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Manin's conjecture on a nonsingular quartic del Pezzo surface

by

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1. Introduction. Let V be the nonsingular del Pezzo surface of degree four defined by the zero locus of the equations

$$0 = x_1x_2 - x_3x_4,$$
$$0 = x_1^2 + x_2^2 + x_3^2 - x_4^2 - 2x_5^2.$$

Let $U \subseteq V$ be formed by deleting the lines from V. Given a rational point $\mathbf{x} = [x_1, \ldots, x_5] \in \mathbb{P}^4(\mathbb{Q})$ with $x_1, \ldots, x_5 \in \mathbb{Z}$ and $\gcd(x_1, \ldots, x_5) = 1$, we define the *height* of $\mathbf{x}$ to be $\|\mathbf{x}\| = \max(|x_1|, \ldots, |x_5|)$. Given $B \geq 1$, the density of rational points on U is specified by the cardinality

$$N_U(B) = \#\{\mathbf{x} \in U \cap \mathbb{P}^4(\mathbb{Q}) : \|\mathbf{x}\| \leq B\}.$$

Manin's conjecture, proposed in [4] for Fano varieties in general, predicts that if the set of rational points on V is nonempty, then

$$N_U(B) = c_V B(\log B)^{\varrho-1}(1 + o(1))$$

as $B \to \infty$, where c_V is a positive constant and ϱ is the rank of the Picard group of V. Our principal result is the following:

Theorem 1.1. $B(\log B)^{\varrho-1} \ll N_U(B) \ll B(\log B)^{\varrho-1}$.

An overview of progress in proving Manin's conjecture for del Pezzo surfaces can be found in [2]. In general, singular del Pezzo surfaces of low degree have proven more tractable than their nonsingular counterparts.

For nonsingular quartic del Pezzo surfaces, the best result until now is due to Salberger, who proved $N_U(B) \ll B^{1+\varepsilon}$ for any $\varepsilon > 0$, provided V contains a rational conic; this work was presented at the 2001 Budapest conference *Higher dimensional varieties and rational points*. Our result refines Salberger's.

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Both bounds comprising Theorem 1.1 involve fibering V into a family of conics; this allows us to reduce the problem of estimating $N_U(B)$ to the problem of estimating the density of certain rational points on these conics. The same idea is central to Salberger's result; our improved bound stems from tighter control on the uniformity of bounds for rational points on the conics. The method appears to be applicable in a far more general setting, and we intend to explore this in a future paper.

2. The constant ϱ. We begin by recounting some geometry of quartic del Pezzo surfaces. We refer the reader to [5] for a comprehensive exposition.

In general, a nonsingular quartic del Pezzo surface X contains 16 lines, each of which intersects exactly five others. Given any subset of five pairwise skew lines $L_1, \dots, L_5$, X is isomorphic to $\mathbb{P}^2$ blown up along five points $P_1, \dots, P_5$ in general position such that $L_1, \dots, L_5$ are the preimages of those points under the blowup. Moreover, there exists a unique line L_0 intersecting $L_1, \dots, L_5$; L_0 is the preimage of the unique conic on $\mathbb{P}^2$ through $P_1, \dots, P_5$.

Let $K_0, \dots, K_6$ denote the linear equivalence classes of $L_0, \dots, L_6$, respectively, and K denote the class of the preimage of a line on $\mathbb{P}^2$. Then

$$K_0 \sim 2K - (K_1 + \dots + K_5). \tag{2.1}$$

The *geometric* Picard group of X—that is, the Picard group of X defined over an extension E of minimal degree over $\mathbb{Q}$ such that all the lines on X are defined over E—has a basis $\{K, K_1, \dots, K_5\}$. The Picard group of X is that part of the geometric Picard group invariant under the action of $\mathrm{Gal}(E/\mathbb{Q})$.

The 16 lines on V have the following parametrizations:

$$\begin{array}{llll}
L_1: & [a,b,a,b,a], & L_2: & [a,b,a,b,-a],\\
L_3: & [a,b,-a,-b,a], & L_4: & [a,b,-a,-b,-a],\\
L_5: & [a,b,b,a,b], & L_6: & [a,b,b,a,-b],\\
L_7: & [a,b,-b,-a,b], & L_8: & [a,b,-b,-a,-b],\\
L_9: & [a,b,ia,-ib,b], & L_{10}: & [a,b,ia,-ib,-b],\\
L_{11}: & [a,b,-ia,ib,b], & L_{12}: & [a,b,-ia,ib,-b],\\
L_{13}: & [a,b,-ib,ia,a], & L_{14}: & [a,b,-ib,ia,-a],\\
L_{15}: & [a,b,ib,-ia,a], & L_{16}: & [a,b,ib,-ia,-a].
\end{array}$$

Note that all the lines are defined over $\mathbb{Q}(i)$. Let $K_0, \dots, K_5$ denote the classes of L_5, L_1, L_4, L_6, L_9 and L_{11}, respectively. Note that the latter five lines are pairwise skew, and that they are intersected by L_5. Let K denote the class of the preimage of a line on $\mathbb{P}^2$. In view of (2.1), since K_0, K_1, K_2, K_3 and $K_4 + K_5$ are invariant under the action of $\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})$, so too is K; and since $\{K, K_1, \ldots, K_5\}$, being a basis, is a linearly independent set, the set $\{K, K_1, K_2, K_3, K_4 + K_5\}$ is also linearly independent. Therefore the Picard group of V has rank at least 5. Since not all the lines on V are invariant under the action of $\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})$, we conclude that the Picard group of V has rank exactly 5.

3. The lower bound

3.1. *Preliminaries.* Let $B > 0$ be given and

$$P = \{(r, s) : s \text{ is even}, \gcd(r, s) = 1 \text{ and } 1 \le r, s \le B^{1/100}\}.$$

Given $(r, s) \in P$, the first quadric of V is satisfied by taking $x_1 = rX_1$, $x_2 = sX_2$, $x_3 = sX_1$ and $x_4 = rX_2$; and setting $x_5 = X_3$, the second quadric of V is a ternary quadric $0 = Q_{r,s}(\mathbf{X})$, where

$$Q_{r,s}(\mathbf{X}) = (r^2 + s^2)X_1^2 - (r^2 - s^2)X_2^2 - 2X_3^2.$$

If $\gcd(X_1, X_2, X_3) = 1$, then $\|\mathbf{x}\| \le B$ is implied by the bounds

$$|X_1|, |X_2| \le \frac{B}{\max(r, s)}, \qquad |X_3| \le B. \tag{3.1}$$

Let

$$N_{r,s} = \#\{\mathbf{X} : 0 = Q_{r,s}(\mathbf{X}), \gcd(X_1, X_2, X_3) = 1 \text{ and (3.1) holds}\},$$

and let P_i denote the set of pairs $(r, s) \in P$ in the dyadic ranges

$$2^{i-1} = R_i < r \le 2R_i = 2^i, \qquad 2^i = S_i < s \le 2S_i = 2^{i+1}.$$

(Note that, given $(r, s) \in P_i$ for any i, we have $r < s$.) Then

$$N_U(B) \gg \sum_i \sum_{(r,s)\in P_i} N_{r,s}, \tag{3.2}$$

where the i are summed over those values such that the sets P_i are nonempty.

3.2. *The cardinality $N_{r,s}$.* Let $(r, s) \in P_i$ be given. We estimate $N_{r,s}$ by parametrizing a subset of rational points on the quadric $0 = Q_{r,s}(\mathbf{X})$.

We begin by observing that $[1, 1, s]$ is a point on $0 = Q_{r,s}(\mathbf{X})$. We fix a nonzero integer constant c, and consider all points on the quadric of the form

$$\mathbf{X} = [c + x + 2sy, c + x, cs],$$

where (x, y) is an integer pair satisfying the coprimality condition

$$\gcd(x, 2sy) = 1. \tag{3.3}$$

Note that distinct pairs (x, y) parametrize distinct points $\mathbf{X}$. We proceed to eliminate the constant c. Substituting $\mathbf{X}$ back into $0 = Q_{r,s}(\mathbf{X})$, we get

$$0 = (r^2 + s^2)((x + 2sy)^2 + 2c(x + 2sy)) - (r^2 - s^2)(x^2 + 2cx).$$

We rearrange this to get

$$cf_{r,s}(x, y) = -(r^2 + s^2)(x + 2sy)^2 + (r^2 - s^2)x^2,$$

where

$$f_{r,s}(x, y) = 2(r^2 + s^2)(x + 2sy) - 2x(r^2 - s^2).$$

We simplify $\mathbf{X}$ by multiplying each of its components by $f_{r,s}(x, y)$ and then dividing out by s^2, getting $\mathbf{X} = [f_{1,r,s}(x, y), f_{2,r,s}(x, y), f_{3,r,s}(x, y)]$, where

$$\begin{aligned} f_{1,r,s}(x, y) &= x^2 + 4sxy + 2(r^2 + s^2)y^2, \\ f_{2,r,s}(x, y) &= x^2 - 2(r^2 + s^2)y^2, \\ f_{3,r,s}(x, y) &= -sx^2 - 2(r^2 + s^2)xy - 2s(r^2 + s^2)y^2. \end{aligned}$$

Now given an integer pair (x, y) satisfying (3.3), the forms $f_{1,r,s}(x, y)$, $f_{2,r,s}(x, y)$ and $f_{3,r,s}(x, y)$ may have a nontrivial common divisor:

LEMMA 3.1. *Let (x, y) be an integer pair satisfying* (3.3). *Then the greatest common divisor of $f_{1,r,s}(x, y)$, $f_{2,r,s}(x, y)$ and $f_{3,r,s}(x, y)$ is equal to*

$$\gcd(x, r^2 + s^2) \gcd(x + 2sy, r^2 - s^2).$$

Proof. Note that

$$f_{1,r,s}(x, y) + f_{2,r,s}(x, y) = 2x(x + 2sy).$$

Now 2, x and $x + 2sy$ are pairwise coprime; hence the greatest common divisor of $f_{1,r,s}(x, y)$, $f_{2,r,s}(x, y)$ and $f_{3,r,s}(x, y)$ is equal to the product of the factors $\gcd(2, f_{2,r,s}(x, y), f_{3,r,s}(x, y))$, $\gcd(x, f_{2,r,s}(x, y), f_{3,r,s}(x, y))$ and $\gcd(x + 2sy, f_{2,r,s}(x, y), f_{3,r,s}(x, y))$. We denote these factors F_1, F_2 and F_3, respectively, and simplify each in turn. For the first, (3.3) implies that x, hence $f_{2,r,s}(x, y)$, is odd; thus $F_1 = 1$. For the second, we again apply (3.3), getting

$$F_2 = \gcd(x, 2(r^2 + s^2)y^2, 2s(r^2 + s^2)y^2) = \gcd(x, r^2 + s^2).$$

For the third, note that $f_{2,r,s}(x, y) = (x + 2sy)(x - 2sy) - 2(r^2 - s^2)y^2$ and $f_{3,r,s}(x, y) = -(x + 2sy)(sx + 2r^2y) + 2s(r^2 - s^2)y^2$; hence

$$F_3 = \gcd(x + 2sy, 2(r^2 - s^2)y^2, 2s(r^2 - s^2)y^2) = \gcd(x + 2sy, r^2 - s^2),$$

which completes the proof. ∎

Suppose we have $\gcd(x, r^2 + s^2) \gcd(x + 2sy, r^2 - s^2) = n$. Then, given a point $\mathbf{X} = [f_{1,r,s}(x, y), f_{2,r,s}(x, y), f_{3,r,s}(x, y)]$, the bounds (3.1) are implied by the bounds

$$\frac{|f_{1,r,s}(x,y)|}{n}, \frac{|f_{2,r,s}(x,y)|}{n} \le \frac{B}{s}, \qquad \frac{|f_{3,r,s}(x,y)|}{n} \le B,$$

which are themselves implied by the bounds

$$1 \le x \le X = \left(\frac{Bn}{4s}\right)^{1/2}, \qquad |y| \le Y = \left(\frac{Bn}{16s^3}\right)^{1/2}.$$

For convenience, we let $z = x + 2sy$, which allows us to replace the above bounds with $1 \le x, z \le X$.

We estimate $N_{r,s}$ by indexing the pairs (x,z) contributing to $N_{r,s}$ according to the greatest common divisor of the components of $\mathbf{X}$. Let

$$N_{n,r,s} = \#\left\{(x,z) : \begin{array}{l} \gcd(x,sz)=1,\ 2s \mid x-z,\ 1 \le x,z \le X, \\ \gcd(x,r^2+s^2)\gcd(z,r^2-s^2) = n \end{array}\right\}.$$

Then

$$N_{r,s} \ge \sum_{n \ge 1} N_{n,r,s}.$$

The most cumbersome condition on $N_{n,r,s}$ is the last. In order to keep track of it, we redefine $N_{n,r,s}$ in terms of positive integer pairs (a,b), where $\gcd(x,r^2+s^2) = a$, $\gcd(z,r^2-s^2) = b$ and $ab = n$. We write $x = au$, $r^2+s^2 = ac$, $z = bv$ and $r^2-s^2 = bd$, where

$$\gcd(u,c) = 1 \quad \text{and} \quad \gcd(v,d) = 1. \tag{3.4}$$

The last condition on $N_{n,r,s}$ is implicit in these definitions. The coprimality condition $\gcd(x,sz) = 1$ is implied by

$$\gcd(a,v) = \gcd(u,b) = 1 = \gcd(u,v) = 1 = \gcd(u,s) = 1; \tag{3.5}$$

the divisibility condition $2s \mid x-z$ is simply restated as

$$2s \mid au - bv; \tag{3.6}$$

and the bounds $1 \le x, z \le X$ are implied by the bounds

$$1 \le u \le U = \left(\frac{Bb}{4as}\right)^{1/2}, \qquad 1 \le v \le V = \left(\frac{Ba}{4bs}\right)^{1/2}. \tag{3.7}$$

Thus, defining

$$N_{a,b,r,s} = \#\{(u,v) : \text{(3.4)–(3.7) hold}\},$$

we have

$$N_{r,s} \ge \sum_{a \mid r^2+s^2} \sum_{b \mid r^2-s^2} N_{a,b,r,s}. \tag{3.8}$$

3.3. *The cardinality* $N_{a,b,r,s}$. Let $(r,s) \in P_i$, $a \mid r^2+s^2$ and $b \mid r^2-s^2$ be given. We estimate $N_{a,b,r,s}$ by fixing u and then estimating the number of v such that (u,v) contributes to $N_{a,b,r,s}$. Given u such that $\gcd(u,s)=1$, let

$$N_{u,a,b,r,s} = \#\left\{ v : \begin{array}{l} \gcd(v,d) = \gcd(v,a) = \gcd(v,u) = 1, \\ 2s \mid au - bv,\ 1 \le v \le V \end{array} \right\}.$$

Then

$$N_{a,b,r,s} = \sum_{u} N_{u,a,b,r,s},$$

where the sum is taken over a suitable set of u. We shall define this set below.

We use the Möbius function to pick out the coprimality conditions on $N_{u,a,b,r,s}$. Let

$$N'_{u,a,b,r,s}(n_1,n_2,n_3) = \#\{v : \operatorname{lcm}(n_1,n_2,n_3) \mid v,\ 2s \mid au - bv,\ 1 \le v \le V\}.$$

Then

$$N_{u,a,b,r,s} = \sum_{n_1 \mid d} \sum_{n_2 \mid a} \sum_{n_3 \mid u} \mu(n_1)\mu(n_2)\mu(n_3) N'_{u,a,b,r,s}(n_1,n_2,n_3).$$

Let n_1, n_2 and n_3 be in the range of summation above. Then $\gcd(2s,n_1)=1$, since $n_1 \mid r^2-s^2$ and $\gcd(2s,r^2-s^2)=1$; $\gcd(2s,n_2)=1$, since $n_2 \mid r^2+s^2$ and $\gcd(2s,r^2+s^2)=1$; and $\gcd(2s,n_3)=1$, since $\gcd(u,s)=1$ and s is even. Moreover, $\gcd(2s,b)=1$, since $b \mid r^2-s^2$. Thus

$$N'_{u,a,b,r,s}(n_1,n_2,n_3) = \frac{V}{2s \cdot \operatorname{lcm}(n_1,n_2,n_3)} + O(1),$$

and

$$N_{u,a,b,r,s} = \sum_{n_1 \mid d} \sum_{n_2 \mid a} \sum_{n_3 \mid u} \mu(n_1)\mu(n_2)\mu(n_3) \left(\frac{V}{2s \cdot \operatorname{lcm}(n_1,n_2,n_3)} + O(1) \right).$$

We estimate $N_{a,b,r,s}$ by summing $N_{u,a,b,r,s}$ over the set

$$P_{a,b,r,s}(n_3) = \{u : \gcd(u,c) = \gcd(u,b) = \gcd(u,s) = 1,\ n_3 \mid u,\ 1 \le u \le U\};$$

that is, $N_{a,b,r,s}$ is equal to

$$\sum_{n_1 \mid d} \sum_{n_2 \mid a} \sum_{n_3 \le U} \sum_{u \in P_{a,b,r,s}(n_3)} \mu(n_1)\mu(n_2)\mu(n_3) \left(\frac{V}{2s \cdot \operatorname{lcm}(n_1,n_2,n_3)} + O(1) \right).$$

Since the cardinality of $P_{a,b,r,s}(n_3)$ has an upper bound U/n_3, the contribution to $N_{a,b,r,s}$ of the error term above is of order at most

$$U \sum_{n_1 \mid d} \sum_{n_2 \mid a} \sum_{n_3 \le U} \frac{1}{n_3} \le U(R_i S_i U)^{\varepsilon}$$

for any $\varepsilon > 0$, provided i and B are sufficiently large; that is,

$$N_{a,b,r,s} = \frac{V}{2s} \sum_{n_1 | d} \sum_{n_2 | a} \sum_{n_3 \le U} \sum_{u \in P_{a,b,r,s}(n_3)} \frac{\mu(n_1)\mu(n_2)\mu(n_3)}{\operatorname{lcm}(n_1, n_2, n_3)} + O(U(R_i S_i U)^{\varepsilon}).$$

We now estimate the cardinality of $P_{a,b,r,s}(n_3)$ more precisely. As in the case of $N_{u,a,b,r,s}$, we use the Möbius function to pick out coprimality conditions on the set. Let

$$P'_{a,b,r,s}(n_3, m_1, m_2, m_3) = \{u : \operatorname{lcm}(n_3, m_1, m_2, m_3) \mid u,\ 1 \le u \le U\}.$$

Then

$$\# P_{a,b,r,s} = \sum_{m_1 | c} \sum_{m_2 | b} \sum_{m_3 | s} \mu(m_1)\mu(m_2)\mu(m_3) \# P'_{a,b,r,s}(n_3, m_1, m_2, m_3).$$

Now

$$\# P'_{a,b,r,s}(n_3, m_1, m_2, m_3) = \frac{U}{\operatorname{lcm}(n_3, m_1, m_2, m_3)} + O(1).$$

The contribution to $N_{a,b,r,s}$ of the error term is of order at most

$$\frac{V}{s} \sum_{n_1 | d} \sum_{n_2 | a} \sum_{n_3 \le U} \sum_{m_1 | c} \sum_{m_2 | b} \sum_{m_3 | s} \frac{1}{\operatorname{lcm}(n_1, n_2, n_3)} \le \frac{V}{s} (R_i S_i U)^{\varepsilon}$$

for any $\varepsilon > 0$, provided i and B are sufficiently large; that is, $N_{a,b,r,s}$ equals

$$\frac{UV}{2s} \sum_{n_1 | d} \sum_{n_2 | a} \sum_{n_3 \le U} \sum_{m_1 | c} \sum_{m_2 | b} \sum_{m_3 | s} \frac{\mu(n_1)\mu(n_2)\mu(n_3)\mu(m_1)\mu(m_2)\mu(m_3)}{\operatorname{lcm}(n_1, n_2, n_3) \cdot \operatorname{lcm}(n_3, m_1, m_2, m_3)} + O\biggl(\biggl(U + \frac{V}{s}\biggr)(R_i S_i U)^{\varepsilon}\biggr).$$

Finally, we estimate the main term above. Let $T_{a,b,r,s}$ denote this term, and $T'_{a,b,r,s}$ denote $T_{a,b,r,s}$ but with the difference that, in $T'_{a,b,r,s}$, we sum over all positive integers n_3 rather than over the range $n_3 \le U$. Now the difference between $T'_{a,b,r,s}$ and $T_{a,b,r,s}$ is of order at most

$$\frac{UV}{s} \sum_{n_1 | d} \sum_{n_2 | a} \sum_{n_3 > U} \sum_{m_1 | c} \sum_{m_2 | b} \sum_{m_3 | s} \frac{1}{n_3^2} \le \frac{UV}{s} (R_i S_i)^{\varepsilon} \sum_{n_3 > U} \frac{1}{n_3^2} \le \frac{V}{s} (R_i S_i)^{\varepsilon}$$

for any $\varepsilon > 0$, provided i and B are sufficiently large; that is,

$$N_{a,b,r,s} = T'_{a,b,r,s} + O\biggl(\biggl(U + \frac{V}{s}\biggr)(R_i S_i U)^{\varepsilon}\biggr).$$

In order to estimate $T'_{a,b,r,s}$, we define the condition

$$n_1 \mid d, \quad n_2 \mid a, \quad m_1 \mid c, \quad m_2 \mid b \quad \text{and} \quad m_3 \mid s, \tag{3.9}$$

and the function $f_{a,b,r,s}(n_1,n_2,n_3,m_1,m_2,m_3)$ to be equal to

$$\begin{cases} \dfrac{\mu(n_1)\mu(n_2)\mu(n_3)\mu(m_1)\mu(m_2)\mu(m_3)}{\operatorname{lcm}(n_1,n_2,n_3)\cdot\operatorname{lcm}(n_3,m_1,m_2,m_3)} & \text{if (3.9) holds,} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$T'_{a,b,r,s} = \frac{UV}{2s} \sum_{\substack{n_i,m_i \geq 1 \\ \text{for } 1 \leq i \leq 3}} f_{a,b,r,s}(n_1,n_2,n_3,m_1,m_2,m_3).$$

Because $f_{a,b,r,s}$ is multiplicative and we have

$$\sum_{\substack{n_i,m_i \geq 1 \\ \text{for } 1 \leq i \leq 3}} |f_{a,b,r,s}(n_1,n_2,n_3,m_1,m_2,m_3)| \leq \sum_{n_1|d} \sum_{n_2|a} \sum_{m_1|c} \sum_{m_2|b} \sum_{m_3|2s} \sum_{n_3 \geq 1} \frac{1}{n_3^2},$$

which converges, we may write

$$T'_{a,b,r,s} = \frac{UV}{2s} \prod_p f_{p,a,b,r,s},$$

where the product is taken over all primes p, and the local factors $f_{p,a,b,r,s}$ are defined by

$$f_{p,a,b,r,s} = \sum_{\substack{e_i,e'_i \in \{0,1\} \\ \text{for } 1 \leq i \leq 3}} f_{a,b,r,s}(p^{e_1},p^{e_2},p^{e_3},p^{e'_1},p^{e'_2},p^{e'_3}).$$

We evaluate $f_{p,a,b,r,s}$ directly, in three cases. If p does not divide any element in the set $\{a,b,c,d,s\}$, then $f_{p,a,b,r,s} = 1-p^{-2}$; if p divides exactly one element in the set $\{a,b,c,d,s\}$, then $f_{p,a,b,r,s} = 1-p^{-1}$; and if p divides exactly two elements in the set $\{a,b,c,d,s\}$—that is, either $p \mid a$ and $p \mid c$, or $p \mid b$ and $p \mid d$—then $f_{p,a,b,r,s} = (1-p^{-1})^2$. Hence

$$\begin{aligned} T'_{a,b,r,s} &\geq \frac{UV}{2s} \prod_{p \nmid s\Delta_{r,s}} \left(1-\frac{1}{p^2}\right) \prod_{p|s} \left(1-\frac{1}{p}\right) \prod_{p|\Delta_{r,s}} \left(1-\frac{1}{p}\right)^2 \\ &\gg \frac{UV}{s} \prod_{p|s} \left(1-\frac{1}{p}\right) \prod_{p|\Delta_{r,s}} \left(1-\frac{1}{p}\right)^2, \end{aligned}$$

where $\Delta_{r,s}$ denotes $|r^4-s^4|$, and the relation $\gg$ does not depend on our choice of a, b, r or s. (For the remainder of this section we assume that all relations $\gg$ are thus independent.) Thus

$$N_{a,b,r,s} \gg \frac{UV}{s} \prod_{p|s} \left(1-\frac{1}{p}\right) \prod_{p|\Delta_{r,s}} \left(1-\frac{1}{p}\right)^2 + O\left(\left(U+\frac{V}{s}\right)(R_i S_i U)^{\varepsilon}\right)$$

for any $\varepsilon > 0$, provided i and B are sufficiently large. We conclude that

$$N_{a,b,r,s} \gg \frac{B}{s^2} \prod_{p|s} \left(1 - \frac{1}{p}\right) \prod_{p|\Delta_{r,s}} \left(1 - \frac{1}{p}\right)^2. \tag{3.10}$$

3.4. *The cardinality $N_U(B)$.* For convenience we define the multiplicative function

$$f(n) = \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

for any $n \in \mathbb{N}$, with $f(1) = 1$. With this notation, and in view of the bounds (3.2), (3.8) and (3.10), we have

$$\begin{aligned} N_U(B) &\gg B \sum_i \frac{1}{S_i^2} \sum_{(r,s)\in P_i} \sum_{\substack{a|r^2+s^2 \\ b|r^2-s^2}} f(s) f(\Delta_{r,s})^2 \\ &\geq B \sum_i \frac{1}{S_i^2} \sum_{(r,s)\in P_i} d(\Delta_{r,s}) f(s) f(\Delta_{r,s})^2, \end{aligned}$$

where we sum over those i such that the P_i are nonempty. We may restrict the range of summation on the right-hand side above without invalidating the bound, and it will be useful to impose the condition that, for any pair (r, s) in that range of summation, s is not only even but divisible by 6; that is,

$$N_U(B) \gg B \sum_i \frac{1}{S_i^2} \sum_{\substack{S_i<s\leq 2S_i \\ 6|s}} f(s) \sum_{\substack{R_i<r\leq 2R_i \\ \gcd(r,s)=1}} d(\Delta_{r,s}) f(\Delta_{r,s})^2. \tag{3.11}$$

We estimate the inner sum on the right-hand side of (3.11). Let s be in the range of summation. By the Möbius inversion formula, we have

$$d(n) f(n)^2 = \sum_{m|n} f'(m)$$

for any $n \in \mathbb{N}$ if, and only if,

$$f'(n) = \sum_{m|n} \mu\left(\frac{n}{m}\right) d(m) f(m)^2$$

for any $n \in \mathbb{N}$. Now f' is multiplicative, and given a prime power p^e with $e \geq 1$, we have

$$f'(p^e) = \begin{cases} 2(1 - 1/p)^2 - 1 & \text{if } e = 1, \\ (1 - 1/p)^2 & \text{otherwise;} \end{cases}$$

that is, $f'(p^e) > 0$ for any $e \in \mathbb{N}$ provided $p \geq 5$. No primes smaller than 5 divide $\Delta_{r,s}$, since 2 and 3 both divide s; hence

$$d(\Delta_{r,s})f(\Delta_{r,s})^2 = \sum_{m\mid\Delta_{r,s}} f'(m) \geq \sum_{\substack{m\mid\Delta_{r,s}\\ m\leq R_i^{1/2}}} f'(m),$$

where $f'(m)$ is nonnegative over the range of summation. (It will shortly become clear why we impose a bound on m.) Thus

$$\sum_{\substack{R_i<r\leq 2R_i\\ \gcd(r,s)=1}} d(\Delta_{r,s})f(\Delta_{r,s})^2 \geq \sum_{\substack{R_i<r\leq 2R_i\\ \gcd(r,s)=1}} \sum_{\substack{m\mid\Delta_{r,s}\\ m\leq R_i^{1/2}}} f'(m).$$

We use the Möbius function to pick out the coprimality condition on the right-hand side. As an intermediate step, we define

$$N_{m,s} = \#\{r : R_i < r \leq 2R_i,\ \gcd(r,s) = 1,\ m \mid \Delta_{r,s}\}.$$

Then

$$\sum_{\substack{R_i<r\leq 2R_i\\ \gcd(r,s)=1}} d(\Delta_{r,s})f(\Delta_{r,s})^2 \geq \sum_{\substack{m\leq R_i^{1/2}\\ \gcd(m,s)=1}} f'(m)N_{m,s}.$$

We impose the condition that $\gcd(m,s) = 1$ on the range of summation on the right-hand side to ensure that the $N_{m,s}$ we sum are nonzero.

Let $N_{m,s}$ be nonzero; then the congruence $r^4 \equiv s^4 \pmod{m}$ is soluble in r, with $F(m)$ solutions $(\mathrm{mod}\ m)$, where F is a multiplicative function with

$$F(2^e) = \begin{cases} 2^{e-1} & \text{if } e \leq 4,\\ 8 & \text{if } e > 4,\end{cases} \qquad F(p^e) = \begin{cases} 2 & \text{if } p \equiv 3 \pmod{4},\\ 4 & \text{if } p \equiv 1 \pmod{4}.\end{cases}$$

Given a solution $r \equiv c \pmod{m}$, we define

$$N_{c,m,s} = \#\{r : R_i < r \leq 2R_i,\ \gcd(r,s) = 1 \text{ and } r \equiv c \pmod{m}\},$$
$$N_{c,s}(n) = \#\{r : R_i < r \leq 2R_i,\ n \mid r \text{ and } r \equiv c \pmod{m}\};$$

then

$$N_{c,m,s} = \sum_{n\mid s} \mu(n)N_{c,s}(n).$$

Let $n \mid s$. Then $\gcd(n,m) = 1$, since $\gcd(m,s) = 1$. Thus

$$N_{c,s}(n) = \frac{R_i}{nm} + O(1).$$

The contribution to $N_{c,m,s}$ of the error term above is of order at most $d(s)$; that is,

$$N_{c,m,s} = \frac{R_i}{m}\sum_{n\mid s}\frac{\mu(n)}{n} + O(d(s)) = \frac{R_i f(s)}{m} + O(d(s)) \gg \frac{R_i f(s)}{m}.$$

(The above bound follows from the fact that $m \le R_i^{1/2}$.) Thus

$$N_{m,s} \gg \frac{F(m) R_i f(s)}{m},$$

and

$$\sum_{\substack{R_i < r \le 2R_i \\ \gcd(r,s)=1}} d(\Delta_{r,s}) f(\Delta_{r,s})^2 \gg R_i f(s) \sum_{\substack{m \le R_i^{1/2} \\ \gcd(m,s)=1}} \frac{F(m) f'(m)}{m}.$$

In view of the bound (3.11) and the fact that R_i and S_i are of the same order, we conclude that

$$N_U(B) \gg B \sum_i \frac{1}{S_i} \sum_{\substack{m \le R_i^{1/2} \\ \gcd(m,6)=1}} \frac{F(m) f'(m)}{m} \sum_{\substack{S_i < s \le 2S_i \\ 6 \mid s \\ \gcd(m,s)=1}} f(s)^2, \tag{3.12}$$

where we sum over those i such that the sets P_i are nonempty. (We impose the condition $\gcd(m,6) = 1$ for convenience.)

We proceed to estimate the inner sum on the right-hand side of (3.12). Let $s = 6t$ and $S_i/6 = T_i$. Then

$$\sum_{\substack{S_i < s \le 2S_i \\ 6 \mid s \\ \gcd(m,s)=1}} f(s)^2 \gg \sum_{\substack{T_i < t \le 2T_i \\ \gcd(m,t)=1}} f(t)^2.$$

By the Cauchy–Schwarz inequality, we have

$$\sum_{\substack{T_i < t \le 2T_i \\ \gcd(m,t)=1}} f(t)^2 \ge \Big(\sum_{\substack{T_i < t \le 2T_i \\ \gcd(m,t)=1}} 1 \Big)^{-1} \Big(\sum_{\substack{T_i < t \le 2T_i \\ \gcd(m,t)=1}} f(t) \Big)^2.$$

We estimate the two sums on the right-hand side, using the following two standard relations: first, given any positive integer constant c, we have

$$\#\{n : n \le N \text{ and } \gcd(n,c) = 1\} = \frac{N\phi(c)}{c} + O(c^{\varepsilon}) \tag{3.13}$$

for any $\varepsilon > 0$; and second,

$$\sum_{n \le N} \phi(n) = \frac{3N^2}{\pi^2} + O(N \log N). \tag{3.14}$$

For the first sum on the right-hand side of the Cauchy–Schwarz inequality, we have, by (3.13),

$$\sum_{\substack{T_i < t \le 2T_i \\ \gcd(m,t)=1}} 1 \ll \frac{T_i \phi(m)}{m} = T_i f(m).$$

For the second sum, we have

$$\sum_{\substack{T_i<t\le 2T_i\\ \gcd(m,t)=1}} f(t) = \sum_{\substack{T_i<t\le 2T_i\\ \gcd(m,t)=1}} \frac{\phi(t)}{t} \gg \frac{1}{T_i} \sum_{\substack{T_i<t\le 2T_i\\ \gcd(m,t)=1}} \phi(t).$$

By (3.13), we get

$$\sum_{\substack{T_i<t\le 2T_i\\ \gcd(m,t)=1}} \phi(t) \ge \sum_{\substack{T_i<t\le 2T_i\\ \gcd(ms,t)=1}} \phi(t) \gg T_i \sum_{s\le T_i} \frac{\phi(ms)}{ms} \ge T_i f(m) \sum_{s\le T_i} f(s);$$

that is,

$$\sum_{\substack{T_i<t\le 2T_i\\ \gcd(m,t)=1}} f(t) \gg f(m) \sum_{s\le T_i} f(s) \gg T_i f(m),$$

where the second inequality follows from (3.14). Thus

$$\sum_{\substack{T_i<t\le 2T_i\\ \gcd(m,t)=1}} f(t)^2 \gg T_i f(m) \gg S_i f(m);$$

and, in view of (3.12),

$$N_U(B) \gg B \sum_i \sum_{\substack{m\le R_i^{1/2}\\ \gcd(m,6)=1}} \frac{F(m)f(m)f'(m)}{m}, \tag{3.15}$$

where we sum over i such that $P_i \neq \emptyset$.

We now estimate the inner sum on the right-hand side of (3.15). Since F, f and f' are all multiplicative, we consider the corresponding Dirichlet series

$$D(z) = \sum_{\substack{m\ge 1\\ \gcd(m,6)=1}} \frac{F(m)f(m)f'(m)}{m^z},$$

which admits an Euler product

$$D(z) = \prod_{p\ge 5} \left(1 + \frac{F(p)f(p)f'(p)}{p^z} + \sum_{e\ge 2} \frac{F(p^e)f(p^e)f'(p^e)}{p^{ez}}\right),$$

where the product is taken over all primes $p \ge 5$. It is straightforward to rewrite this as $D(z) = \zeta(z)^3 L(z,\chi) F'(z)$, where F' is a holomorphic function bounded on the half-plane $\mathrm{Re}(z) > 3/4$. Hence, by Perron's formula, the inner sum on the right-hand side of (3.15) is equal to

$$\frac{1}{2\pi i} \int_{\varepsilon-iT}^{\varepsilon+iT} \zeta(1+w)^3 L(1+w,\chi) F'(1+w) \frac{M^w}{w}\, dw + O(1).$$

The integrand has a pole of order 4 at $w = 0$. We apply the residue theorem to the rectangular contour with corners at $\varepsilon - iT$, $\varepsilon + iT$, $-1/8 + iT$ and $-1/8 - iT$, and use the bounds

$$\zeta(w), L(w,\chi) \ll |w|^{1/8},$$

which hold provided $\mathrm{Re}(w) \geq 7/8$ and $|w-1| \geq 1/8$. These bounds imply that the integrand along the horizontal segments is of order at most

$$(T^{1/8})^3 T^{1/8}\,\frac{M^{\mathrm{Re}(w)}}{T},$$

where $-1/8 \leq \mathrm{Re}(w) \leq \varepsilon$; that is, the contribution of the integral along the horizontal segments of our contour is of order at most 1. Similarly, the integrand along the vertical segment joining $-1/8 + iT$ to $-1/8 - iT$ is of order at most

$$\frac{(T^{1/8})^3 T^{1/8}}{M^{1/8}};$$

that is, the contribution of the integral along that segment is of order at most

$$\frac{T^{3/2}}{M^{1/8}} = M^{3\varepsilon - 1/8} \ll 1$$

provided $\varepsilon < 1/24$. Hence

$$\sum_{\substack{m \leq M\\ \gcd(m,6)=1}} \frac{F(m)f(m)f'(m)}{m} \gg (\log M)^3 = (\log R_i^{1/2})^3.$$

We insert the above bound into (3.15), getting

$$N_U(B) \gg B \sum_i (\log R_i^{1/2})^3,$$

where we sum over i such that $P_i \neq \emptyset$. Now a set P_i is nonempty provided $2^{i+1} \leq B^{1/100}$, that is, provided we have $i \leq k \log B$ for some fixed constant $k > 0$. Thus

$$N_U(B) \gg B \sum_{i \leq k \log B} (\log R_i^{1/2})^3 \gg B \sum_{i \leq k \log B} (i-1)^3 \gg B(\log B)^4.$$

4. The upper bound

4.1. *Preliminaries.* We define the following projections from V onto $\mathbb{P}^1$:

$$f^{(1)} : [x_1, \ldots, x_5] \mapsto \begin{cases} [x_1, x_3] & \text{if } (x_1, x_3) \neq (0,0),\\ [x_4, x_2] & \text{otherwise,}\end{cases}$$

$$f^{(2)} : [x_1, \ldots, x_5] \mapsto \begin{cases} [x_1, x_4] & \text{if } (x_1, x_4) \neq (0,0),\\ [x_3, x_2] & \text{otherwise.}\end{cases}$$

We have the following lemma:

LEMMA 4.1. $\|f^{(1)}(\mathbf{x})\| \cdot \|f^{(2)}(\mathbf{x})\| \le \|\mathbf{x}\|$ *for all* $\mathbf{x} \in V$.

Proof. Let $\gcd(x_1, x_2, x_3, x_4) = n$, and let m_{ij} denote $\gcd(x_i, x_j)n^{-1}$ for $1 \le i, j \le 4$. Then

$$\frac{x_1}{n} = m_{13}m_{14}, \qquad \frac{x_2}{n} = m_{23}m_{24}, \qquad \frac{x_3}{n} = m_{13}m_{23}, \qquad \frac{x_4}{n} = m_{14}m_{24}.$$

Now either $\|f^{(1)}(\mathbf{x})\| = \|[x_1, x_3]\|$ or $\|f^{(1)}(\mathbf{x})\| = \|[x_4, x_2]\|$; in both cases we get $\|f^{(1)}(\mathbf{x})\| = \|[m_{14}, m_{23}]\|$. Similarly, $\|f^{(2)}(\mathbf{x})\| = \|[m_{13}, m_{24}]\|$. ∎

We define, for $i = 1, 2$,

$$N_U^{(i)}(B) = \#\{\mathbf{x} \in U : \|\mathbf{x}\| \le B \text{ and } \|f^{(i)}(\mathbf{x})\| \le B^{1/2}\}.$$

By Lemma 4.1, we have

$$N_U(B) \le N_U^{(1)}(B) + N_U^{(2)}(B).$$

We will bound the $N_U^{(i)}(B)$. In fact, it suffices to bound $N_U^{(1)}(B)$; the bound for $N_U^{(2)}(B)$ follows by symmetry.

Suppose $\mathbf{x}$ contributes to $N_U^{(1)}(B)$; say $f^{(1)}(\mathbf{x}) = [r, s]$ with $\gcd(r, s) = 1$. Then $\mathbf{x}$ is of the form $[rX_1, sX_2, sX_1, rX_2, x_5]$, where $X_1 = \gcd(x_1, x_3)$ and $X_2 = \gcd(x_2, x_4)$; and, upon setting $x_5 = X_3$, the second quadric of V is a ternary quadric $0 = Q_{r,s}^{(1)}(\mathbf{X})$, where

$$Q_{r,s}^{(1)}(\mathbf{X}) = (r^2 + s^2)X_1^2 - (r^2 - s^2)X_2^2 - 2X_3^2.$$

(Note that the condition $\mathbf{x} \in U$ ensures that $r^4 - s^4 \ne 0$.) The condition $\|\mathbf{x}\| \le B$ implies

$$(4.1) \qquad |X_1|, |X_2| \le \frac{B}{\max(r, s)}, \qquad |X_3| \le B.$$

Thus, defining

$$N_{r,s} = \#\{\mathbf{X} : \gcd(X_1, X_2, X_3) = 1,\ 0 = Q_{r,s}^{(1)}(\mathbf{X}) \text{ and (4.1) holds}\},$$

we have

$$N_U^{(1)}(B) \ll \sum_{\substack{\gcd(r,s)=1 \\ 1 \le r,s \le B^{1/2}}} N_{r,s}.$$

We split the set of suitable pairs (r, s) into dyadic ranges, letting $P_{i,j}$ denote the set of coprime pairs (r, s) in the range

$$2^{i-1} = R_i < r \le 2R_i = 2^i \quad \text{and} \quad 2^{j-1} = S_j < s \le 2S_j = 2^j.$$

The bounds $1 \le r, s \le B^{1/2}$ imply that the indices i and j have an upper bound $i, j \le 2 \log B$. Thus

$$(4.2) \qquad N_U^{(1)}(B) \ll \sum_{i \le 2\log B} \sum_{j \le i} \sum_{(r,s) \in P_{i,j}} N_{r,s} + \sum_{j \le 2\log B} \sum_{i \le j} \sum_{(r,s) \in P_{i,j}} N_{r,s}.$$

We bound the first of the terms on the right-hand side; the second term is dealt with similarly.

4.2. *Tools.* Our first tool, used to estimate $N_{r,s}$, may be found in [3]:

LEMMA 4.2. *Let $f \in \mathbb{Z}[\mathbf{X}]$ be a ternary quadratic form. Let M denote its matrix representation M, let $\Delta = |\det M| \neq 0$, and let Δ_0 denote the highest common factor of the 2×2 minors of M. Let*

$$N = \#\{\mathbf{X} : \gcd(X_1, X_2, X_3) = 1,\ 0 = f(\mathbf{x}) \text{ and } |x_i| \le B_i \text{ for } i = 1,2,3\}.$$

Then

$$N \ll \left(1 + \left(\frac{B_1 B_2 B_3 \Delta_0^2}{\Delta}\right)^{1/3}\right) d(\Delta).$$

We require some notation for our next result. Given $f \in \mathbb{Z}[x]$ with no fixed prime divisors, the multiplicative function $\varrho_f(m)$ denotes the number of solutions $n \pmod m$ of $f(n) \equiv 0 \pmod m$. We collect below some useful results on this function. The first three are classical, and may be found in [6], for example. The last is attributed in [1] to unpublished work by Stephan Daniel; a proof is given in [1, Lemma 2].

LEMMA 4.3. *Let $f \in \mathbb{Z}[x]$ be of degree g, have no fixed prime divisors, and be such that* $\mathrm{Disc}(f) \neq 0$. *Then*:

(a) $\varrho_f(p) \le g$;
(b) $\varrho_f(p^e) \le g p^{e-1}$ *for all* $e \in \mathbb{N}$;
(c) $\varrho_f(p^e) = \varrho_f(p)$ *for all* $e \in \mathbb{N}$, *provided* $p \nmid \mathrm{Disc}(f)$;
(d) $\varrho_f(p^e) \le 2g^3 p^{e(1-1/g)}$ *for all* $e \in \mathbb{N}$.

We are now ready to prove the following:

LEMMA 4.4. *Let $f \in \mathbb{Z}[x]$ be of degree* 4, *have no fixed prime divisors, and be such that* $\mathrm{Disc}(f) \neq 0$. *Let $\alpha, \beta \in (0,1)$ and $N_1, N_2 \ge 2$ be such that $N_2^{\alpha} \le N_2 - N_1 \le N_2$ and $\|f\|^{\beta} \le N_2$. Then the sum*

$$\sum_{N_1 < n \le N_2} d(|f(n)|)$$

is of order at most

$$(N_2 - N_1) \prod_{p \le N_2} \left(1 - \frac{\varrho_f(p)}{p}\right) \exp\left(\sum_{p \le N_2} \frac{d(p)\varrho_f(p)}{p} + c \sum_{p \mid \mathrm{Disc}(f)} \frac{1}{p}\right)$$

for a constant $c > 0$, where the implied constant depends only on α and β.

Proof. This is a special case of the main theorem in [7]. Nair's bound depends implicitly on the discriminant $\mathrm{Disc}(f)$. This dependence arises in two places in [7]. In both instances we may make it explicit or remove it.

The first instance is in [7, Lemma 2], in the implied constant of the bound

$$\sum_{n\le N}\left(\frac{n}{\phi(n)}\right)^4\frac{d(n)\varrho_f(n)}{n}\ll\exp\biggl(\sum_{p\le N}\left(\frac{p}{\phi(p)}\right)^4\frac{d(p)\varrho_f(p)}{p}\biggr).\tag{4.3}$$

We make this dependence explicit. We begin with the fact that

$$\sum_{n\le N}\left(\frac{n}{\phi(n)}\right)^4\frac{d(n)\varrho_f(n)}{n}\le\exp\biggl(\sum_{p\le N}\left(\frac{p}{\phi(p)}\right)^4\sum_{e\ge1}\frac{d(p^e)\varrho_f(p^e)}{p^e}\biggr).$$

We shall make use of the bound

$$\sum_{e\ge E}\frac{e+1}{n^e}\ll\frac{1}{n^E}\left(\frac{n}{n-1}\right)^2,\tag{4.4}$$

which holds for all $n\in\mathbb{N}$. (Here the relation $\ll$ depends only on E.) Now given p such that $p\nmid\mathrm{Disc}(f)$, by Lemma 4.3(a), Lemma 4.3(c) and (4.4), we have

$$\sum_{e\ge2}\frac{d(p^e)\varrho_f(p^e)}{p^e}\le4\sum_{e\ge2}\frac{e+1}{p^e}\ll\frac{1}{p^2}.$$

Likewise, given p such that $p\mid\mathrm{Disc}(f)$, by Lemma 4.3(d) and (4.4), we have

$$\sum_{e\ge8}\frac{d(p^e)\varrho_f(p^e)}{p^e}\le128\sum_{e\ge8}\frac{e+1}{p^{e/4}}\ll\frac{1}{p^2}.$$

Finally, given p such that $p\mid\mathrm{Disc}(f)$, by Lemma 4.3(b), we have

$$\sum_{2\le e<8}\frac{d(p^e)\varrho_f(p^e)}{p^e}\le4\sum_{e<8}\frac{e+1}{p}\ll\frac{1}{p}.$$

These bounds combine to give

$$\sum_{n\le N}\left(\frac{n}{\phi(n)}\right)^4\frac{d(n)\varrho_f(n)}{n}\ll\exp\biggl(\sum_{p\le N}\left(\frac{p}{\phi(p)}\right)^4\frac{d(p)\varrho_f(p)}{p}+c\sum_{p\mid\mathrm{Disc}(f)}\frac{1}{p}\biggr)$$

for a constant $c>0$, where the relation $\ll$ does not depend on $\mathrm{Disc}(f)$. The difference between this bound and (4.3) accounts for the difference between Lemma 4.4 and the main result in [7].

The second place in [7] in which a dependence on $\mathrm{Disc}(f)$ arises is in the author's reduction of the bound [7, (6.3)], where, given a positive integer n such that $N^{1/2}<n\le N$, the bound $\varrho_f(n)\ll N^{1/8}$ is invoked; $\mathrm{Disc}(f)$ figures in the implied constant. We remove the dependence on $\mathrm{Disc}(f)$ by invoking the bound $\varrho_f(n)\ll n^{4/5}$ for all $n\in\mathbb{N}$, where the relation $\ll$ does not depend on $\mathrm{Disc}(f)$; this proves to be sufficient. ∎

We use Lemma 4.4 to prove our version of a result due to Browning and de la Bretèche, which we use to sum our estimates for $N_{r,s}$ over the pairs

$(r,s) \in P_{i,j}$. We require a generalization of the function ϱ_f to binary forms. Let $f \in \mathbb{Z}[x_1, x_2]$ have no fixed prime divisors. Then $\varrho_{f(1,x)}(m)$ denotes the number of solutions $n \pmod m$ of $f(1,n) \equiv 0 \pmod m$, and we define for any prime p the function

$$\varrho_f^*(p) = \begin{cases} \varrho_{f(1,x)}(p) + 1 & \text{if } p \mid f(0,1), \\ \varrho_{f(1,x)}(p) & \text{otherwise.} \end{cases}$$

THEOREM 4.5. *Let $f \in \mathbb{Z}[x_1, x_2]$ be of degree 4, have no fixed prime divisors, and be such that* $\mathrm{Disc}(f) \neq 0$ *and* $f(1,0)f(0,1) \neq 0$. *Let $\alpha, \beta \in (0,1)$ and $N, N_1, N_2 \geq 2$ be such that $N_2^\alpha \leq N_2 - N_1 \leq N_2$ and* $\min(N, N_2) \geq a \max(N, N_2)^{4\beta} \|f\|^\beta$ *for a constant $a > 0$ dependent only on β. Then*

$$\sum_{1 \leq n_1 \leq N} \sum_{N_1 < n_2 \leq N_2} d(|f(n_1, n_2)|) \ll N(N_2 - N_1)T,$$

where

$$T = \prod_{p \mid \mathrm{Disc}(f)} \left(1 + \frac{1}{p}\right)^b \exp\left(c \sum_{p \mid \mathrm{Disc}(f)} \frac{1}{p}\right) \exp\left(\sum_{p \leq \max(N, N_2)} \frac{\varrho_f^*(p)}{p}\right)$$

for constants $b, c > 0$, and the relation $\ll$ depends only on α and β.

Proof. This theorem is an adaptation of [1, Theorem 1]. There, the authors take $n_2 \leq N_2$; we take a shorter range of summation and appeal to Lemma 4.4. The condition that f have no fixed prime divisors is required in the proof of the main theorem in [7], and implicit in the proof of Lemma 4.4. As in [1, §3], we fix n_1 and consider the sum

$$\sum_{N_1 < n_2 \leq N_2} d(|f(n_2)|).$$

By Lemma 4.4, the above sum has an upper bound of order at most

$$(N_2 - N_1) \prod_{p \leq N_2} \left(1 - \frac{\varrho_f(p)}{p}\right) \exp\left(\sum_{p \leq N_2} \frac{d(p)\varrho_f(p)}{p} + c \sum_{p \mid n_1 \mathrm{Disc}(f)} \frac{1}{p}\right)$$

for a constant $c > 0$. In comparison, in [1, §3], the authors conclude that

$$\sum_{n_2 \leq N_2} d(|f(n_2)|) \ll N_2 \prod_{p \leq N_2} \left(1 - \frac{\varrho_f(p)}{p}\right) \sum_{n_2 \leq N_2} \frac{d(n_2)\varrho_f(n_2)}{n_2}.$$

This difference accounts for the discrepancy between Theorem 4.5 and [1, Theorem 1]. Proceeding according to the argument of [1, §3], we have

$$\sum_{1 \leq n_1 \leq N} \sum_{N_1 < n_2 \leq N_2} d(|f(n_1, n_2)|) \ll N(N_2 - N_1)t_1 t_2,$$

where

$$t_1 = \prod_{4<p\le N_2} \left(1 - \frac{\varrho_{f(1,x)}(p)}{p}\right) \exp\left(\sum_{\substack{p\le N_2\\ p\nmid n_1}} \frac{d(p)\varrho_f(p)}{p}\right),$$

$$t_2 = \prod_{p|\mathrm{Disc}(f)} \left(1 + \frac{1}{p}\right)^b \exp\left(c \sum_{p|\mathrm{Disc}(f)} \frac{1}{p}\right)$$

for constants $b, c > 0$. It is straightforward to show that t_1 is of order at most

$$\prod_{4<p\le N_2} \left(1 - \frac{\varrho_{f(1,x)}(p)}{p}\right) \exp\left(\sum_{4<p\le N_2} \frac{2\varrho_{f(1,x)}(p)}{p}\right) \ll \exp\left(\sum_{p\le N_2} \frac{\varrho_{f(1,x)}(p)}{p}\right),$$

which, combined with t_2, yields the theorem. ∎

Our third main tool is a classical result due to Dedekind and Landau:

Lemma 4.6. *Let $f \in \mathbb{Z}[x]$ be irreducible and of degree $g \ge 1$. Then*

$$\sum_{p\le B} \varrho_f(p) = \mathrm{Li}(B) + \left(\frac{B}{\exp(c(\log B)^{1/2})}\right)$$

for a constant $c > 0$ dependent only on the splitting field of f over $\mathbb{Q}$.

Proof. Let L be the splitting field of f over $\mathbb{Q}$. For all but finitely many p, $f \pmod p$ has factorization $F_1 \cdots F_n \pmod p$, where each $F_i \in \mathbb{Z}_p[x]$ is irreducible and of degree g_i, if and only if the principal ideal (p) has factorization $P_1 \cdots P_n$, where each P_i is a prime ideal over L with norm p^{g_i}. Now

$$\varrho_f(p) = \#\{i : F_i \text{ is linear}\} = \#\{i : \mathrm{norm}(P_i) = p\},$$

and by Landau's Prime Ideal Theorem,

$$\#\{\text{prime ideals } P : \mathrm{norm}(P) = p \le B\} = \mathrm{Li}(B) + O\left(\frac{B}{\exp(c(\log B)^{1/2})}\right)$$

for a constant $c > 0$ dependent only on L. ∎

Now the Prime Ideal Theorem is simply the generalization to number fields of the Prime Number Theorem; given $n \in \mathbb{N}$, we have

$$\pi(n) = \mathrm{Li}(n) + O\left(\frac{n}{\exp(c'(\log n)^{1/2})}\right)$$

for a constant $c' > 0$. This symmetry between $\pi(t)$ and the average order of $\varrho_f(p)$ will be useful. We also record the following bound, due to Rosser and Schoenfeld [8]:

Lemma 4.7. *Let $n \ge 67$. Then*

$$\frac{n}{\log n - 1/2} < \pi(n) < \frac{n}{\log n - 3/2}.$$

4.3. *The proof of the upper bound.* As in §3, we let $\Delta_{r,s}$ denote $|r^4 - s^4|$. We shall also write P, R and S for $P_{i,j}$, R_i and S_j, respectively.

We begin by applying Lemma 4.2 to get

$$\sum_{(r,s)\in P} N_{r,s} \ll B\biggl(\frac{1}{R^{2/3}} \sum_{(r,s)\in P} \frac{d(\Delta_{r,s})}{\Delta_{r,s}^{1/3}}\biggr). \tag{4.5}$$

We evaluate the sum on the right-hand side according to the size of $\Delta_{r,s}$. Let the linear factors of $\Delta_{r,s}$ be denoted $|s - \alpha_i r|$ for $i = 1, 2, 3, 4$. We consider three cases:

Case I: R and S are not of the same order;

Case II: R and S are of the same order, and $|s - \alpha_i r| > R/4$ for $i = 1, 2, 3, 4$; and

Case III: R and S are of the same order, and $|s - \alpha_i r| \le R/4$ for some $i \in \{1, 2, 3, 4\}$. (We may assume moreover that $\alpha_i = 1$ or -1, for otherwise we have $|s - \alpha_i r| > R/4$.)

Note that, since we are in search of an upper bound, we may apply selectively the coprimality condition on P.

In Case I, $\Delta_{r,s}$ is dominated by the r^4 term, and we have

$$\sum_{(r,s)\in P} N_{r,s} \ll B\biggl(\frac{1}{R^2} \sum_{(r,s)\in P} d(\Delta_{r,s})\biggr).$$

Now

$$\sum_{(r,s)\in P} d(\Delta_{r,s}) \le \sum_{s\le 2\max(S,R^{1/2})} \sum_{r\le 2R} d(\Delta_{r,s}).$$

We apply Theorem 4.5 to the right-hand side, getting

$$\sum_{(r,s)\in P} d(\Delta_{r,s}) \ll \max(S, R^{1/2}) R T,$$

where

$$T = \prod_{p\mid \mathrm{Disc}(\Delta_{r,s})} \biggl(1 + \frac{1}{p}\biggr)^b \exp\biggl(c \sum_{p\mid \mathrm{Disc}(\Delta_{r,s})} \frac{1}{p}\biggr) \exp\biggl(\sum_{p\le 2R} \frac{\varrho^*_{\Delta_{r,s}}(p)}{p}\biggr)$$

for some constants $b, c > 0$. The fact that $\mathrm{Disc}(\Delta_{r,s}) = 128i$ implies that the first two terms are $\ll 1$, whence

$$T \ll \exp\biggl(\sum_{p\le 2R} \frac{\varrho^*_{\Delta_{r,s}}(p)}{p}\biggr) \ll \exp\biggl(\sum_{p\le 2R} \frac{\varrho_{\Delta_{1,x}}(p)}{p}\biggr).$$

We appeal to Lemmas 4.6 and 4.7. The sum on the far right-hand side is equal to

$$\frac{1}{2R}\sum_{p\le 2R}\varrho_{\Delta_{1,x}}(p)+\int_1^{2R}\sum_{p\le t}\varrho_{\Delta_{1,x}}(p)\,\frac{dt}{t^2}+O\biggl(\int_1^{2R}\sum_{p\le t}\varrho_{\Delta_{1,x}}(p)\,\frac{dt}{t^3}\biggr).$$

The first term is small. Indeed, let $f_1(x)=1+x^2$, $f_2(x)=1+x$ and $f_3(x)=1-x$. Then, by Lemma 4.6, the first term is equal to

$$\frac{1}{2R}\Bigl(\sum_{p\le 2R}\varrho_{f_1(x)}(p)+\sum_{p\le 2R}\varrho_{f_2(x)}(p)+\sum_{p\le 2R}\varrho_{f_3(x)}(p)\Bigr)=O(1).$$

The error term is also small: by Lemma 4.3(a) we have $\varrho_{\Delta_{1,x}}(p)\le 4$ for all primes p; that is,

$$O\biggl(\int_1^{2R}\sum_{p\le t}\varrho_{\Delta_{1,x}}(p)\,\frac{dt}{t^3}\biggr)=O\biggl(\int_1^{2R}\frac{dt}{t^2}\biggr)=O(1).$$

Thus

$$T\ll\exp\biggl(\int_{67}^{2R}\sum_{p\le t}\varrho_{\Delta_{1,x}}(p)\,\frac{dt}{t^2}+O(1)\biggr).$$

By Lemma 4.7, we have

$$\int_{67}^{2R}\sum_{p\le t}\varrho_{\Delta_{1,x}}(p)\,\frac{dt}{t^2}<\int_{67}^{2R}\frac{1}{\pi(t)}\sum_{p\le t}\varrho_{\Delta_{1,x}}(p)\,\frac{dt}{t(\log t-3/2)}. \tag{4.6}$$

Lemma 4.6 gives

$$\begin{aligned}\sum_{p\le t}\varrho_{\Delta_{1,x}}(p)&=\sum_{p\le t}\varrho_{f_1(x)}(p)+\sum_{p\le t}\varrho_{f_2(x)}(p)+\sum_{p\le t}\varrho_{f_3(x)}(p)\\&=3\biggl(\mathrm{Li}(t)+O\biggl(\frac{t}{\exp(c(\log t)^{1/2})}\biggr)\biggr)\end{aligned}$$

for a constant $c>0$. We also have

$$\pi(t)=\mathrm{Li}(t)+O\biggl(\frac{t}{\exp(c'(\log t)^{1/2})}\biggr)$$

for a constant $c'>0$. Let $C=\min(c,c')$. Then

$$\frac{1}{\pi(t)}\sum_{p\le t}\varrho_{\Delta_{x,1}}(p)=3+O\biggl(\frac{1}{\log t-3/2}\biggr).$$

Substituting back into (4.6), we get

$$\begin{aligned}\int_{67}^{2R}\sum_{p\le t}\varrho_{\Delta_{1,x}}(p)\,\frac{dt}{t^2}&<\int_{67}^{2R}\frac{3\,dt}{t(\log t-3/2)}+O\biggl(\int_{67}^{2R}\frac{dt}{t(\log t-3/2)^2}\biggr)\\&=\log(\log(2R)-2/3)^3+O(1)<\log(\log B)^3+O(1).\end{aligned}$$

Thus $T \ll (\log B)^3$, and

$$\sum_{(r,s)\in P} d(\Delta_{r,s}) \ll \max(S, R^{1/2})R(\log B)^3;$$

that is,

$$\sum_{(r,s)\in P} N_{r,s} \ll \max\left(\frac{S}{R}, \frac{1}{R^{1/2}}\right)B(\log B)^3 \tag{4.7}$$

for Case I.

Case II is handled identically: as in Case I, we have $\Delta_{r,s} \gg R^4$, and the same bound (4.7) results.

In Case III, suppose $\alpha_1 \in \mathbb{R}$ and $|s - \alpha_1 r| \le R/4$. Then the bounds

$$r|\alpha_1 - \alpha_i| - |s - \alpha_1 r| \le |s - \alpha_i r| \le r|\alpha_1 - \alpha_i| + |s - \alpha_1 r|$$

for $i = 2, 3, 4$ imply that $\Delta_{r,s}$ is of order $|s - \alpha_1 r|R^3$. We split the set of values for $|s - \alpha_1 r|$ into dyadic ranges

$$2^{i-1} = B_i < |s - \alpha_1 r| \le 2B_i = 2^i,$$

where the index i has an upper bound

$$I = \left\lceil \frac{\log(R/4)}{\log 2} \right\rceil = \frac{\log R}{\log 2} + O(1).$$

In view of (4.5), we have

$$\sum_{(r,s)\in P} N_{r,s} \ll B\left(\frac{1}{R^{5/3}} \sum_{i\le I} \frac{1}{B_i^{1/3}} \sum_{\substack{(r,s)\in P \\ B_i<|s-\alpha_1 r|\le 2B_i}} d(\Delta_{r,s})\right).$$

Now

$$\sum_{\substack{(r,s)\in P \\ B_i<|s-\alpha_1 r|\le 2B_i}} d(\Delta_{r,s}) \le \sum_{1\le s\le 2S} \sum_{K_i\le r\le L_i} d(\Delta_{r,s}), \tag{4.8}$$

where

$$K_i = \max(1, s - 2\max(B_i, S^{1/3})), \qquad L_i = \min(2R, s + 2\max(B_i, S^{1/3})).$$

We apply Theorem 4.5 to the right-hand side of (4.8), getting

$$\sum_{\substack{(r,s)\in P \\ B_i<|s-\alpha_1 r|\le 2B_i}} d(\Delta_{r,s}) \ll \max(B_i, S^{1/3})S(\log B)^3,$$

hence

$$\sum_{(r,s)\in P} N_{r,s} \ll \left(\frac{S}{R^{5/3}} \sum_{i\le I} \frac{\max(B_i, S^{1/3})}{B_i^{1/3}}\right)B(\log B)^3.$$

If $\max(B_i, S^{1/3}) = B_i$, then

$$\sum_{i\le I} \frac{\max(B_i, S^{1/3})}{B_i^{1/3}} \ll 2^{2I/3} = \exp\left(\frac{2I}{3}\log 2\right) \ll R^{2/3},$$

and if $\max(B_i, S^{1/3}) = S^{1/3}$, then

$$\sum_{i\ll \log R} \frac{\max(B_i, S^{1/3})}{B_i^{1/3}} \ll S^{1/3}.$$

Thus we have, for Case III, the bound

$$\sum_{(r,s)\in P} N_{r,s} \ll \max\left(\frac{S}{R}, \frac{S^{4/3}}{R^{5/3}}\right) B(\log B)^3 = \frac{S}{R} B(\log B)^3. \tag{4.9}$$

Comparing the bounds (4.7) and (4.9), we conclude that

$$\sum_{(r,s)\in P} N_{r,s} \ll \max\left(\frac{S}{R}, \frac{1}{R^{1/2}}\right) B(\log B)^3. \tag{4.10}$$

5. The cardinality $N_U(B)$. By the bounds (4.2) and (4.10), we have

$$\sum_{i\le 2\log B}\sum_{j\le i}\sum_{(r,s)\in P_{i,j}} N_{r,s} \ll B(\log B)^3 \sum_{i\le 2\log B}\sum_{j\le i} \max\left(\frac{S_j}{R_i}, \frac{1}{R_i^{1/2}}\right).$$

If $S_j \ge R_i^{1/2}$, the sum on the right-hand side is equal to

$$\sum_{i\le 2\log B}\sum_{j\le i} \frac{1}{2^{i-j}} \le \sum_{i\le 2\log B}\sum_{j\ge 0} \frac{1}{2^j} \ll \log B;$$

otherwise, it is equal to

$$\sum_{i\le 2\log B}\sum_{j\le i} \frac{1}{2^{(i-1)/2}} \ll 1.$$

Thus we have

$$\sum_{i\le 2\log B}\sum_{j\le i}\sum_{(r,s)\in P_{i,j}} N_{r,s} \ll B(\log B)^4$$

as required.

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