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Lower bounds for the number of zeros of cosine polynomials in the period: a problem of Littlewood

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1. Introduction. Let $0 \le n_1 < n_2 < \cdots < n_N$ be integers. A cosine polynomial of the form $T_N(\theta) = \sum_{j=1}^{N} \cos(n_j\theta)$ must have at least one real zero in a period. This is obvious if $n_1 \neq 0$, since then the integral of the sum on a period is 0. The above statement is less obvious if $n_1 = 0$, but for sufficiently large N it follows from Littlewood's conjecture simply. Here we mean the conjecture proved by S. Konyagin [5] and independently by McGehee, Pigno, and Smith [11] in 1981. See also [4] for a book proof. It is not difficult to prove the statement in general even in the case $n_1 = 0$. One possible way is to use the identity

$$\sum_{j=1}^{n_N} T_N((2j-1)\pi/n_N) = 0.$$

See [6], for example. Another way is to use Theorem 2 of [12]. So there is certainly no shortage of possible approaches to prove the starting observation of this paper even in the case $n_1 = 0$.

It seems likely that the number of zeros of the above sums in a period must tend to ∞ with N. In a private communication B. Conrey asked how fast the number of zeros of the above sums in a period tends to ∞ as a function of N. In [3] the authors observed that for an odd prime p the Fekete polynomial $f_p(z) = \sum_{k=0}^{p-1} \left(\frac{k}{p}\right) z^k$ (the coefficients are Legendre symbols) has $\sim \kappa_0 p$ zeros on the unit circle, where $0.500813 > \kappa_0 > 0.500668$. Conrey's question in general does not appear to be easy.

Littlewood in his 1968 monograph "Some Problems in Real and Complex Analysis" [10, problem 22] poses the following research problem, which appears to be still open: "If the n_m are integral and all different, what is the lower bound on the number of real zeros of $\sum_{m=1}^{N} \cos(n_m\theta)$? Possibly

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$N-1$, or not much less." Here real zeros are counted in a period. In fact no progress appears to have been made on this in the last half-century. In a recent paper [2] we showed that this is false. There exists a cosine polynomial $\sum_{m=1}^{N}\cos(n_m\theta)$ with the n_m integral and all different so that the number of its real zeros in the period is $O(N^{9/10}(\log N)^{1/5})$ (here the frequencies $n_m = n_m(N)$ may vary with N). However, there are reasons to believe that a cosine polynomial $\sum_{m=1}^{N}\cos(n_m\theta)$ always has many zeros in the period. In this paper we prove the following.

2. New result

THEOREM 1. *Suppose the set* $\{a_j : j \in \mathbb{N}\} \subset \mathbb{R}$ *is finite and the set* $\{j \in \mathbb{N} : a_j \neq 0\}$ *is infinite. Let*

$$T_n(t) = \sum_{j=0}^{n} a_j \cos(jt).$$

Then $\lim_{n\to\infty} \mathcal{N}(T_n) = \infty$, *where* $\mathcal{N}(T)$ *is the number of zeros of a trigonometric polynomial* T *in the period* $[-\pi,\pi)$.

3. Lemmas and proofs. To prove the new result we need a few lemmas. The first two are straightforward from [4, pp. 285–288] which offers an elegant book proof of the Littlewood conjecture first shown in [5] and [11]. The book [1] deals with a number of related topics. Littlewood [7]–[10] was interested in many closely related problems.

LEMMA 3.1. *Let* $\lambda_0 < \lambda_1 < \cdots < \lambda_m$ *be nonnegative integers and let*

$$S_m(t) = \sum_{j=0}^{m} A_j \cos(\lambda_j t), \qquad A_j \in \mathbb{R},\ j = 0,1,\ldots,m.$$

Then

$$\int_{-\pi}^{\pi} |S_m(t)|\,dt \ge \frac{1}{60}\sum_{j=0}^{m}\frac{|A_{m-j}|}{j+1}.$$

LEMMA 3.2. *Let* $\lambda_0 < \lambda_1 < \cdots < \lambda_m$ *be nonnegative integers and let*

$$S_m(t) = \sum_{j=0}^{m} A_j \sin(\lambda_j t), \qquad A_j \in \mathbb{R},\ j = 0,1,\ldots,m.$$

Then

$$\int_{-\pi}^{\pi} |S_m(t)|\,dt \ge \frac{1}{60}\sum_{j=0}^{m}\frac{|A_{m-j}|}{j+1}.$$

LEMMA 3.3. *Let* $\lambda_0 < \lambda_1 < \cdots < \lambda_m$ *be nonnegative integers and let*

$$S_m(t) = \sum_{j=0}^{m} A_j \cos(\lambda_j t), \qquad A_j \in \mathbb{R},\ j = 0,1,\ldots,m.$$

Let $A := \max\{|A_j| : j = 0, 1, \dots, m\}$. *Suppose* S_m *has at most* $K - 1 \ge 0$ *zeros in the period* $[-\pi, \pi)$. *Then*

$$\int_{-\pi}^{\pi} |S_m(t)|\,dt \le 2KA\bigg(\pi + \sum_{j=1}^{m} \frac{1}{\lambda_j}\bigg) \le 2KA(5 + \log m).$$

Proof. We may assume that $\lambda_0 = 0$; the case $\lambda_0 > 0$ can be handled similarly. Associated with S_m in the lemma let

$$R_m(t) := A_0 t + \sum_{j=0}^{m} \frac{A_j}{\lambda_j} \sin(\lambda_j t).$$

Clearly

$$\max_{t\in[-\pi,\pi]} |R_m(t)| \le A\bigg(\pi + \sum_{j=1}^{m} \frac{1}{\lambda_j}\bigg).$$

Also, for every $c \in \mathbb{R}$ the function $R_m - c$ has at most K zeros in the period $[-\pi, \pi)$, otherwise Rolle's theorem implies that $S_m = (R_m - c)'$ has at least K zeros in the period $[-\pi, \pi)$. Hence

$$\begin{aligned}\int_{-\pi}^{\pi} |S_m(t)|\,dt &= V_{-\pi}^{\pi}(R_m) \le 2K \max_{t\in[-\pi,\pi]} |R_m(t)| \\ &\le 2KA\bigg(\pi + \sum_{j=1}^{m} \frac{1}{\lambda_j}\bigg) \le 2KA(5 + \log m),\end{aligned}$$

and the lemma is proved. ∎

Proof of the theorem when $(a_n)_{n=0}^{\infty}$ *is NOT eventually periodic.* Suppose the theorem is false. Then there are $k \in \mathbb{N}$, a sequence $(n_\nu)_{\nu=1}^{\infty}$ of positive integers $n_1 < n_2 < \cdots$, and even trigonometric polynomials $Q_{n_\nu} \in \mathcal{T}_k$ with maximum norm 1 on the period such that

$$T_{n_\nu}(t) Q_{n_\nu}(t) \ge 0, \qquad t \in \mathbb{R}. \tag{3.1}$$

We can pick a subsequence of $(n_\nu)_{\nu=1}^{\infty}$ (without loss of generality we may assume that it is the sequence $(n_\nu)_{\nu=1}^{\infty}$ itself) that converges to a $Q \in \mathcal{T}_k$ uniformly on the period $[-\pi, \pi)$. That is,

$$\lim_{\nu\to\infty} \varepsilon_\nu = 0 \quad \text{with} \quad \varepsilon_\nu := \max_{t\in[-\pi,\pi]} |Q(t) - Q_{n_\nu}(t)|. \tag{3.2}$$

We introduce the formal trigonometric series

$$\sum_{j=0}^{\infty} b_j \cos(\beta_j t) := \Big(\sum_{j=0}^{\infty} a_j \cos(jt)\Big) Q(t)^3, \qquad b_j \ne 0,\ j = 0, 1, \dots,$$

$$\sum_{j=0}^{\infty} d_j \cos(\delta_j t) := \Big(\sum_{j=0}^{\infty} a_j \cos(jt)\Big) Q(t)^4, \qquad d_j \ne 0,\ j = 0, 1, \dots,$$

where $\beta_0 < \beta_1 < \cdots$ and $\delta_0 < \delta_1 < \cdots$ are nonnegative integers. Since the set $\{a_j : j \in \mathbb{N}\} \subset \mathbb{R}$ is finite, the sets

$$\{b_j : j \in \mathbb{N}\} \subset \mathbb{R} \quad \text{and} \quad \{d_j : j \in \mathbb{N}\} \subset \mathbb{R}$$

are finite as well. Hence there are $\varrho, M \in (0, \infty)$ such that

$$|a_j| \le M, \qquad \varrho \le |b_j|, |d_j| \le M, \qquad j = 0, 1, \dots. \tag{3.3}$$

Let

$$K_\nu := |\{j \in \mathbb{N} : 0 \le \beta_j \le n_\nu\}|, \qquad L_\nu := |\{j \in \mathbb{N} : 0 \le \delta_j \le n_\nu\}|.$$

Since the sequence $(a_n)_{n=0}^\infty$ is not eventually periodic, we have

$$\lim_{\nu \to \infty} K_\nu = \infty \quad \text{and} \quad \lim_{\nu \to \infty} L_\nu = \infty. \tag{3.4}$$

We claim that

$$K_\nu \le c_1 L_\nu \tag{3.5}$$

with some $c_1 > 0$ independent of $\nu \in \mathbb{N}$. Indeed, using Parseval's formula and (3.2)–(3.4), we deduce

$$\begin{aligned} \frac{1}{\pi} \int_{-\pi}^{\pi} T_{n_\nu}(t)^2 Q(t)^4 Q_{n_\nu}(t)^2 \, dt &= \frac{1}{\pi} \int_{-\pi}^{\pi} (T_{n_\nu}(t) Q(t)^2 Q_{n_\nu}(t))^2 \, dt \\ &\ge (K_\nu - 3k) \, \frac{1}{2} \, \varrho^2 \ge \frac{1}{4} \, \varrho^2 K_\nu \end{aligned} \tag{3.6}$$

for every sufficiently large $\nu \in \mathbb{N}$. Also, (3.1)–(3.4) imply

$$\begin{aligned} &\frac{1}{\pi} \int_{-\pi}^{\pi} T_{n_\nu}(t)^2 Q(t)^4 Q_{n_\nu}(t)^2 \, dt \\ &\quad = \frac{1}{\pi} \int_{-\pi}^{\pi} (T_{n_\nu}(t) Q_{n_\nu}(t))(T_{n_\nu}(t) Q(t)^4) Q_{n_\nu}(t) \, dt \\ &\quad \le \frac{1}{\pi} \Big(\int_{-\pi}^{\pi} T_{n_\nu}(t) Q_{n_\nu}(t) \, dt \Big) \big(\max_{t \in [-\pi, \pi]} |T_{n_\nu}(t) Q(t)^4| \big) \big(\max_{t \in [-\pi, \pi]} |Q_{n_\nu}(t)| \big) \\ &\quad \le \frac{1}{\pi} \Big(\int_{-\pi}^{\pi} T_{n_\nu}(t) Q_{n_\nu}(t) \, dt \Big) (L_\nu M + 4kM) \big(\max_{t \in [-\pi, \pi]} |Q_{n_\nu}(t)| \big) \le c_2 L_\nu \end{aligned} \tag{3.7}$$

with a constant $c_2 > 0$ independent of ν for every sufficiently large $\nu \in \mathbb{N}$. Now (3.5) follows from (3.6) and (3.7).

From Lemma 3.1 we deduce

$$\int_{-\pi}^{\pi} |T_{n_\nu}(t) Q(t)^4| \, dt \ge c_3 \varrho \log L_\nu - c_4 \tag{3.8}$$

with some constants $c_3 > 0$ and $c_4 > 0$ independent of $\nu \in \mathbb{N}$. On the other hand, using (3.1), Lemma 3.3, (3.2), (3.3), (3.5), and (3.4), we obtain

$$
\begin{aligned}
(3.9)\qquad & \int_{-\pi}^{\pi} |T_{n_\nu}(t)Q(t)^4|\,dt \\
&\le \int_{-\pi}^{\pi} T_{n_\nu}(t)Q_{n_\nu}(t)|Q(t)|^3\,dt + \int_{-\pi}^{\pi} |T_{n_\nu}(t)Q(t)^3|\,|Q(t)-Q_{n_\nu}(t)|\,dt \\
&\le \Big(\int_{-\pi}^{\pi} T_{n_\nu}(t)Q_{n_\nu}(t)\,dt\Big)\big(\max_{t\in[-\pi,\pi]}|Q(t)|^3\big) \\
&\quad + \Big(\int_{-\pi}^{\pi} |T_{n_\nu}(t)Q(t)^3|\,dt\Big)\big(\max_{t\in[-\pi,\pi]}|Q(t)-Q_{n_\nu}(t)|\big) \\
&\le c_5 + c_6(\log K_\nu)\varepsilon_\nu \le c_5 + c_6(\log(c_1 L_\nu))\varepsilon_\nu \\
&\le c_7 + c_6(\log L_\nu)\varepsilon_\nu = o(\log L_\nu),
\end{aligned}
$$

where c_1, c_5, c_6, and c_7 are constants independent of $\nu \in \mathbb{N}$. Since (3.9) contradicts (3.8), the proof of the theorem is finished in the case when the sequence $(a_n)_{n=0}^{\infty}$ is not eventually periodic. ■

Proof of the theorem when $(a_n)_{n=0}^{\infty}$ is eventually periodic. The theorem now follows from Lemma 3.4 below. ■

LEMMA 3.4. *Let $(a_j)_{j=0}^{\infty}$ be an eventually periodic sequence of real numbers. Suppose the set $\{j \in \mathbb{N} : a_j \ne 0\}$ is infinite. Then, for all sufficiently large n, the trigonometric polynomials*

$$T_n(t) := \sum_{j=0}^{n} a_j \cos(jt)$$

have at least $c_8 \log n$ zeros in the period $[-\pi, \pi)$ with a constant $c_8 > 0$ depending only on $(a_j)_{j=0}^{\infty}$.

Proof. It is a well known classical result that for the trigonometric polynomials

$$Q_n(t) := \sum_{j=1}^{n} \frac{\sin(jt)}{j}$$

we have

$$|Q_n(t)| \le 1 + \pi, \qquad t \in \mathbb{R},\ n = 1, 2, \ldots.$$

Using the standard way to prove this, it can be easily shown that if $(a_j)_{j=0}^{\infty}$

is an eventually periodic sequence of real numbers, then for the functions

$$S_n(t) := a_0 t + \sum_{j=1}^{n} \frac{a_j \sin(jt)}{j}$$

we have

$$|S_n(t)| \le M, \qquad t \in [-\pi, \pi),\ n = 1, 2, \dots, \tag{3.10}$$

with a constant $M > 0$ depending only on $(a_j)_{j=0}^{\infty}$. Observe that $S_n'(t) = T_n(t)$, so Lemma 3.1 (a consequence of the resolution of the Littlewood conjecture) implies that for all sufficiently large n,

$$V_{-\pi}^{\pi}(S_n) = \int_{-\pi}^{\pi} |S_n'(t)|\,dt = \int_{-\pi}^{\pi} |T_n(t)|\,dt \ge \eta \log n \tag{3.11}$$

with a constant $\eta > 0$ depending only on $(a_j)_{j=0}^{\infty}$. Combining (3.10) and (3.11) we can easily deduce that there is a $c \in [-M, M]$ such that for all sufficiently large n, the function $S_n - c$ has at least $(2M)^{-1}\eta \log n$ distinct zeros in the period $[-\pi, \pi)$. Hence by Rolle's theorem $T_n = (S_n - c)'$ has at least $(2M)^{-1}\eta \log n - 1$ distinct zeros in the period $[-\pi, \pi)$ for all sufficiently large n. ∎

We prove one more result, Theorem 3.6, closely related to Lemma 3.4. In the proof of Theorem 3.6 we need the following observation.

LEMMA 3.5. *Suppose $k > 2m \ge 0$, k is even. Let*

$$z_j := \exp\left(\frac{2\pi j i}{k}\right), \qquad j = 0, 1, \dots, k-1,$$

be the kth roots of unity. Suppose $0 \notin \{b_0, b_1, \dots, b_{k-1}\} \subset \mathbb{R}$ and let

$$Q(z) := z^m \sum_{j=0}^{k-1} b_j z^j.$$

Then there is a value of $j \in \{0, 1, \dots, k-1\}$ for which $\operatorname{Im}(Q(z_j)) \ne 0$.

Proof. If the statement of the lemma were false, then

$$z^{m+k-1}(Q(z) - Q(1/z)) = (z^k - 1) \sum_{\nu=0}^{2m+k-2} \alpha_\nu z^\nu.$$

Obviously

$$\begin{aligned} z^{m+k-1}(Q(z) - Q(1/z)) = {} & -b_{k-1} - b_{k-2}z - b_{k-3}z^2 - \dots - b_0 z^{k-1} \\ & + b_0 z^{2m+k-1} + b_1 z^{2m+k} + b_2 z^{2m+k+1} + \dots + b_{k-1} z^{2m+2k-2}. \end{aligned}$$

Hence

$$\alpha_\nu = -b_{k-1-\nu}, \qquad \alpha_{2m+k-2-\nu} = b_{k-1-\nu}, \qquad \nu = 0, 1, \dots, k-1.$$

Then for $\nu := m + k/2 - 1 < k - 1$ we have

$$-b_{k-1-\nu} = b_{k-1-\nu}, \quad \text{that is,} \quad b_{k-1-\nu} = 0,$$

a contradiction. ∎

THEOREM 3.6. *Let* $0 \notin \{b_0, b_1, \dots, b_{k-1}\} \subset \mathbb{R}$, $\{a_0, a_1, \dots, a_{m-1}\} \subset \mathbb{R}$ *and*

$$a_{m+lk+j} = b_j, \qquad l = 0, 1, \dots, \ j = 0, 1, \dots, k-1.$$

Suppose $k > 2m \geq 0$, k *is even. Let* $n = m + lk + u$ *with integers* $m \geq 0$, $l \geq 0$, $k \geq 1$, *and* $0 \leq u \leq k-1$. *Then for every sufficiently large* n,

$$T_n(t) := \operatorname{Im}\Big(\sum_{j=0}^{n} a_j e^{ijt}\Big)$$

has at least $c_9 n$ *zeros in* $[-\pi, \pi)$, *where* $c_9 > 0$ *is independent of* n.

Proof. Note that

$$\begin{aligned}\sum_{j=0}^{n} a_j z^j &= \sum_{j=0}^{m-1} a_j z^j + z^m \Big(\sum_{j=0}^{k-1} b_j z^j\Big) \frac{z^{(l+1)k} - 1}{z^k - 1} + z^{m+lk} \sum_{j=0}^{u} b_j z^j \\ &= P_1(z) + P_2(z),\end{aligned}$$

where

$$P_1(z) := \sum_{j=0}^{m-1} a_j z^j + z^{m+lk} \sum_{j=0}^{u} b_j z^j,$$

$$P_2(z) := z^m \sum_{j=0}^{k-1} b_j z^j \, \frac{z^{(l+1)k} - 1}{z^k - 1} = Q(z) \, \frac{z^{(l+1)k} - 1}{z^k - 1},$$

with

$$Q(z) := z^m \sum_{j=0}^{k-1} b_j z^j.$$

By Lemma 3.5 there is a kth root of unity $\xi = e^{i\tau}$ such that $\operatorname{Im}(Q(\xi)) \neq 0$. Then for every $K > 0$ there is a $\delta \in (0, 2\pi/k)$ such that $\operatorname{Im}(P_2(e^{it}))$ oscillates between $-K$ and K at least $c_{10}(l+1)k\delta$ times, where $c_{10} > 0$ is a constant independent of n. Now we choose $\delta \in (0, 2\pi/k)$ for

$$K := 1 + \sum_{j=0}^{m-1} |a_j| + \sum_{j=0}^{k-1} |b_j|.$$

Then

$$T_n(t) := \operatorname{Im}\Big(\sum_{j=0}^{n} a_j e^{ijt}\Big) = \operatorname{Im}(P_1(e^{it})) + \operatorname{Im}(P_2(e^{it}))$$

has at least one zero on each interval on which $\mathrm{Im}(P_2(e^{it}))$ oscillates between $-K$ and K, and hence it has at least $c_{10}(l+1)k\delta > c_9 n$ zeros on $[-\pi, \pi)$, where $c_9 > 0$ is a constant independent of n. ■

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