ACTA ARITHMETICA
136.4 (2009)

Univoque numbers and an avatar of Thue–Morse

by

JEAN-PAUL ALLOUCHE (Orsay) and CHRISTIANE FROUGNY (Paris)

1. Introduction. Komornik and Loreti determined in [17] the smallest *univoque* real number in the interval $(1,2)$, i.e., the smallest number $\lambda \in (1,2)$ such that 1 has a unique expansion $1 = \sum_{j\ge 0} a_j/\lambda^{j+1}$ with $a_j \in \{0,1\}$ for every $j \ge 0$. The word "univoque" in this context seems to have been introduced (with a slightly different meaning) by Daróczy and Kátai in [12, 13], while unique expansions of the real number 1 were characterized by Erdős, Joó, and Komornik in [14]. The first author and Cosnard showed in [4] how the result of [17] parallels (and can be deduced from) their study of a certain set of binary sequences arising in the study of iterations of unimodal continuous functions on the unit interval (see [11, 2, 1]). The relevant sets of binary sequences occurring in [2, 1], resp. [17], can be defined by

$$\Gamma := \{A \in \{0,1\}^{\mathbb{N}} : \forall k \ge 0,\ \overline{A} \le \sigma^k A \le A\},$$
$$\Gamma_{\text{strict}} := \{A \in \{0,1\}^{\mathbb{N}} : \forall k \ge 1,\ \overline{A} < \sigma^k A < A\},$$

where σ is the shift on sequences and the bar operation replaces 0's by 1's and 1's by 0's, i.e., if $A = (A_n)_{n\ge 0}$, then $\sigma A := (a_{n+1})_{n\ge 0}$ and $\overline{A} := (1-a_n)_{n\ge 0}$; furthermore, $\le$ denotes the lexicographical order on sequences induced by $0 < 1$, the notation $A < B$ meaning as usual that $A \le B$ and $A \ne B$. The smallest univoque number in $(1,2)$ and the smallest nonperiodic sequence in Γ both involve the Thue–Morse sequence (see for example [6] for more on this sequence).

It is tempting to generalize these sets to alphabets with more than two letters.

DEFINITION 1. For b a positive integer, we will say that the real number $\lambda > 1$ is $\{0,1,\dots,b\}$-*univoque* if the number 1 has a unique expansion as $1 = \sum_{j\ge 0} a_j\lambda^{-(j+1)}$, where $a_j \in \{0,1,\dots,b\}$ for all $j \ge 0$. Furthermore, if $\lambda > 1$ is $\{0,1,\dots,\lceil\lambda\rceil-1\}$-univoque, we will simply say that λ is *univoque*.

2000 *Mathematics Subject Classification*: 11A63, 11B83, 11B85, 68R15, 11J81.
Key words and phrases: beta-expansion, univoque numbers, iteration of continuous functions, Thue–Morse sequence, uniform morphism, automatic sequence, transcendence.

DOI: 10.4064/aa136-4-2

Remark 1. If $\lambda > 1$ is $\{0, 1, \ldots, b\}$-univoque for some positive integer b, then $\lambda \leq b + 1$. Also note that any integer $q \geq 2$ is univoque, since there is exactly one expansion $1 = \sum_{j\geq 0} a_j q^{-(j+1)}$ with $a_j \in \{0, 1, \ldots, q-1\}$, namely $1 = \sum_{j\geq 0}(q-1)q^{-(j+1)}$.

Komornik and Loreti studied in [18] the reals $\lambda \in (1, b+1]$ that are $\{0, 1, \ldots, b\}$-univoque. For their study, they introduced *admissible sequences* on the alphabet $\{0, 1, \ldots, b\}$. Denote, as above, by σ the shift on sequences, and by bar the operation that replaces every $t \in \{0, 1, \ldots, b\}$ by $b - t$, i.e., if $A = (a_n)_{n\geq 0}$, then $\overline{A} := (b - a_n)_{n\geq 0}$. Also denote by $\leq$ the lexicographical order on sequences induced by the natural order on $\{0, 1, \ldots, b\}$. Then a sequence $A = (a_n)_{n\geq 0}$ on $\{0, 1, \ldots, b\}$ is *admissible* if

$$\forall k \geq 0 \text{ such that } a_k < b, \quad \sigma^{k+1}A < A,$$
$$\forall k \geq 0 \text{ such that } a_k > 0, \quad \sigma^{k+1}A > \overline{A}.$$

(Note that our notation is not exactly the notation of [18], since our sequences are indexed by $\mathbb{N}$ and not $\mathbb{N} \setminus \{0\}$.) These sequences have the following property: the map that associates with a real $\lambda \in (1, b+1]$ the sequence of coefficients $(a_j)_{j\geq 0} \in \{0, 1, \ldots, b\}$ of the greedy (i.e., lexicographically largest) expansion of 1, $1 = \sum_{j\geq 0} a_j \lambda^{-(j+1)}$, is a bijection from the set of $\{0, 1, \ldots, b\}$-univoque λ's to the set of admissible sequences on $\{0, 1, \ldots, b\}$ (see [18]).

Now there are two possible generalizations of the result of [17] on the smallest univoque number in $(1, 2)$, i.e., the smallest admissible binary sequence. One is to look at the smallest (if any) admissible sequence on the alphabet $\{0, 1, \ldots, b\}$, as did Komornik and Loreti in [18], the other is to look at the smallest (if any) univoque number in $(b, b+1)$, as did de Vries and Komornik in [22].

It so happens that the first author has already studied a generalization of the set Γ to the case of more than two letters (see [1, Part 3]). Interestingly enough, unlike the study of Γ, this study was unrelated to iterations of continuous functions, being just a tempting formal arithmetico-combinatorial generalization of the study of the set Γ of binary sequences to a similar set of sequences with more than two values.

The purpose of the present paper is threefold:

(1) to show how the 1983 study [1, Part 3, pp. 63–90] gives both the result of Komornik and Loreti in [18] on the smallest admissible sequence on $\{0, 1, \ldots, b\}$, and the result of de Vries and Komornik in [22] on the smallest univoque number $\lambda \in (b, b+1)$ where b is any positive integer;

(2) to bring to light a *universal* morphism that governs the smallest elements in (1) above, and to show that the infinite sequence generated by this morphism is an avatar of the Thue–Morse sequence;

(3) to prove that the smallest univoque number in $(b, b+1)$ (where b is any positive integer) is transcendental.

The paper consists of five sections. In Section 2 we recall some results of [1, Part 3, pp. 63–90] on the generalization of the set Γ to a $(b+1)$-letter alphabet. Then we give some properties of the lexicographically least nonperiodic sequence of this set, completing the results of [1, Part 3, pp. 63–90]. In Section 3 we give two corollaries of the properties of this least sequence: one gives the result in [18], the other gives the result in [22]. The transcendence results are proven in the last section.

2. The generalized Γ and Γ_{strict} sets

Definition 2. Let b be a positive integer, and $\mathcal{A}$ be a finite ordered set with $b+1$ elements $\alpha_0 < \alpha_1 < \cdots < \alpha_b$. The *bar* operation is defined on $\mathcal{A}$ by $\overline{\alpha}_j = \alpha_{b-j}$. We extend this operation to $\mathcal{A}^{\mathbb{N}}$ by $\overline{(a_n)_{n\geq 0}} := (\overline{a}_n)_{n\geq 0}$. Let σ be the *shift* on $\mathcal{A}^{\mathbb{N}}$, defined by $\sigma((a_n)_{n\geq 0}) := (a_{n+1})_{n\geq 0}$.

We define

$$\Gamma(\mathcal{A}) := \{A = (a_n)_{n\geq 0} \in \mathcal{A}^{\mathbb{N}} : a_0 = \max \mathcal{A},\ \forall k \geq 0,\ \overline{A} \leq \sigma^k A \leq A\},$$
$$\Gamma_{\text{strict}}(\mathcal{A}) := \{A = (a_n)_{n\geq 0} \in \mathcal{A}^{\mathbb{N}} : a_0 = \max \mathcal{A},\ \forall k \geq 1,\ \overline{A} < \sigma^k A < A\}.$$

Remark 2. The set $\Gamma(\mathcal{A})$ was introduced by the first author in [1, Part 3, p. 63]. Note that there is a misprint in the definition on p. 66 in [1]: $a_{\beta-i}$ should be changed into $a_{\beta-1-i}$ as confirmed by the rest of the text.

Remark 3. A sequence belongs to $\Gamma_{\text{strict}}(\mathcal{A})$ if and only if it belongs to $\Gamma(\mathcal{A})$ and is nonperiodic. Indeed, $\sigma^k A = A$ if and only if A is k-periodic; if $\sigma^k A = \overline{A}$, then $\sigma^{2k} A = A$, and the sequence A is $2k$-periodic.

Remark 4. If the set $\mathcal{A} := \{i, i+1, \ldots, i+z\}$, where i and z are integers, is equipped with the natural order, then for any $x \in \mathcal{A}$, we have $\overline{x} = 2i+z-x$. Indeed, following Definition 2 above, we write $\alpha_0 := i, \alpha_1 := i+1, \ldots, \alpha_z := i+z$. Hence, for any $j \in [0, z]$, we have $\overline{\alpha}_j = \alpha_{z-j}$, which can be rewritten $\overline{i+j} = i+z-j$, i.e., for any x in $\mathcal{A}$, we have $\overline{x} = i+z-(x-i) = 2i+z-x$.

A first result is that the sets $\Gamma_{\text{strict}}(\mathcal{A})$ are closely linked to the set of admissible sequences whose definition was recalled in the introduction.

Proposition 1. *Let $A = (a_n)_{n\geq 0}$ be a sequence in $\{0, 1, \ldots, b\}^{\mathbb{N}}$ such that $a_0 = t \in [0, b]$ and $A \neq b\ b\ b\ \ldots$. Then A is admissible if and only if $2t > b$ and $A \in \Gamma_{\text{strict}}(\{b-t, b-t+1, \ldots, t\})$. (The order on $\{b-t, b-t+1, \ldots, t\}$ is induced by the order on $\mathbb{N}$. From Remark 4 the bar operation is given by $\overline{j} = b-j$.)*

Proof. First suppose that $2t > b$ and $A \in \Gamma_{\text{strict}}(\{b-t, b-t+1, \dots, t\})$. Then, for all $k \ge 1$, $\overline{A} < \sigma^k A < A$, which clearly implies that A is admissible.

Conversely, suppose that A is admissible. We thus have

$$\forall k \ge 1 \text{ such that } a_{k-1} < b, \quad \sigma^k A < A,$$
$$\forall k \ge 1 \text{ such that } a_{k-1} > 0, \quad \sigma^k A > \overline{A}.$$

We first prove that if A is not a constant sequence, then

$$\forall k \ge 1, \quad \overline{A} < \sigma^k A < A.$$

We only prove that $\sigma^k A < A$; the remaining inequalities are proved in a similar way. If $a_{k-1} < b$, then $\sigma^k A < A$. If $a_{k-1} = b$, there are two cases: either

- $a_0 = a_1 = \dots = a_{k-1} = b$; then if $a_k < b$ we clearly have $\sigma^k A < A$; if $a_k = b$, then the sequence $\sigma^k A$ begins with some block of b's followed by a letter $< b$, thus it begins with a block of b's shorter than the initial block of b's in A, hence $\sigma^k A < A$; or
- there exists an index ℓ with $1 < \ell < k$ such that $a_{\ell-1} < b$ and $a_\ell = a_{\ell+1} = \dots = a_{k-1} = b$. As A is admissible, we have $\sigma^\ell A < A$. It thus suffices to prove that $\sigma^k A \le \sigma^\ell A$. This is clearly the case if $a_k < b$. On the other hand, if $a_k = b$, the sequence $\sigma^k A$ begins with a block of b's which is shorter than the initial block of b's in $\sigma^\ell A$, hence $\sigma^k A \le \sigma^\ell A$.

Now, since $a_0 = t$ and $\sigma^k A < A$ for all $k \ge 1$, we have $a_k \le t$ for all $k \ge 0$. Similarly, since $\sigma^k A > \overline{A}$ for all $k \ge 1$, we have $a_k \ge b - t$ for all $k \ge 1$. Finally, $A > \overline{A}$ implies that $t = a_0 \ge b - t$. Thus $2t \ge b$ and $A \in \Gamma(\{b-t, b-t+1, \dots, t\})$. Now, if $b = 2t$, then $\{b-t, b-t+1, \dots, t\} = \{t\}$ and $\bar{t} = t$. This implies that $A = t\ t\ t\ \dots$, which is not an admissible sequence. ∎

Remark 5. For $b = 1$, this (easy) result is given without proof in [14] and proved in [4].

We need another definition from [1].

Definition 3. Let b be a positive integer, and $\mathcal{A}$ be a finite ordered set with $b+1$ elements $\alpha_0 < \alpha_1 < \dots < \alpha_b$. We suppose that $\mathcal{A}$ is equipped with a bar operation as in Definition 2. Let $A = (a_n)_{n \ge 0}$ be a periodic sequence of *smallest* period T, and with $a_{T-1} < \max \mathcal{A}$. Let $a_{T-1} = \alpha_j$ (thus $j < b$). Then $\Phi(A)$ is the $2T$-periodic sequence beginning with $a_0\ a_1\ \dots\ a_{T-2}\ \alpha_{j+1}\ \overline{a}_0\ \overline{a}_1\ \dots\ \overline{a}_{T-2}\ \alpha_{b-j-1}$, i.e.,

$$\Phi((a_0\ a_1\ \dots\ a_{T-2}\ \alpha_j)^\infty) := (a_0\ a_1\ \dots\ a_{T-2}\ \alpha_{j+1}\ \overline{a}_0\ \overline{a}_1\ \dots\ \overline{a}_{T-2}\ \alpha_{b-j-1})^\infty.$$

We first prove the following easy claim.

Proposition 2. *The smallest element of* $\Gamma(\{b-t, b-t+1, \dots, t\})$ (*where* $2t > b$) *is the* 2*-periodic sequence* $(t\ (b-t))^\infty = (t\ (b-t)\ t\ (b-t)\ t\ \dots)$.

Proof. Since any sequence $A = (a_n)_{n\ge 0}$ in $\Gamma(\{b-t, b-t+1, \dots, t\})$ begins in t, and satisfies $\sigma A \ge \overline{A}$, it must satisfy $a_0 = t$ and $a_1 \ge b-t$. Now if $a_0 = t$ and $a_1 = b-t$, then A must be the 2-periodic sequence $(t\ (b-t))^\infty$ ([1, Lemma 2b, p. 73]). Since this periodic sequence trivially belongs to $\Gamma(\{b-t, b-t+1, \dots, t\})$, it is its smallest element. ∎

Denoting as usual by Φ^s the sth iterate of Φ, we state the following theorem which is a particular case of the theorem on pp. 72–73 of [1] about the smallest elements in certain subintervals of $\Gamma(\{0, 1, \dots, b\})$, and of the definition of q-mirror sequences given in [1, Section II, 1, p. 67] (here $q = 2$).

Theorem 1 ([1]). *Define* $P := (t\ (b-t))^\infty = (t\ (b-t)\ t\ (b-t)\ t\ \dots)$. *The smallest nonperiodic sequence in* $\Gamma(\{b-t, b-t+1, \dots, t\})$ (*i.e., the smallest element of* $\Gamma_{\text{strict}}(\{b-t, b-t+1, \dots, t\})$) *is the sequence*

$$M := \lim_{s\to\infty} \Phi^s(P),$$

that actually takes the (*not necessarily distinct*) *values* $b-t$, $b-t+1$, $t-1$, t. *Furthermore*, *this sequence*

$$M = (m_n)_{n\ge 0} = t\quad b-t+1\quad b-t\quad t\quad b-t\quad t-1\quad \dots$$

can be recursively defined by

$$\begin{aligned} &\forall k \ge 0, \quad m_{2^{2k}-1} = t,\\ &\forall k \ge 0, \quad m_{2^{2k+1}-1} = b+1-t,\\ &\forall k \ge 0,\ \forall j \in [0, 2^{k+1}-2], \qquad m_{2^{k+1}+j} = \overline{m}_j. \end{aligned}$$

It was proven in [1] that the sequence $\lim_{s\to\infty} \Phi^s((t\ (b-t))^\infty$ is 2-automatic (for more about automatic sequences, see [7]). The second author noted that this sequence is actually a fixed point of a uniform morphism of length 2 as soon as the cardinality of the set $\{b-t, b-t+1, \dots, b\}$ is at least 4, i.e., $2t \ge b+3$. (Recall that we always have $t \ge b-t$, i.e., $2t \ge b$.) More precisely, we have Theorem 2 below, where the Thue–Morse sequence pops up, as in [1] and in [18], but also as in [2] and [17]. Before stating this theorem we give a definition.

Definition 4. The "universal" morphism Θ is defined on $\{e_0, e_1, e_2, e_3\}$ by

$$\Theta(e_3) := e_3e_1, \qquad \Theta(e_2) := e_3e_0, \qquad \Theta(e_1) := e_0e_3, \qquad \Theta(e_0) := e_0e_2.$$

Note that this morphism has an infinite fixed point beginning in e_3,

$$\Theta^\infty(e_3) = \lim_{k\to\infty} \Theta^k(e_3) = e_3\ e_1\ e_0\ e_3\ e_0\ e_2\ e_3\ e_1\ e_0\ e_2 \dots.$$

THEOREM 2. *Let $(\varepsilon_n)_{n\ge 0}$ be the Thue–Morse sequence defined by $\varepsilon_0 = 0$ and $\varepsilon_{2k} = \varepsilon_k$ and $\varepsilon_{2k+1} = 1-\varepsilon_k$ for all $k \ge 0$. Then the smallest nonperiodic sequence $M = (m_n)_{n\ge 0}$ in $\Gamma(\{b-t, b-t+1, \dots, t\})$ satisfies*

$$\forall n \ge 0, \qquad m_n = \varepsilon_{n+1} - (2t-b-1)\varepsilon_n + t - 1.$$

Using the morphism Θ introduced in Definition 4 above we thus have:

- *if $2t \ge b+3$, then M is the fixed point beginning in t of the morphism deduced from Θ by renaming e_0, e_1, e_2, e_3 respectively $b-t, b-t+1, t-1, t$ (note that the condition $2t \ge b+3$ implies that these four numbers are distinct)*;
- *if $2t = b+2$ (thus $b-t+1 = t-1$), then M is the pointwise image of the fixed point beginning in e_3 of the morphism Θ under the map g defined by $g(e_3) := t$, $g(e_2) = g(e_1) := t-1$, $g(e_0) := b-t$;*
- *if $2t = b+1$ (thus $b-t = t-1$ and $b-t+1 = t$), then M is the pointwise image of the fixed point beginning in e_3 of the morphism Θ under the map h defined by $h(e_3) = h(e_1) := t$, $h(e_2) = h(e_0) := t-1$.*

Proof. Let us first prove that the sequence $M = (m_n)_{n\ge 0}$ is equal to the sequence $(u_n)_{n\ge 0}$, where $u_n := \varepsilon_{n+1} - (2t-b-1)\varepsilon_n + t - 1$. It suffices to prove that $(u_n)_{n\ge 0}$ satisfies the recursive relations defining $(m_n)_{n\ge 0}$ that are given in Theorem 1. Recall that ε_n is equal to the parity of the sum of the binary digits of n (see [6] for example). Hence, for all $k \ge 0$, $\varepsilon_{2^{2k}-1} = 0$, $\varepsilon_{2^{2k+1}-1} = 1$, and $\varepsilon_{2^{2k}} = \varepsilon_{2^{2k+1}} = 1$. This implies that for all $k \ge 0$, $u_{2^{2k}-1} = t$ and $u_{2^{2k+1}-1} = b+1-t$. Furthermore, for all $k \ge 0$ and $j \in [0, 2^{k+1}-2]$, we have $\varepsilon_{2^{k+1}+j} = 1 - \varepsilon_j$ and $\varepsilon_{2^{k+1}+j+1} = 1 - \varepsilon_{j+1}$. Hence $u_{2^{k+1}+j} = b - u_j = \overline{u}_j$.

To show how the "universal" morphism Θ enters the picture, we study the sequence $(v_n)_{n\ge 0}$ with values in $\{0,1\}^2$ defined by $v_n := (\varepsilon_n, \varepsilon_{n+1})$ for all $n \ge 0$. Since $v_{2n} = (\varepsilon_n, 1-\varepsilon_n)$ and $v_{2n+1} = (1-\varepsilon_n, \varepsilon_{n+1})$ for all $n \ge 0$, we clearly have

$$\begin{aligned}
&\text{if} \quad v_n = (0,0), \quad \text{then} \quad v_{2n} = (0,1) \quad \text{and} \quad v_{2n+1} = (1,0),\\
&\text{if} \quad v_n = (0,1), \quad \text{then} \quad v_{2n} = (0,1) \quad \text{and} \quad v_{2n+1} = (1,1),\\
&\text{if} \quad v_n = (1,0), \quad \text{then} \quad v_{2n} = (1,0) \quad \text{and} \quad v_{2n+1} = (0,0),\\
&\text{if} \quad v_n = (1,1), \quad \text{then} \quad v_{2n} = (1,0) \quad \text{and} \quad v_{2n+1} = (0,1).
\end{aligned}$$

This exactly means that $(v_n)_{n\ge 0}$ is the fixed point beginning in $(0,1)$ of the 2-morphism

$$\begin{aligned}
(0,0) &\to (0,1)(1,0),\\
(0,1) &\to (0,1)(1,1),\\
(1,0) &\to (1,0)(0,0),\\
(1,1) &\to (1,0)(0,1).
\end{aligned}$$

We may define $e_0 := (1,0)$, $e_1 := (1,1)$, $e_2 := (0,0)$, $e_3 := (0,1)$. Then the above morphism can be written

$$e_3 \to e_3e_1, \quad e_2 \to e_3e_0, \quad e_1 \to e_0e_3, \quad e_0 \to e_0e_2,$$

which is the morphism Θ. The above construction shows that the sequence $(v_n)_{n\geq 0}$ is a fixed point of Θ.

Now, define the map ω on $\{0,1\}^2$ by

$$\omega((x,y)) := y - (2t-b-1)x + t - 1.$$

We have $\omega(v_n) = m_n$ for all $n \geq 0$. Thus

- if $2t \geq b+3$, the sequence $(m_n)_{n\geq 0}$ takes exactly four distinct values, namely $b-t, b-t+1, t-1, t$. This implies that $(m_n)_{n\geq 0}$ is the fixed point beginning in t of the morphism obtained from Θ by renaming the letters, i.e., $e_3 \to t$, $e_2 \to (t-1)$, $e_1 \to (b-t+1)$, $e_0 \to (b-t)$. The morphism can thus be written $t \to t\ (b-t+1)$, $(t-1) \to t\ (b-t)$, $(b-t+1) \to (b-t)\ t$, $(b-t) \to (b-t)\ (t-1)$;
- if $2t = b+2$ (resp. $2t = b+1$) the sequence $(m_n)_{n\geq 0}$ takes exactly three (resp. two) values, namely $b-t, t-1, t$ (resp. $t-1, t$). It is still the pointwise image under Θ of the sequence $(v_n)_{n\geq 0}$. Renaming the fixed point of Θ under g (resp. h) as in the statement of Theorem 2 only takes into account that the integers $b-t, b-t+1, t-1, t$ are not distinct. ∎

Remark 6. The reason for the choice of indices for e_3, e_2, e_1, e_0 is that the order of indices is the same as the natural order on the integers $t, t-1$, $b-t+1, b-t$ to which they correspond when $2t \geq b+3$. In particular, if $b = t = 3$, the morphism reads: $3 \to 31$, $2 \to 30$, $1 \to 03$, $0 \to 02$. Interestingly enough, though not surprisingly, this morphism also occurs (up to renaming the letters once more) in the study of infinite square-free sequences on a 3-letter alphabet. Namely, in [9], Berstel proves that the square-free Istrail sequence [15], originally defined (with no mention of the Thue–Morse sequence) as the fixed point of the (nonuniform) morphism $0 \to 12$, $1 \to 102$, $2 \to 0$, is actually the pointwise image of the fixed point beginning in 1 of a 2-morphism Θ' on the 4-letter alphabet $\{0,1,2,3\}$ under the map $0 \to 0$, $1 \to 1$, $2 \to 2$, $3 \to 0$. The morphism Θ' is given by

$$\Theta'(0) = 12, \quad \Theta'(1) = 13, \quad \Theta'(2) = 20, \quad \Theta'(3) = 21.$$

The reader will note immediately that Θ' is another avatar of Θ obtained by renaming letters as follows: $0 \to 2$, $1 \to 3$, $2 \to 0$, $3 \to 1$. This, in particular, shows that *the sequence* $(m_n)_{n\geq 0}$, *in the case where* $2t = b+2$, *is the fixed point of the nonuniform morphism* $t \to t\ (t-1)\ (b-t)$, $(t-1) \to t\ (b-t)$, $(b-t) \to (t-1)$, *i.e., an avatar of Istrail's square-free sequence. Furthermore, it follows from* [9] *that this sequence on three letters cannot be the fixed point* *of a uniform morphism.* A last remark is that the square-free Braunholtz sequence on three letters given in [10] (see also [9, p. 18-07]) is exactly our sequence $(m_n)_{n\geq 0}$ when $t = b = 2$, i.e., the sequence 2 1 0 2 0 1 2 1 0 1 2 0 ...

3. Small admissible sequences and small univoque numbers with given integer part

3.1. *Small admissible sequences with values in* $\{0, 1, \ldots, b\}$. In [18] the authors are interested in the smallest admissible sequence with values in $\{0, 1, \ldots, b\}$, where b is an integer ≥ 1. They prove in particular the following result, which is an immediate corollary of our Theorem 2.

Corollary 1 (Theorems 4.3 and 5.1 of [18]). *Let b be an integer ≥ 1. The smallest admissible sequence with values in $\{0, 1, \ldots, b\}$ is the sequence $(z + \varepsilon_{n+1})_{n\geq 0}$ if $b = 2z + 1$, and $(z + \varepsilon_{n+1} - \varepsilon_n)_{n\geq 0}$ if $b = 2z$.*

Proof. Let $A = (a_n)_{n\geq 0}$ be the smallest (nonconstant) admissible sequence with values in $\{0, 1, \ldots, b\}$. Since $A > \overline{A}$, we must have $a_0 \geq \overline{a}_0 = b - a_0$.

Thus, if $b = 2z + 1$ we have $a_0 \geq z + 1$. We also have, for all $i \geq 0$, $\overline{a}_0 \leq a_i \leq a_0$. Now the smallest element of $\Gamma(\{b-z-1, b-z, \ldots, z-1, z+1\})$ is the smallest admissible sequence on $\{0, 1, \ldots, b\}$ that begins in $z + 1$. Hence this is the smallest admissible sequence with values in $\{0, 1, \ldots, b\}$. Theorem 2 shows that this sequence is $(m_n)_{n\geq 0}$ with $m_n = \varepsilon_{n+1} + z$ for all $n \geq 0$.

If $b = 2z$, we have $a_0 \geq z$. But if $a_0 = z$, then $\overline{a}_0 = z$, and the condition of admissibility implies that $a_n = z$ for all $n \geq 0$ and $(a_n)_{n\geq 0}$ would be the constant sequence $(z\ z\ z \ldots)$. Hence we must have $a_0 \geq z + 1$. Now the smallest element of $\Gamma(\{b - z - 1, b - z, \ldots, z - 1, z + 1\})$ is the smallest admissible sequence on $\{0, 1, \ldots, b\}$ that begins in $z + 1$. Hence this is the smallest admissible sequence with values in $\{0, 1, \ldots, b\}$. Theorem 2 implies that this sequence is $(m_n)_{n\geq 0}$ with $m_n = \varepsilon_{n+1} - \varepsilon_n + z$ for all $n \geq 0$. ∎

3.2. *Small univoque numbers with given integer part.* We are interested here in the univoque numbers λ in an interval $(b, b + 1]$ with b a positive integer. This set was studied in [16], where it was proved to have Lebesgue measure 0. Since $1 = \sum_{j\geq 0} a_j \lambda^{-(j+1)}$ and $\lambda \in (b, b + 1]$, and $a_0 \leq b$, the fact that the expansion of 1 is unique, hence equal to the greedy expansion, implies that $a_0 = b$. In other words, we study the admissible sequences with values in $\{0, 1, \ldots, b\}$ that begin in b, i.e., the set $\Gamma_{\text{strict}}(\{0, 1, \ldots, b\})$. A corollary of Theorem 2 is that, for any positive integer b, there exists a smallest univoque number belonging to $(b, b + 1]$. This result was obtained in [22] (see the penultimate remark in that paper); it generalizes the result obtained for $b = 1$ in [17].

Corollary 2. *For any positive integer* b, *there exists a smallest univoque number in* $(b, b+1]$. *It is the solution of the equation* $1 = \sum_{n\geq 0} d_n \lambda^{-n-1}$, *where* $d_n := \varepsilon_{n+1} - (b-1)\varepsilon_n + b - 1$ *for all* $n \geq 0$.

Proof. It suffices to apply Theorem 2 with $t = b$. ■

4. Transcendence results. We now prove, mimicking the proof given in [3], that numbers λ such that the λ-expansion of 1 is given by the sequence $(m_n)_{n\geq 0}$ are transcendental. This generalizes the transcendence results of [3] and [18].

Theorem 3. *Let* b *be an integer* ≥ 1 *and* $t \in [0, b]$ *be an integer such that* $2t \geq b + 1$. *Define the sequence* $(m_n)_{n\geq 0}$ *as in Theorem 2 by* $m_n := \varepsilon_{n+1} - (2t - b - 1)\varepsilon_n + t - 1$ *for all* $n \geq 0$, *thus* $(m_n)_{n\geq 0}$ *begins with* $t\ b-t+1\ b-t\ t\ b-t\ t-1\ \ldots$. *Then the number* $\lambda \in (1, b+1)$ *defined by* $1 = \sum_{n\geq 0} m_n \lambda^{-n-1}$ *is transcendental.*

Proof. Define the ± 1 Thue–Morse sequence (r_n) by $r_n := (-1)^{\varepsilon_n}$. We clearly have $r_n = 1 - 2\varepsilon_n$ (recall that ε_n is 0 or 1). It is also immediate that the function F defined for the complex numbers X with $|X| < 1$ by $F(X) = \sum_{n\geq 0} r_n X^n$ satisfies $F(X) = \prod_{k\geq 0}(1 - X^{2^k})$ (see, e.g., [6]). Since

$$2m_n = 2\varepsilon_{n+1} - 2(2t - b - 1)\varepsilon_n + 2t - 2 = b - r_{n+1} + (2t - b - 1)r_n$$

we have, for $|X| < 1$,

$$2X \sum_{n\geq 0} m_n X^n = ((2t - b - 1)X - 1)F(X) + 1 + \frac{bX}{1 - X}\cdot$$

Taking $X = 1/\lambda$ where $1 = \sum_{n\geq 0} m_n \lambda^{-n-1}$, we get the equation

$$2 = ((2t - b - 1)\lambda^{-1} - 1)F(1/\lambda) + 1 + \frac{b}{\lambda - 1}\cdot$$

Now, if λ were algebraic, then this equation shows that $F(1/\lambda)$ would be an algebraic number. But, since $1/\lambda$ would then be an algebraic number in $(0, 1)$, the quantity $F(1/\lambda)$ would be transcendental from a result of Mahler [19], giving a contradiction. ■

Remark 7. In particular the $\{0, 1, \ldots, b\}$-univoque number corresponding to the smallest admissible sequence with values in $\{0, 1, \ldots, b\}$ is transcendental, as proved in [18] (Theorems 4.3 and 5.9). Also the smallest univoque number belonging to $(b, b+1)$ is transcendental.

5. Conclusion. There are many papers dealing with univoque numbers. We just mention here the study of univoque Pisot numbers. The authors together with K. G. Hare determined in [5] the smallest univoque Pisot number, which happens to have algebraic degree 14. Note that the number corresponding to the sequence of Proposition 2 is the larger real root of the polynomial $X^2 - tX - (b - t + 1)$, hence a Pisot number (which is unitary if $t = b$). Also note that for any $b \geq 2$, the real number β such that the β-expansion of 1 is $b1^\infty$ is a univoque Pisot number in $(b, b+1)$. It would be interesting to determine the smallest univoque Pisot number in $(b, b+1)$: the case $b = 1$ was addressed in [5], but the proof uses heavily the fine structure of Pisot numbers in $(1, 2)$ (see [8, 20, 21]). A similar study of Pisot numbers in $(b, b + 1)$ would certainly help.

Acknowledgments. The authors thank M. de Vries and V. Komornik for their remarks on a previous version of this paper.

References

[1] J.-P. Allouche, *Théorie des Nombres et Automates*, Thèse d'État, Bordeaux, 1983, http://tel.archives-ouvertes.fr/tel-00343206/fr/.

[2] J.-P. Allouche et M. Cosnard, *Itérations de fonctions unimodales et suites engendrées par automates*, C. R. Acad. Sci. Paris Sér. I 296 (1983), 159–162.

[3] —, —, *The Komornik–Loreti constant is transcendental*, Amer. Math. Monthly 107 (2000), 448–449.

[4] —, —, *Non-integer bases, iteration of continuous real maps, and an arithmetic self-similar set*, Acta Math. Hungar. 91 (2001), 325–332.

[5] J.-P. Allouche, Ch. Frougny and K. Hare, *On univoque Pisot numbers*, Math. Comp. 76 (2007), 1639–1660.

[6] J.-P. Allouche and J. Shallit, *The ubiquitous Prouhet–Thue–Morse sequence*, in: C. Ding *et al.* (eds.), Sequences and Their Applications (Singapore, 1998), Springer, 1999, 1–16.

[7] —, —, *Automatic Sequences. Theory, Applications, Generalizations*, Cambridge Univ. Press, Cambridge, 2003.

[8] M. Amara, *Ensembles fermés de nombres algébriques*, Ann. Sci. École Norm. Sup. 83 (1966), 215–270.

[9] J. Berstel, *Sur la construction de mots sans carré*, Sém. Théor. Nombres Bordeaux (1978–1979), Exp. 18, 18-01–18-15.

[10] C. H. Braunholtz, *An infinite sequence on 3 symbols with no adjacent repeats* (*Solution to Problem 439 posed by H. Noland*), Amer. Math. Monthly 70 (1963), 675–676.

[11] M. Cosnard, *Étude de la classification topologique des fonctions unimodales*, Ann. Inst. Fourier (Grenoble) 35 (1985), 59–77.

[12] Z. Daróczy and I. Kátai, *Univoque sequences*, Publ. Math. Debrecen 42 (1993), 397–407.

[13] —, —, *On the structure of univoque numbers*, ibid. 46 (1995), 385–408.

[14] P. Erdős, I. Joó and V. Komornik, *Characterization of the unique expansions* $1 = \sum_{i=1}^{\infty} q^{-n_i}$, *and related problems*, Bull. Soc. Math. France 118 (1990), 377–390.

[15] S. Istrail, *On irreductible* [sic] *languages and nonrational numbers*, Bull. Math. Soc. Sci. Math. R. S. Roumanie 21 (1977), 301–308.

[16] I. Kátai and G. Kallós, *On the set for which 1 is univoque*, Publ. Math. Debrecen 58 (2001), 743–750.

[17] V. Komornik and P. Loreti, *Unique developments in non-integer bases*, Amer. Math. Monthly 105 (1998), 636–639.

[18] V. Komornik and P. Loreti, *Subexpansions, superexpansions and uniqueness properties in non-integer bases*, Period. Math. Hungar. 44 (2002), 197–218.

[19] K. Mahler, *Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen*, Math. Ann. 101 (1929), 342–366; Corrigendum, ibid. 103 (1930), 532.

[20] F. L. Talmoudi, *Sur les nombres de* $S \cap [1, 2]$, C. R. Acad. Sci. Paris Sér. Math. 285 (1977), 969–971.

[21] —, *Sur les nombres de* $S \cap [1, 2[$, ibid. 287 (1978), 739–741.

[22] M. de Vries and V. Komornik, *Unique expansions of real numbers*, preprint, 2007, http://arxiv.org/abs/math/0609708v3.

CNRS, LRI, UMR 8623
Université Paris-Sud, Bâtiment 490
F-91405 Orsay Cedex, France
E-mail: allouche@lri.fr

LIAFA, CNRS UMR 7089
Case 7014
F-75205 Paris Cedex 13, France
and
Université Paris 8
E-mail: Christiane.Frougny@liafa.jussieu.fr

Received on 9.11.2007
and in revised form on 5.10.2008 (5571)