Quartic diophantine chains

by

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Dedicated to Professor A. Schinzel on his 70th birthday

Introduction. Very little is known about quartic diophantine chains of the type

$$f(x_1, y_1) = f(x_2, y_2) = f(x_3, y_3), \tag{1}$$

where $f(x, y)$ is a binary quartic form in the variables x and y. For instance, only one numerical solution of the diophantine chain

$$X_1^4 - Y_1^4 = X_2^4 - Y_2^4 = X_3^4 - Y_3^4 \tag{2}$$

has been published [7, p. 652].

This paper is concerned with diophantine chains of the type

$$(3) \quad x_1 y_1(x_1^2 + r x_1 y_1 + y_1^2) = x_2 y_2(x_2^2 + r x_2 y_2 + y_2^2) = x_3 y_3(x_3^2 + r x_3 y_3 + y_3^2),$$

where r is an arbitrary rational number. The case $r = 0$ is of special interest since then the chain (3) reduces to

$$x_1 y_1(x_1^2 + y_1^2) = x_2 y_2(x_2^2 + y_2^2) = x_3 y_3(x_3^2 + y_3^2), \tag{4}$$

which is equivalent to the chain of biquadrates

$$(5) \quad (x_1 + y_1)^4 - (x_1 - y_1)^4 = (x_2 + y_2)^4 - (x_2 - y_2)^4 = (x_3 + y_3)^4 - (x_3 - y_3)^4.$$

We will find a parametric solution of (3) for all non-zero values of r, and infinitely many integer solutions when $r = 0$. In view of the equivalence of (4) and (5), the infinitely many solutions of (4) lead to infinitely many solutions of the chain of biquadrates (2).

We will regard a diophantine chain of type (3) as trivial if the common value $x_i y_i(x_i^2 + r x_i y_i + y_i^2)$, $i = 1, 2, 3$, is 0. Such chains are readily determined and will not be considered further in this paper. When the common value $x_i y_i(x_i^2 + r x_i y_i + y_i^2)$ is not 0, we have the following two types of chains:

(i) if the absolute value of one of the numbers x_i, y_i is the same positive number for each of the three values of i, and the absolute values of the three remaining numbers out of the six numbers $x_i, y_i, i = 1, 2, 3$, are distinct, the chain will be called *semi-trivial*;

(ii) if the absolute values of all the six numbers $x_i, y_i, i = 1, 2, 3$, are distinct, the chain will be called *non-trivial*.

In Section 2 below we obtain a necessary and sufficient condition for the existence of semi-trivial chains of type (3), and show how the complete solution may be obtained when this condition is satisfied. In Section 3 we obtain non-trivial diophantine chains for all rational values of r, and in Section 4 we give numerical solutions of the chain of biquadrates (2). As an application of this chain of biquadrates, in Section 5 we obtain infinitely many triads of biquadrates with equal sums and equal products.

We note that as equations (3) and (4) are both homogeneous, it suffices to obtain their solutions in rational numbers since any such solution may be multiplied through by a suitable integer to yield a solution in integers.

2. Semi-trivial quartic diophantine chains. In this section we will determine the complete solution of semi-trivial chains of type (3). Since equation (3) is of even degree, and in view of its symmetry, there is no loss of generality in assuming that for semi-trivial chains we may take $x_1 = x_2 = x_3$. With this assumption, (3) reduces to

(6) $\quad y_1(x_1^2 + rx_1y_1 + y_1^2) = y_2(x_1^2 + rx_1y_1 + y_2^2) = y_3(x_1^2 + rx_1y_3 + y_3^2).$

The first part of equation (6) reduces to

(7) $$x_1^2 + rx_1(y_1 + y_2) + y_1^2 + y_1y_2 + y_2^2 = 0,$$

and we similarly get

(8) $$x_1^2 + rx_1(y_1 + y_3) + y_1^2 + y_1y_3 + y_3^2 = 0.$$

Now (7) may be rewritten as

(9) $$(4 - r^2)X^2 + Y^2 + (3 - r^2)Z^2 = 0,$$

where

$$X = 2x_1 + r(y_1 + y_2), \quad Y = (4 - r^2)y_1 + (2 - r^2)y_2, \quad Z = 2y_2.$$

The solvability of the quadratic equation (9) is determined by the well-known Legendre theorem [5, pp. 272–274], and when the condition of solvability is satisfied, the complete solution of (9), and hence of (7), can be effectively determined. We note in particular that when $r < \sqrt{3}$, the left-hand side of (9) is always positive for real values of X, Y, Z, not all 0, and hence there are no non-trivial solutions of (9).

Next we assume that equation (7) has non-trivial solutions. In this case the remaining condition for the existence of the chain (6), that is, equation (8), needs to be solved together with (7). Subtracting (7) from (8), and removing the non-zero factor $y_2 - y_3$, we obtain a linear equation which is readily solved to get the unique solution $y_3 = -rx_1 - y_1 - y_2$.

Thus for any given value of r, we can determine whether or not there exist semi-trivial solutions of (3) by determining the solvability of (9). When this condition is satisfied, all semi-trivial chains of type (3) are given by the complete solution of the quadratic equation (7) together with $x_2 = x_1$, $x_3 = x_1$, $y_3 = -rx_1 - y_1 - y_2$.

As a specific example, when $r = 4$, we get the following parametric solution of (3):

$$x_1 = x_2 = x_3 = p^2 + 3pq + 3q^2,$$
$$y_1 = -3p^2 - 6pq - 2q^2, \quad y_2 = p^2 + 4pq + q^2, \quad y_3 = -2p^2 - 10pq - 11q^2.$$

3. Non-trivial quartic diophantine chains. We first prove a preliminary lemma that will be used repeatedly to obtain solutions of the chain (3).

LEMMA. *If a, b, c, d are non-zero rational numbers satisfying the simultaneous diophantine equations*

$$(10) \qquad a(a^2 + abr + b^2) - cd = 0,$$
$$(11) \qquad 3a^2 + 2abr + b^2 - c^2 + d^2 + 1 = 0,$$

then the rational numbers $x_1, y_1, x_2, y_2, x_3, y_3$ defined by

$$(12) \quad x_1 = bc, \quad y_1 = ac - d, \quad x_2 = bd, \quad y_2 = ad + c, \quad x_3 = b, \quad y_3 = a + cd$$

satisfy the diophantine chain (3).

Proof. When x_1, y_1, x_2, y_2 are defined by (12), we have

$$(13) \quad x_1 y_1 (x_1^2 + rx_1 y_1 + y_1^2) - x_2 y_2 (x_2^2 + rx_2 y_2 + y_2^2)$$
$$= b(c^2 + d^2)[(c^2 - d^2)\{a(a^2 + abr + b^2) - cd\}$$
$$- cd(3a^2 + 2abr + b^2 - c^2 + d^2 + 1)] = 0,$$

in view of (10) and (11). Similarly,

$$(14) \quad x_1 y_1 (x_1^2 + rx_1 y_1 + y_1^2) - x_3 y_3 (x_3^2 + rx_3 y_3 + y_3^2)$$
$$= b(c^2 + 1)[(c^2 - 1)\{a(a^2 + abr + b^2) - cd\}$$
$$- cd(3a^2 + 2abr + b^2 - c^2 + d^2 + 1)] = 0.$$

Combining (13) and (14), we get the chain (3). This completes the proof.

In view of the above lemma, to obtain rational solutions of the diophantine chain (3), it suffices to solve the simultaneous diophantine equations (10) and (11). This leads to two cases according as r is non-zero or zero. We consider the case when $r \neq 0$ in Subsection 3.1 below and the case $r = 0$ in Subsection 3.2.

3.1. To obtain rational solutions of equations (10) and (11) when $r \neq 0$, we solve (10) for d to get

$$(15) \qquad d = a(a^2 + abr + b^2)/c,$$

and substituting this value of d in (11), we get the following quartic equation in c:

$$(16) \qquad c^4 - (3a^2 + b^2 + 2abr + 1)c^2 - a^2(a^2 + abr + b^2)^2 = 0.$$

This may be considered as a quadratic equation in c^2, and for it to have a rational solution for c^2, its discriminant, namely,

$$(17) \quad (4a^2 + 1)b^4 + 4ar(2a^2 + 1)b^3 + \{4a^2r^2(a^2 + 1) + 8a^4 + 6a^2 + 2\}b^2 \\ + 4ar(2a^2 + 1)(a^2 + 1)b + (4a^2 + 1)(a^2 + 1)^2,$$

must be made a perfect square. By choosing $a = (t^2 - 1)/(4t)$, the discriminant (17) becomes a quartic function of b in which the constant term is a perfect square, and hence following the usual procedure described by Dickson [3, p. 639], we can obtain a value of b which makes this quartic function a perfect square. With this value of b, equation (16) gives rational values for c^2 but not for c. If, however, we take $t = q^2/p^2$, then (16) has a rational solution for c. Thus, by taking

$$(18) \qquad a = -(p^4 - q^4)/(4p^2q^2),$$

when we get the following value of b which makes (17) a perfect square:

$$(19) \qquad b = \{(p^4 - q^4)^4p^4q^4r^2 + (p^4 + q^4)^2(p^8 + 6p^4q^4 + q^8)^2\} \\ \times \{4p^2q^2r(p^4 - q^4)(p^4 + q^4)^2(p^8 + 6p^4q^4 + q^8)\}^{-1},$$

we can solve (16) to obtain the following rational solution for c:

$$(20) \quad c = \{(p^8 - p^6q^2 + 4p^4q^4 - p^2q^6 + q^8)(p^2 - q^2)^4(p^2 + q^2)^2p^2q^2r^2 \\ + (p^4 + q^4)^2(p^8 + 6p^4q^4 + q^8)^2\} \\ \times \{8p^3q^3r(p^2 - q^2)(p^4 + q^4)^2(p^8 + 6p^4q^4 + q^8)\}^{-1}.$$

Substituting the values of a, b, c given by (18)–(20) in (15), we get

$$(21) \quad d = \{(p^8 + p^6q^2 + 4p^4q^4 + p^2q^6 + q^8)(p^2 + q^2)^4(p^2 - q^2)^2p^2q^2r^2 \\ - (p^4 + q^4)^2(p^8 + 6p^4q^4 + q^8)^2\} \\ \times \{8p^3q^3r(p^2 + q^2)(p^4 + q^4)^2(p^8 + 6p^4q^4 + q^8)\}^{-1}.$$

Thus a solution of (10) and (11) when r is an arbitrary non-zero rational number is given in terms of arbitrary parameters p and q by (18)–(21). With these values of a, b, c, d, the values of $x_i, y_i, i = 1, 2, 3$, defined by (12) constitute a parametric solution of the diophantine chain (3). It is readily verified that this parametric solution actually generates non-trivial diophantine chains of type (3) for every non-zero value of r.

3.2. We will now obtain infinitely many solutions of the diophantine chain (4). In view of the Lemma, it is sufficient to solve the following two simultaneous diophantine equations obtained by substituting $r = 0$ in equations (10) and (11):

$$(22) \qquad a(a^2 + b^2) - cd = 0,$$

$$(23) \qquad 3a^2 + b^2 - c^2 + d^2 + 1 = 0.$$

We proceed as in Subsection 3.1 and instead of (16), we get the following equation obtained by substituting $r = 0$ in (16):

$$(24) \qquad c^4 - (3a^2 + b^2 + 1)c^2 - a^2(a^2 + b^2)^2 = 0.$$

As in Subsection 3.1, we must make (17) a perfect square when $r = 0$, that is, we must solve the following quartic equation in b and s:

$$(25) \qquad s^2 = (4a^2 + 1)b^4 + (8a^4 + 6a^2 + 2)b^2 + (4a^2 + 1)(a^2 + 1)^2.$$

As before, we take

$$(26) \qquad a = (t^2 - 1)/(4t)$$

in (25), and we get

$$(27) \qquad s^2 = \{256(t^2 + 1)^2 t^4 b^4 + 32(t^8 + 8t^6 + 46t^4 + 8t^2 + 1)t^2 b^2 \\ + (t^2 + 1)^2(t^4 + 14t^2 + 1)^2\}/(1024t^6).$$

The birational transformation defined by the relations

$$(28) \qquad \begin{aligned} b &= Y/(4t(t^2 + 1)X), \\ s &= \{-4t^2(t^2 - 1)^2(t^4 + 6t^2 + 1)^2 - X^2\}/\{32t^3(t^2 + 1)X\}, \end{aligned}$$

and

$$(29) \qquad \begin{aligned} X &= 8t^2(t^2 + 1)^2 b^2 - 16t^3(t^2 + 1)s \\ &\quad + (t^8 + 8t^6 + 46t^4 + 8t^2 + 1)/2, \\ Y &= 32t^3(t^2 + 1)^3 b^3 - 64t^4(t^2 + 1)^2 bs \\ &\quad + 2t(t^2 + 1)(t^8 + 8t^6 + 46t^4 + 8t^2 + 1)b, \end{aligned}$$

reduces the quartic equation (27) to the parametrized elliptic curve

$$(30) \qquad Y^2 = X\{X + 4t^2(t^2 - 1)^2\}\{X - (t^4 + 6t^2 + 1)^2\}.$$

Now on using (27), (28) and (30), equation (24) gives

$$
\begin{aligned}
(31) \quad c^2 &= (3a^2 + b^2 + 1 \pm s)/2 \\
&= \mp(t \mp 1)^2\{X - 4t^6 - 24t^4 - 4t^2 \mp (2t^7 + 14t^5 + 14t^3 + 2t)\}^2 \\
&\quad \times \{64t^3(t^2 + 1)^2 X\}^{-1}.
\end{aligned}
$$

For c to be rational, we must find points on the curve (30) such that $\mp tX$ is a perfect square. It follows from the duplication formula [6, p. 31] that if there exists a rational point P on the elliptic curve (30), then the abscissa of the point $2P$ is a perfect square. It is therefore simplest to take $t = T^2$ in (30) and find a value of T such that the curve (30) has a rational point. While $T = 1$ leads to triviality, it was found using APECS (a package written by Connell [2] in MAPLE for working with elliptic curves) that when $T = 2$ and also when $T = 3$, there exist rational points on the elliptic curve (30). We will take $T = 3$, that is, $t = 9$, since this leads to numerically smaller solutions of the diophantine chains (2) and (4). With this value of t, the curve (30) becomes

$$(32) \qquad Y^2 = X^3 - 47600704X^2 - 103004636774400X,$$

and a rational point P_1 on this curve, found using APECS, is given by

$$(33) \qquad (X, Y) = (-3087587840/1521, 126935576104960/59319).$$

As this rational point does not have integer co-ordinates, it follows from the Nagell–Lutz theorem [6, p. 56] on elliptic curves that this is not a point of finite order. Thus, there exist infinitely many rational points on the curve (32) and these can be determined by the group law. By using the duplication formula, we can find on this curve infinitely many rational points whose abscissae are perfect squares, and these rational points yield infinitely many rational solutions of equations (22) and (23) by using the relations (26), (28), (31) and (15) with $r = 0$. We note that in all these solutions of (22) and (23), the value of a remains fixed as 20/9. By the Lemma, the infinitely many solutions of (22) and (23) yield infinitely many rational solutions of the diophantine chain (4) and these, in turn, lead to infinitely many integer solutions of (4) as well as of (2).

We now prove that the above method actually generates infinitely many non-trivial chains of type (4). It is readily seen that the chains generated satisfy $x_i y_i(x_i^2 + y_i^2) \neq 0, i = 1, 2, 3$. The only remaining possibility for a chain to be trivial arises when we have $x_i y_i = x_j y_j$ for some $i \neq j$. In each of these cases the numbers a, b, c, d must satisfy another equation in addition to the equations (22) and (23). Eliminating c and d from the three equations in a, b, c, d, we get an equation in a and b which for a fixed non-zero

a has only finitely many solutions in b. Since there are only three such sets of three equations in a, b, c, d, we can simply exclude the resulting finitely many possibilities from the infinite set of solutions with $a = 20/9$, and thus obtain infinitely many non-trivial diophantine chains of type (4). The possibility that these infinitely many non-trivial diophantine chains are actually multiples of a finite number of such chains is also similarly ruled out. We are thus assured of generating infinitely many non-trivial solutions of (4), and hence also of (2).

While the above procedure guarantees the existence of infinitely many integer solutions of (2) and (4), the solutions generated involve large integers. For instance, using the rational point P_1 already found on the curve (32), we determine the point $2P_1$ which is given by

$$(X, Y) = \left(\frac{6867760262600^2}{274373463^2}, \frac{3113333938772941966988917299982967296000}{20655053024783974477163847} \right).$$

As expected, the abscissa of $2P_1$ is a perfect square, and this yields the following solution of the chain of biquadrates (2):

$$
\begin{aligned}
X_1 &= 481414016786139336958126813632656903003004, \\
X_2 &= 452693557121427726005012503163912114942985, \\
X_3 &= 591920656152766167797077466343528334914140, \\
Y_1 &= 336113147372142763632798008407011624578004, \\
Y_2 &= 179887595380427256472663477672099005579255, \\
Y_3 &= 534811611453397202401641358319895724884620.
\end{aligned}
$$

(34)

Solutions that are much smaller than the above do exist, and can be obtained by numerical methods. These methods are discussed in the next section.

We have already noted that when $t = 4$ and $t = 9$, rational points on the elliptic curve (30) lead to solutions of the diophantine chain (2). There are several other elliptic curves, arising from equations (22) and (23), that similarly lead to integer solutions of (2). The existence of several such elliptic curves, together with the frequency of numerical solutions obtained in the next section, suggests the possibility of a parametric solution of (2). This, however, remains an open problem.

4. Chains of biquadrates. To obtain small solutions of (2) we search for small solutions of the equation

(35) $$A^4 + B^4 = C^4 + D^4.$$

One method to find small solutions of (35) is an exhaustive search. The key observation to speed it up is to consider two cases.

  (i) If there is one odd and one even term on each side of (35) then we can assume A, C to be odd and B, D to be even. It can be shown that in this case $A \equiv \pm C \pmod{64}$ and $B \equiv D \pmod 4$.

 (ii) If there are two odd terms on each side of (35) then we can assume $A^4 \equiv C^4 \pmod{13}$ and $B^4 \equiv D^4 \pmod{13}$.

In both cases pairs (A, C) are stored in memory and checked against pairs (B, D). We fix a prime p and for each $0 \le R < p$ we store at once only the pairs (A, C) with $A^4 - C^4 \equiv R \pmod p$. We will not discuss the further details of the particular implementation used.

A search using a C program determined that there are 1420 primitive solutions of (35) with $A, B, C, D < 10^7$. Case (i) took under 19 days on Pentium 4 1.7 GHz and case (ii) took about 16 days with a 2-threaded program on Pentium 4 HT 2.4 GHz.

An easy check up of the results determined that the chain (2) has four primitive solutions with positive integers $X_1, Y_1, X_2, Y_2, X_3, Y_3 < 10^7$, and in the same range there is no chain

$$(36) \qquad\qquad X_1^4 + Y_1^4 = X_2^4 + Y_2^4 = X_3^4 + Y_3^4.$$

Another algorithm used to produce solutions of (35) is the method of Pythagorean triplets described in [7]. An exhaustive search for solutions to [7, eq. (10), p. 638] in the range $x, y, z \le 40000$ resulted in 1592 solutions of (35), and a selective search in a higher range produced many more.

Every solution of (35) leads to numbers x_1, y_1, x_2, y_2 satisfying the left equation of (4) and we can use (12) to get complex numbers x_3, y_3 satisfying (4). If x_3, y_3 happen to be rational we get a rational chain (4) which leads to a chain (2).

This method produced 40 primitive integer chains (2), the largest having 14-digit terms.

We give 23 chains with terms below 10^{10} in Table I.

It was verified that among primitive solutions of (35) with $A, B, C, D < 10^7$ there are no two solutions with the same $\max(A, B, C, D)$.

Using the 14th chain in Table I we get two primitive solutions of (35) with $\max(A, B, C, D) = 657153271$.

5. An application of the chains of biquadrates. Using chains (2) we can obtain an infinite family of solutions to the diophantine system

$$(37) \qquad\qquad A^4 B^4 C^4 = D^4 E^4 F^4,$$
$$(38) \qquad\qquad A^4 + B^4 + C^4 = D^4 + E^4 + F^4,$$

mentioned in [4, p. 142] and discussed in [1].

Table I. Solutions of the chain $X_1^4 - Y_1^4 = X_2^4 - Y_2^4 = X_3^4 - Y_3^4$

X_1	Y_1	X_2	Y_2	X_3	Y_3
335084	296668	265076	93436	264047	1169
421296	273588	415137	248289	401168	17228
854688	813396	747633	682161	614656	465236
3138156	2840232	2377876	500296	2376783	249999
27220940	24543080	20830065	6730545	20773068	39432
43134160	39597700	37607745	31601025	32721072	19453572
49734032	47450804	32004351	5106879	31999248	1829556
49888344	39566652	44400113	19439153	43986552	1772484
57218008	56255396	28999496	7923364	28962047	4144961
80325288	41563476	79714568	36280748	79087329	26275617
91785840	89603460	50547505	11071345	50518608	3231108
140326844	139033552	61413863	18878503	61306948	12960784
197237095	150215015	191610280	136159640	178105688	37694312
657153271	613063351	469658376	241846092	461665368	117778356
751888607	509011231	728659604	414173824	708852076	31047392
1180872001	1161828737	616132528	383720564	601308944	301870196
1596600137	1582784777	716243652	450436344	702011844	380055768
3071712177	2980091889	1860358392	1161850644	1805476184	828030484
4275254036	4274662012	660603913	271493129	660079796	263615324
4779990264	4779753552	628280072	478478672	625923087	473081841
6427460484	5799560808	5594408068	4483641784	4989936609	2584078431
6621888824	6399150812	4017771432	1956085212	3968658927	1208210001
9865447832	8250756572	8782377144	5772976932	8348157633	2076294399

If we take

(39)
$$A = X_1 Y_2, \quad B = X_2 Y_3, \quad C = X_3 Y_1,$$
$$D = X_1 Y_3, \quad E = X_2 Y_1, \quad F = X_3 Y_2,$$

then (37) is obvious, while (38) can be written as

(40)
$$\det \begin{pmatrix} X_1^4 & X_2^4 & X_3^4 \\ Y_1^4 & Y_2^4 & Y_3^4 \\ 1 & 1 & 1 \end{pmatrix} = 0,$$

which is true since by (2) the difference of the first two rows is a multiple of the third one.

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