ACTA ARITHMETICA
128.4 (2007)

# On the height of algebraic numbers with real conjugates

by

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**1. Introduction.** Mahler's measure of a polynomial $f$, denoted by $M(f)$, is defined as the product of the absolute values of those roots of $f$ that lie outside the unit disk, multiplied by the absolute value of the leading coefficient. If $f(x) = b\prod_{i=1}^{d}(x-\alpha_i)$, then $M(f) = |b|\prod_{i=1}^{d}\max\{1,|\alpha_i|\}$. For an algebraic number $\alpha$, let $M(\alpha) \equiv M(f)$ where $f$ is the minimal polynomial of $\alpha$ over $\mathbb{Z}$. If $f \in \mathbb{Z}[x]$, then $M(f) \geq 1$, and it is a theorem of Kronecker that for $f \in \mathbb{Z}[x]$, $M(f) = 1$ if and only if $\pm f$ is a product of a power of $x$ and cyclotomic polynomials. It follows from a result of Schinzel ([2, Corollary 1′]) that if $\alpha \neq 0, \pm 1$ is a totally real algebraic number of degree $d$ then

$$M(\alpha) \geq \left(\frac{1+\sqrt{5}}{2}\right)^{d/2}.$$

This article establishes the following generalization of the last inequality.

THEOREM 1. *Let $\alpha$ be an algebraic number, different from* 0 *and* $\pm 1$. *Let $\Lambda$ be the set of Galois conjugates of $\alpha$ that are real and suppose that $|\Lambda| \neq 0$. Let $d = [\mathbb{Q}(\alpha):\mathbb{Q}]$ and let $R_\alpha \equiv |\Lambda|/d$. Let $\beta = 1 - 1/R_\alpha$. Then*

$$M(\alpha) \geq \log\left(\frac{2^{\beta}+\sqrt{4^{\beta}+4}}{2}\right)^{dR_\alpha/2}.$$

It is a natural question to ask whether the full Corollary 1′ of [2] can be generalized in the same way. We mention that in the case $0 < R_\alpha < (\log 2)/(3\log d)$, $\alpha$ an integer, and $d > d_0$, Theorem 2 of Blanksby and Montgomery [1] gives a stronger result.

Amongst the absolute values in a place $v$ of an algebraic number field, $\mathbb{K}$, two play a role in this article. If $v$ is archimedean, let $\|\cdot\|_v$ denote the unique absolute value in $v$ which restricts to the usual absolute value on $\mathbb{Q}$. If $v$ is non-archimedean and $v\,|\,p$, let $\|\cdot\|_v$ denote the unique absolute value

2000 *Mathematics Subject Classification*: Primary 11R04.
*Key words and phrases*: Mahler measure, Weil height, real roots.

 

in $v$ restricting to the usual $p$-adic absolute value on $\mathbb{Q}$. For each place $v$ of $\mathbb{K}$, let $\mathbb{K}_v$ and $\mathbb{Q}_v$ denote the completions of $\mathbb{K}$ and $\mathbb{Q}$ with respect to $v$ and define the local degree as $d_v \equiv [\mathbb{K}_v : \mathbb{Q}_v]$. Let $|\cdot|_v = \|\cdot\|_v^{d_v/d}$.

The absolute values $|\cdot|_v$ satisfy the product rule: if $\alpha \in \mathbb{K}^\times$, then $\prod_v |\alpha|_v = 1$. The *absolute* (*logarithmic*) *Weil height* of $\alpha$ is defined as $h(\alpha) = \sum_v \log^+ |\alpha|_v$ where the sum is over all places $v$ of $\mathbb{K}$. Because of the way in which the absolute values $|\cdot|_v$ are normalized, the absolute Weil height of $\alpha$ does not depend on the field $\mathbb{K}$ in which $\alpha$ is contained. If $\alpha_i$ and $\alpha_j$ are algebraic numbers, then $h(\alpha_i \cdot \alpha_j) \leq h(\alpha_i) + h(\alpha_j)$; if $\alpha_i$ and $\alpha_j$ are Galois conjugates, then $h(\alpha_i) = h(\alpha_j)$; and for an algebraic number $\alpha$, $h(\alpha) = h(1/\alpha)$. Also, if $\alpha$ is an algebraic integer of degree $d$ then $d \cdot h(\alpha) = \log M(\alpha)$. We provide the following additional result concerning the Weil height of algebraic numbers.

THEOREM 2. *Let $\mathbb{K}/\mathbb{Q}$ be a Galois extension of finite degree. Let $G \equiv \mathrm{Aut}(\mathbb{K}/\mathbb{Q})$. Let $\alpha \in \mathbb{K}^\times$ have a Galois conjugate not on the archimedean unit circle. Let $\sigma : \mathbb{K} \hookrightarrow \mathbb{C}$ be an embedding. Let $\xi \in G$ correspond to complex conjugation with respect to $\sigma$. Let $C_G(\xi) = \{x \in G : x\xi = \xi x\}$. Let $n = [G : C_G(\xi)]$. Let $\theta(\alpha) = 1$ if $\alpha$ has a real Galois conjugate and let $\theta(\alpha) = 2$ if $\alpha$ does not have a real Galois conjugate. Then*

$$h(\alpha) \geq \log\left(\frac{2^{1-n} + \sqrt{4^{1-n} + 4}}{2}\right)^{1/(2\theta(\alpha)n)}.$$

**2. Proof of Theorem 1.** Let $\|\cdot\|_\infty$ be the usual archimedean absolute value on $\mathbb{R}$. Let $\delta \equiv 1 - \alpha^2$. For each place $v$ of $\mathbb{K}$ let

$$b_v \max\{1, \|\alpha^2\|_v\} = \|\delta\|_v.$$

By the ultrametric inequality, for each $v \nmid \infty$ we have $b_v \leq 1$.

For each $\gamma \in \Lambda$ define

$$\|1 - \gamma^2\|_\infty = a_\gamma \max\{1, \|\gamma^2\|_\infty\}.$$

Then

$$a_\gamma = \begin{cases} \|1 - 1/\gamma^2\|_\infty & \text{if } \|\gamma\|_\infty > 1, \\ \|1 - \gamma^2\|_\infty & \text{if } \|\gamma\|_\infty < 1. \end{cases}$$

We define

$$\gamma' = \begin{cases} 1/\gamma & \text{if } \|\gamma\|_\infty > 1, \\ \gamma & \text{if } \|\gamma\|_\infty < 1. \end{cases}$$

We thus have

$$\prod_{\gamma \in \Lambda} (\gamma')^2 \geq \frac{1}{(e^{dh(\alpha)})^4}.$$

Using the arithmetic-geometric mean inequality twice we have

$$\prod_{\gamma\in\Lambda}(1-(\gamma')^2) \le \left(\frac{1}{|\Lambda|}\Big(\sum_{\gamma\in\Lambda}(1-(\gamma')^2)\Big)\right)^{|\Lambda|} = \left(1-\frac{1}{|\Lambda|}\sum_{\gamma\in\Lambda}(\gamma')^2\right)^{|\Lambda|}$$
$$\le \left(1-\Big(\prod_{\gamma\in\Lambda}(\gamma')^2\Big)^{1/|\Lambda|}\right)^{|\Lambda|} \le \left(1-\Big(\frac{1}{(e^{dh(\alpha)})^4}\Big)^{1/dR_\alpha}\right)^{dR_\alpha}.$$

By the triangle inequality, $b_v \le 2$ for all $v \mid \infty$. Let

$$B \equiv \prod_v b_v^{d_v/d}.$$

We recall that $\sum_{v|\infty} d_v = d$. From the Galois action on places we have

$$B \le 2^{1-R_\alpha}\left(1-\Big(\frac{1}{(e^{dh(\alpha)})^4}\Big)^{1/dR_\alpha}\right)^{R_\alpha}.$$

If $dR_\alpha = |\Lambda|$ is sufficiently large in comparison to $e^{dh(\alpha)}$ it follows that $B < 1$.

Fix $v$. We have $\|\delta\|_v = |\delta|_v^{d/d_v} = b_v \max\{1, \|\alpha\|_v^2\}$. Consequently,

$$\log|\delta|_v = (d_v/d)(\log b_v + 2\log^+\|\alpha\|_v).$$

Summing over all places and using the product rule yields

$$0 = \sum_v \log|\delta|_v,$$
$$0 = \sum_v \log b_v^{d_v/d} + 2\sum_v \log^+|\alpha|_v,$$
$$0 = \log B + 2h(\alpha).$$

We thus have

$$h(\alpha) = \frac{1}{2}\log(1/B),$$
$$h(\alpha) \ge \frac{1}{2}\log\left(2^{R_\alpha-1}\left(1-\Big(\frac{1}{(e^{dh(\alpha)})^4}\Big)^{1/dR_\alpha}\right)^{-R_\alpha}\right),$$
$$dh(\alpha) \ge \frac{d}{2}\log\left(2^{R_\alpha-1}\left(1-\Big(\frac{1}{(e^{dh(\alpha)})^4}\Big)^{1/dR_\alpha}\right)^{-R_\alpha}\right),$$
$$dh(\alpha) \ge \log\left(2^{R_\alpha-1}\left(1-\Big(\frac{1}{(e^{dh(\alpha)})^4}\Big)^{1/dR_\alpha}\right)^{-R_\alpha}\right)^{d/2}.$$

We notice that for fixed $d$ and $R_\alpha$, if $h(\alpha)$ decreases the right hand side of the inequality increases. As a result, the inequality implies a lower bound on $h(\alpha)$. We now deduce as follows:

$$
\begin{aligned}
&e^{dh(\alpha)} \geq \left(2^{R_\alpha - 1}\left(1 - \left(\frac{1}{(e^{dh(\alpha)})^4}\right)^{1/dR_\alpha}\right)^{-R_\alpha}\right)^{d/2},\\
&(e^{dh(\alpha)})^{2/d} \geq 2^{R_\alpha - 1}\left(\frac{(e^{dh(\alpha)})^{4/dR_\alpha}}{(e^{dh(\alpha)})^{4/dR_\alpha} - 1}\right)^{R_\alpha},\\
&(e^{dh(\alpha)})^{2/dR_\alpha} \geq 2^{\beta}\,\frac{(e^{dh(\alpha)})^{4/dR_\alpha}}{(e^{dh(\alpha)})^{4/dR_\alpha} - 1},\\
&1 \geq 2^{\beta}\,\frac{(e^{dh(\alpha)})^{2/dR_\alpha}}{(e^{dh(\alpha)})^{4/dR_\alpha} - 1},\\
&(e^{dh(\alpha)})^{4/dR_\alpha} - 1 \geq 2^{\beta}(e^{dh(\alpha)})^{2/dR_\alpha},\\
&((e^{dh(\alpha)})^{2/dR_\alpha})^2 - 2^{\beta}(e^{dh(\alpha)})^{2/dR_\alpha} - 1 \geq 0.
\end{aligned}
$$

From the quadratic formula we deduce that

$$
M(\alpha) = e^{dh(\alpha)} \geq \left(\frac{2^{\beta} + \sqrt{4^{\beta} + 4}}{2}\right)^{dR_\alpha/2}. \ \blacksquare
$$

**3. Proof of Theorem 2.** If $\alpha$ does not have a real Galois conjugate let $\gamma \equiv \alpha\xi(\alpha)$, and if $\alpha$ has a real Galois conjugate, $\tau$, let $\gamma = \tau$. Since $\alpha$ does not have all its conjugates on the archimedean unit circle, we can assume that $\gamma \neq \pm 1$. Let $H_{\mathbb{Q}(\gamma)}$ denote the subgroup of $G$ that fixes the field $\mathbb{Q}(\gamma)$. Let $N_G(H_{\mathbb{Q}(\gamma)}) = \{x \in G : xH_{\mathbb{Q}(\gamma)}x^{-1} = H_{\mathbb{Q}(\gamma)}\}$. From Galois theory we recall that $[G : N_G(H_{\mathbb{Q}(\gamma)})]$ is the number of subfields of $\mathbb{K}$ that are distinct from and conjugate to $\mathbb{Q}(\gamma)$. We have

$$
\left|\frac{C_G(\xi)}{C_G(\xi) \cap N_G(H_{\mathbb{Q}(\gamma)})}\right| \geq \frac{|C_G(\xi)|}{|N_G(H_{\mathbb{Q}(\gamma)})|} = \frac{1}{n} \cdot \frac{|G|}{|N_G(H_{\mathbb{Q}(\gamma)})|}.
$$

Consequently, at least $1/n$ of the elements of the orbit of $\mathbb{Q}(\gamma)$ under $G/N_G(H_{\mathbb{Q}(\gamma)})$ are the images of $\mathbb{Q}(\gamma)$ by elements of $C_G(\xi)$ so that at least $1/n$ of the Galois conjugates of $\gamma$ are real: $R_\gamma \geq 1/n$. It then follows from Theorem 1 that

$$
h(\alpha) \geq \log\left(\frac{2^{1-n} + \sqrt{4^{1-n} + 4}}{2}\right)^{1/(2\theta(\alpha)n)}. \ \blacksquare
$$

## 4. An application to Lehmer's problem

Corollary 3. *For $n \in \mathbb{N}$ let $H_n \equiv (2^{1-n} + \sqrt{4^{1-n} + 4})/2$. Let $\mathbb{K}/\mathbb{Q}$ be a Galois extension of finite degree. Let $C(\mathrm{Aut}(\mathbb{K}/\mathbb{Q}))$ be the center of $\mathrm{Aut}(\mathbb{K}/\mathbb{Q})$. Let $n \equiv [\mathrm{Aut}(\mathbb{K}/\mathbb{Q}) : C(\mathrm{Aut}(\mathbb{K}/\mathbb{Q}))]$. Let $\alpha \in \mathcal{O}_{\mathbb{K}}^{\times}$ be different from the roots of unity such that $\mathbb{K}$ is the Galois closure of $\mathbb{Q}(\alpha)$. Let $a \in (1, \infty)$. If $[\mathbb{K} : \mathbb{Q}] \geq (4n^2 \log a)/(\log H_n)$ then $M(\alpha) \geq a$.*

*Proof.* Let $G \equiv \mathrm{Aut}(\mathbb{K}/\mathbb{Q})$. Let $H_{\mathbb{Q}(\alpha)}$ be the subgroup of $G$ that fixes the field $\mathbb{Q}(\alpha)$. By Galois theory we have $C(G) \cap H_{\mathbb{Q}(\alpha)} = \{1\}$ from which it follows that $[\mathbb{Q}(\alpha) : \mathbb{Q}] \geq |G|/n$. By Theorem 2 we have

$$h(\alpha) \geq \log H_n^{1/4n}.$$

Suppose that

$$[\mathbb{K} : \mathbb{Q}] = |G| \geq \frac{4n^2 \log a}{\log H_n}.$$

Then

$$\log(M(\alpha)) = [\mathbb{Q}(\alpha) : \mathbb{Q}] \cdot h(\alpha) \geq \log H_n^{|G|/4n^2} \geq \log a. \ \blacksquare$$

## References

[1] P. E. Blanksby and H. L. Montgomery, *Algebraic integers near the unit circle*, Acta Arith. 18 (1971), 355–369.

[2] A. Schinzel, *On the product of the conjugates outside the unit circle of an algebraic number*, ibid. 24 (1973), 385–399; Addendum, ibid. 26 (1973), 329–331.

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*Received on 17.12.2006*
*and in revised form on 20.3.2007* (5346)