2, Theorem 3.2] on simultaneous approximation of square roots that are close to 1.

Proof of Theorem 1. Suppose that an irregular $D(4)$-quadruple $\{a, b, c, d\}$ can be extended to a $D(4)$-quintuple $\{a, b, c, d, e\}$, where $a < b < c < d < e$. From Proposition 1 we have two possibilities. First let

$$d \geq 0.087c^{3.5}b^{2.5}, \qquad c > 0.036b^{3.5}.$$

Because this case comes from the irregular quadruple $\{a, b, d_0, c\}$ we know that $c > 4 \cdot 10^7$, which implies $d > 10^{26}$. Furthermore, $d > 7b^{11}$ for $b > 71$. But if $b \leq 71$ we have $c > b^4$, which again implies $d > 7b^{11}$.

Let now

$$d > 0.173c^{6.5}b^{5.5}.$$

It is easy to see $d > 7b^{11}$. Moreover, from $d > 0.173c^6b^6$, we deduce that $d > 10^{26}$ if $bc > 28861$. We can now find all triples $\{a, b, c\}$ such that

$bc \leq 28861$ and $ab^2c > 10^7$. We find 58 of them, and for each we can apply Baker–Davenport reduction using Lemma 5 from [8]. We have done this in Mathematica 5.2. In all cases, in at most four steps of reduction we get $m \leq 3$, which gives us a contradiction. The smallest such triple was $\{4, 143, 195\}$, and the largest $\{81, 85, 332\}$. We needed less than half an hour to finish that with Mathematica. We see that if $d > 10^{26}$ is not satisfied then we cannot extend the triple $\{a, b, c\}$ to an irregular quadruple at all, and we do not consider those cases.

So we have proved $d > \max\{7b^{11}, 10^{26}\}$. Then we can apply Lemma 6 to a quadruple $\{a, b, d, e\}$ and infer that $e = e_+$. But then $e < d(ab + 4) < d^3$. On the other hand, $\{b, c, d, e\}$ is not a regular $D(4)$-quadruple. Then the gap principles from Propositon 1 imply $e > d^3$, a contradiction. ∎

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