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# Small values of the Euler function and the Riemann hypothesis

by

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*À André Schinzel pour son 75ème anniversaire,*
*en très amical hommage*

**1. Introduction.** Let $\varphi$ be the Euler function. In 1903, it was proved by E. Landau (cf. [5, §59] and [4, Theorem 328]) that

$$\limsup_{n\to\infty} \frac{n}{\varphi(n)\log\log n} = e^\gamma = 1.7810724179\ldots$$

where $\gamma = 0.5772156649\ldots$ is Euler's constant.

In 1962, J. B. Rosser and L. Schoenfeld proved (cf. [9, Theorem 15])

$$\frac{n}{\varphi(n)} \le e^\gamma \log\log n + \frac{2.51}{\log\log n} \tag{1.1}$$

for $n \ge 3$ and asked if there exist an infinite number of $n$ such that $n/\varphi(n) > e^\gamma \log\log n$. In [6] (cf. also [7]), I answered this question in the affirmative. Soon after, A. Schinzel told me that he had worked unsuccessfully on this question, which made me very proud to have solved it.

For $k \ge 1$, $p_k$ denotes the $k$th prime and

$$N_k = 2\cdot 3\cdot 5\cdot \ldots \cdot p_k$$

the primorial number of order $k$. In [6], it is proved that the Riemann hypothesis (for short RH) is equivalent to

$$\forall k \ge 1, \qquad \frac{N_k}{\varphi(N_k)} > e^\gamma \log\log N_k.$$

The aim of the present paper is to make the results of [6] more precise by estimating the quantity

$$c(n) = \left(\frac{n}{\varphi(n)} - e^\gamma \log\log n\right)\sqrt{\log n}. \tag{1.2}$$

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Let us denote by $\rho$ a generic root of the Riemann $\zeta$ function satisfying $0 < \Re(\rho) < 1$. Under RH, $1-\rho=\overline{\rho}$. It is convenient to define (cf. [2, p. 159])

$$\beta = \sum_{\rho} \frac{1}{\rho(1-\rho)} = 2+\gamma-\log\pi-2\log 2 = 0.0461914179\ldots. \tag{1.3}$$

We shall prove

Theorem 1.1. *Under the Riemann hypothesis* (*RH*) *we have*

$$\limsup_{n\to\infty} c(n) = e^{\gamma}(2+\beta) = 3.6444150964\ldots, \tag{1.4}$$

$$\forall n \ge N_{120569} = 2\cdot 3\cdot \ldots \cdot 1591883, \qquad c(n) < e^{\gamma}(2+\beta), \tag{1.5}$$

$$\forall n \ge 2, \qquad c(n) \le c(N_{66}) = c(2\cdot 3\cdot\ldots\cdot 317) = 4.0628356921\ldots, \tag{1.6}$$

$$\forall k \ge 1, \qquad c(N_k) \ge c(N_1) = c(2) = 2.2085892614\ldots. \tag{1.7}$$

We keep the notation of [6]. For a real $x \ge 2$, the usual Chebyshev functions are denoted by

$$\theta(x) = \sum_{p\le x} \log p \quad \text{and} \quad \psi(x) = \sum_{p^m \le x} \log p. \tag{1.8}$$

We set

$$f(x) = e^{\gamma}\log\theta(x)\prod_{p\le x}(1-1/p). \tag{1.9}$$

Mertens's formula yields $\lim_{x\to\infty} f(x) = 1$. In [6, Th. 3(c)] it is shown that, if RH fails, there exists $b$, $0<b<1/2$, such that

$$\log f(x) = \Omega_{\pm}(x^{-b}). \tag{1.10}$$

For $p_k \le x < p_{k+1}$, we have $f(x) = e^{\gamma}\log\log(N_k)\frac{\varphi(N_k)}{N_k}\cdot$ When $k\to\infty$, by observing that the Taylor development about 1 yields $\log f(p_k) \sim f(p_k)-1$, we get

$$\log f(p_k) \sim f(p_k) - 1 = \frac{\varphi(N_k)}{N_k}\,\frac{c(N_k)}{\sqrt{\log N_k}} \sim \frac{e^{-\gamma}}{\log\log N_k}\,\frac{c(N_k)}{\sqrt{\log N_k}},$$

and it follows from (1.10) that, if RH does not hold, then

$$\liminf_{n\to\infty} c(n) = -\infty \quad \text{and} \quad \limsup_{n\to\infty} c(n) = +\infty.$$

Therefore, from Theorem 1.1, we deduce:

Corollary 1.1. *Each of the four assertions* (1.4) *to* (1.7) *is equivalent to the Riemann hypothesis.*

**1.1. Notation and results used.** If $\theta(x)$ and $\psi(x)$ are the Chebyshev functions defined by (1.8), we set

$$R(x) = \psi(x) - x \quad \text{and} \quad S(x) = \theta(x) - x. \tag{1.11}$$

Under RH, we shall use the upper bound (cf. [10, (6.3)])

$$x \geq 599 \;\Rightarrow\; |S(x)| \leq T(x) := \frac{1}{8\pi}\sqrt{x}\log^2 x. \tag{1.12}$$

P. Dusart (cf. [1, Table 6.6]) has shown that

$$\theta(x) < x \qquad \text{for } x \leq 8 \cdot 10^{11}, \tag{1.13}$$

thus improving the result of R. P. Brent who has checked (1.13) for $x < 10^{11}$ (cf. [10, p. 360]). We shall also use (cf. [9, Theorem 10])

$$\theta(x) \geq 0.84\, x \geq \tfrac{4}{5}x \qquad \text{for } x \geq 101. \tag{1.14}$$

As in [6], we define the integrals

$$K(x) = \int_x^\infty \frac{S(t)}{t^2}\left(\frac{1}{\log t} + \frac{1}{\log^2 t}\right) dt, \tag{1.15}$$

$$J(x) = \int_x^\infty \frac{R(t)}{t^2}\left(\frac{1}{\log t} + \frac{1}{\log^2 t}\right) dt, \tag{1.16}$$

and, for $\Re(z) < 1$,

$$F_z(x) = \int_x^\infty t^{z-2}\left(\frac{1}{\log t} + \frac{1}{\log^2 t}\right) dt. \tag{1.17}$$

We also set, for $x \geq 1$,

$$W(x) = \sum_\rho \frac{x^{i\,\Im(\rho)}}{\rho(1-\rho)}, \tag{1.18}$$

so that, under RH, from (1.3) we have

$$|W(x)| \leq \beta = \sum_\rho \frac{1}{\rho(1-\rho)}. \tag{1.19}$$

We often implicitly use the following result: for $a$ and $b$ positive, the function

$$t \mapsto \frac{\log^a t}{t^b} \quad \text{is decreasing for } t > e^{a/b} \tag{1.20}$$

and

$$\max_{t\geq 1} \frac{\log^a t}{t^b} = \left(\frac{a}{eb}\right)^a. \tag{1.21}$$

*Organization of the article.* In Section 2, the results of [6] about $f(x)$ are revised so as to get effective upper and lower bounds for both $\log f(x)$ and $1/f(x) - 1$ under RH (cf. Proposition 2.1). In Section 3, we study $c(N_k)$ and $c(n)$ in terms of $f(p_k)$. Section 4 is devoted to the proof of Theorem 1.1.

**2. Estimate of** $\log(f(x))$**.** The following lemma is Proposition 1 of [6].

LEMMA 2.1. *For* $x \ge 121$*, we have*

$$K(x) - \frac{S^2(x)}{x^2 \log x} \le \log f(x) \le K(x) + \frac{1}{2(x-1)}. \tag{2.1}$$

The next lemma is a slight improvement of Lemma 1 of [6].

LEMMA 2.2. *Let* $x$ *be a real number,* $x > 1$*. For* $\Re(z) < 1$*, we have*

$$F_z(x) = \frac{x^{z-1}}{(1-z)\log x} + r_z(x) \quad \textit{with} \quad r_z(x) = \int_x^\infty -\frac{z t^{z-2}}{(1-z)\log^2 t}\, dt \tag{2.2}$$

*and, if* $\Re(z) = 1/2$*,*

$$|r_z(x)| \le \frac{1}{|1-z|\sqrt{x}\log^2 x}\left(1 + \frac{4}{\log x}\right). \tag{2.3}$$

*Moreover, for* $z = 1/2$*, we have*

$$\frac{2}{\sqrt{x}\log x} - \frac{2}{\sqrt{x}\log^2 x} \le F_{1/2}(x) \le \frac{2}{\sqrt{x}\log x} - \frac{2}{\sqrt{x}\log^2 x} + \frac{8}{\sqrt{x}\log^3 x} \tag{2.4}$$

*and, for* $z = 1/3$*,*

$$0 \le F_{1/3}(x) \le \frac{3}{2x^{2/3}\log x}. \tag{2.5}$$

*Proof.* The proof of (2.2) is easy by taking the derivative. By partial summation, we get

$$r_z(x) = -\frac{z}{1-z}\left(\frac{x^{z-1}}{(1-z)\log^2 x} + \int_x^\infty \frac{2\, t^{z-2}}{(z-1)\log^3 t}\, dt\right). \tag{2.6}$$

If we assume $\Re(z) = 1/2$, we have $1 - z = \overline{z}$ and

$$|r_z(x)| \le \frac{1}{|1-z|\sqrt{x}\log^2 x} + \frac{2}{|1-z|\log^3 x}\int_x^\infty t^{-3/2}\, dt,$$

which yields (2.3). The estimates (2.4) follow from (2.2) and (2.6) by choosing $z = 1/2$, while (2.5) follows from (2.2) since $r_{1/3}$ is negative. ∎

To estimate the difference $J(x) - K(x)$, we need Lemma 2.4 below, which, under RH, is an improvement of Propositions 3.1 and 3.2 of [1] (obtained without assuming RH). The following lemma will be useful for proving Lemma 2.4.

LEMMA 2.3. *Let* $\kappa = \kappa(x) = \left\lfloor \frac{\log x}{\log 2} \right\rfloor$ *the largest integer such that* $x^{1/\kappa} \ge 2$*. For* $x \ge 16$*, set*

$$H(x) = 1 + \sum_{k=4}^{\kappa} x^{1/k - 1/3},$$

*and for* $x \geq 4$,

$$L(x) = \sum_{k=2}^{\kappa} \ell_k(x) \quad \textit{with} \quad \ell_k(x) = \frac{T(x^{1/k})}{x^{1/3}} = \frac{\log^2 x}{8\pi\, k^2 x^{1/3-1/(2k)}}.$$

*Then*

(i) $H(x) \leq H(2^j)$ *for* $j \geq 9$ *and* $x \geq 2^j$.
(ii) $L(x) \leq L(2^j)$ *for* $j \geq 35$ *and* $x \geq 2^j$.

*Proof.* The function $H$ is continuous and decreasing on $[2^j, 2^{j+1})$; so, to show (i), it suffices to prove that for $j \geq 9$,

$$H(2^j) \geq H(2^{j+1}). \tag{2.7}$$

If $9 \leq j \leq 19$, we check (2.7) by computation. If $j \geq 20$, we have

$$\begin{aligned} H(2^j) - H(2^{j+1}) &= \sum_{k=4}^{j} 2^{j(\frac{1}{k}-\frac{1}{3})}\bigl(1 - 2^{\frac{1}{k}-\frac{1}{3}}\bigr) - 2^{(j+1)(\frac{1}{j+1}-\frac{1}{3})} \\ &\geq 2^{j(\frac{1}{4}-\frac{1}{3})}\bigl(1 - 2^{\frac{1}{4}-\frac{1}{3}}\bigr) - 2^{(j+1)(\frac{1}{j+1}-\frac{1}{3})} \\ &= 2^{-j/3}[(1-2^{-1/12})2^{j/4} - 2^{2/3}], \end{aligned}$$

which proves (2.7) since the above bracket is $\geq (1-2^{-1/12})2^{20/4} - 2^{2/3} = 0.208\ldots$ and therefore positive.

Let us assume that $j \geq 35$ so that $2^j \geq e^{24}$. From (1.20), for each $k \geq 2$, $x \mapsto \ell_k(x)$ is decreasing for $x \geq 2^j$ so that $L$ is decreasing on $[2^j, 2^{j+1})$, and to show (ii), it suffices to prove

$$L(2^j) \geq L(2^{j+1}). \tag{2.8}$$

We have

$$\begin{aligned} L(2^j) - L(2^{j+1}) &= \sum_{k=2}^{j} \{\ell_k(2^j) - \ell_k(2^{j+1})\} - \ell_{j+1}(2^{j+1}) \\ &\geq \ell_2(2^j) - \ell_2(2^{j+1}) - \ell_{j+1}(2^{j+1}) \\ &= \frac{\log^2 2}{32\pi}\, 2^{-j/3}\{2^{j/4}[j^2 - 2^{-1/12}(j+1)^2] - 4\cdot 2^{1/6}\}. \end{aligned}$$

For $j \geq 1/(2^{1/12}-1) = 16.81\ldots$, the above square bracket is increasing in $j$ and positive for $j = 35$. Therefore, the curly bracket is increasing for $j \geq 35$, and since its value for $j = 35$ is equal to $744.17\ldots$, (2.8) is proved for $j \geq 35$. ∎

Lemma 2.4. *Under RH, we have*

$$\psi(x) - \theta(x) \geq \sqrt{x} \quad \textit{for } x \geq 121, \tag{2.9}$$

*and, for* $x \ge 1$,

$$\frac{\psi(x)-\theta(x)-\sqrt{x}}{x^{1/3}} \le 1.332768\ldots \le \frac{4}{3}. \tag{2.10}$$

*Proof.* For $x < 599^3$, we check (2.9) by computation. Note that 599 is prime. Let $q_0 = 1$, and let $q_1 = 4$, $q_2 = 8$, $q_3 = 9, \ldots, q_{1922} = 599^3$ be the sequence of powers (with exponent $\ge 2$) of primes not exceeding $599^3$. On the intervals $[q_i, q_{i+1})$, the function $\psi - \theta$ is constant and $x \mapsto (\psi(x)-\theta(x))/\sqrt{x}$ is decreasing. For $11 \le i \le 1921$ (i.e. $121 \le q_i < q_{i+1} \le 599^3$), we calculate $\delta_i = (\psi(q_i)-\theta(q_i))/\sqrt{q_{i+1}}$ and find that $\min_{11\le i\le 1921} \delta_i = \delta_{1886} = 1.0379\ldots$ ($q_{1886} = 206468161 = 14369^2$) while $\delta_{10} = 0.9379\ldots < 1$ ($q_{10} = 81$).

Now, we assume $x \ge 599^3$, so that, by (1.12),

$$\psi(x)-\theta(x) \ge \theta(x^{1/2})+\theta(x^{1/3}) \ge x^{1/2}+x^{1/3}-T(x^{1/2})-T(x^{1/3}). \tag{2.11}$$

By using (1.21), we get

$$\frac{T(x^{1/2})}{x^{1/3}}+\frac{T(x^{1/3})}{x^{1/3}} = \frac{1}{8\pi}\left(\frac{\log^2 x}{4x^{1/12}}+\frac{\log^2 x}{9x^{1/6}}\right) \le \frac{20}{\pi e^2} = 0.86157\ldots,$$

which, with (2.11), implies

$$\psi(x)-\theta(x) \ge \sqrt{x}+\left(1-\frac{20}{\pi e^2}\right)x^{1/3} \ge \sqrt{x}. \tag{2.12}$$

The inequality (2.10) is Lemma 3 of [8]. Below we give another proof by considering three cases according to the values of $x$.

Case 1: $1 \le x < 2^{32}$. The largest $q_i$ smaller than $2^{32}$ is $q_{6947} = 4293001441 = 65521^2$. On the intervals $[q_i, q_{i+1})$, the function

$$G(x) := \frac{\psi(x)-\theta(x)-\sqrt{x}}{x^{1/3}}$$

is decreasing. By computing $G(q_0), G(q_1), \ldots, G(q_{6947})$ we get

$$G(x) \le G(q_{103}) = 1.332768\ldots \qquad [q_{103} = 80089 = 283^2].$$

Case 2: $2^{32} \le x < 64\cdot 10^{22}$. By using (1.13), we get

$$\psi(x)-\theta(x) = \sum_{k=2}^{\kappa}\theta(x^{1/k}) \le \sum_{k=2}^{\kappa} x^{1/k}$$

so that Lemma 2.3 implies $G(x) \le H(x) \le H(2^{32}) = 1.31731\ldots$.

Case 3: $x \ge 64\cdot 10^{22} \ge 2^{79}$. By (1.12) and (1.13), we get

$$\psi(x)-\theta(x) = \sum_{k=2}^{\kappa}\theta(x^{1/k}) \le \sum_{k=2}^{\kappa}\{x^{1/k}+T(x^{1/k})\},$$

whence, from Lemma 2.3, $G(x) \le H(x)+L(x) \le H(2^{79})+L(2^{79}) = 1.32386\ldots$. ∎

COROLLARY 2.1. *For $x \geq 121$, we have*

$$F_{1/2}(x) \leq J(x) - K(x) \leq F_{1/2}(x) + \tfrac{4}{3}F_{1/3}(x). \tag{2.13}$$

The following lemma is an improvement of [6, Proposition 2].

LEMMA 2.5. *Assume that RH holds. For $x > 1$, we may write*

$$J(x) = -\frac{W(x)}{\sqrt{x}\log x} - J_1(x) - J_2(x) \tag{2.14}$$

*with*

$$0 < J_1(x) \leq \frac{\log(2\pi)}{x\log x} \quad \textit{and} \quad |J_2(x)| \leq \frac{\beta}{\sqrt{x}\log^2 x}\left(1 + \frac{4}{\log x}\right). \tag{2.15}$$

*Proof.* In [6, (17)–(19)], for $x > 1$, it is proved that

$$J(x) = -\sum_\rho \frac{1}{\rho}F_\rho(x) - J_1(x)$$

with $J_1$ satisfying $0 < J_1(x) \leq \frac{\log(2\pi)}{x\log x}$.

Now, by Lemma 2.2, we have $F_\rho(x) = \frac{x^{\rho-1}}{(1-\rho)\log x} + r_\rho(x)$, which yields (2.14) by setting $J_2(x) = \sum_\rho (1/\rho) r_\rho(x)$. Further, from (2.3) and (1.3), we get the upper bound for $|J_2(x)|$ given in (2.15). ∎

PROPOSITION 2.1. *Under RH, for $x \geq x_0 = 10^9$, we have*

$$-\frac{2+W(x)}{\sqrt{x}\log x} + \frac{0.055}{\sqrt{x}\log^2 x} \leq \log f(x) \leq -\frac{2+W(x)}{\sqrt{x}\log x} + \frac{2.062}{\sqrt{x}\log^2 x} \tag{2.16}$$

*and*

$$\frac{2+W(x)}{\sqrt{x}\log x} - \frac{2.062}{\sqrt{x}\log^2 x} \leq \frac{1}{f(x)} - 1 \leq \frac{2+W(x)}{\sqrt{x}\log x} - \frac{0.054}{\sqrt{x}\log^2 x}. \tag{2.17}$$

*Proof.* By collecting the information from (2.1), (1.12), (2.13), (2.14), (2.15), (2.4) and (2.5), for $x \geq 599$, we get

$$\begin{aligned} \log f(x) \geq &-\frac{W(x)+2}{\sqrt{x}\log x} + \frac{2-\beta}{\sqrt{x}\log^2 x} - \frac{8+4\beta}{\sqrt{x}\log^3 x} \\ &- \frac{\log(2\pi)}{x\log x} - \frac{2}{x^{2/3}\log x} - \frac{\log^3 x}{64\pi^2 x} \end{aligned} \tag{2.18}$$

and

$$\log f(x) \leq -\frac{W(x)+2}{\sqrt{x}\log x} + \frac{2+\beta}{\sqrt{x}\log^2 x} + \frac{4\beta}{\sqrt{x}\log^3 x} + \frac{1}{2(x-1)}. \tag{2.19}$$

Since $x \ge x_0 = 10^9$, (2.18) and (2.19) imply respectively

$$\log f(x) \ge -\frac{W(x)+2}{\sqrt{x}\log x} + \frac{1}{\sqrt{x}\log^2 x}\left(2-\beta-\frac{8+4\beta}{\log x_0} - \frac{\log(2\pi)\log x_0}{\sqrt{x_0}} - \frac{2\log x_0}{x_0^{1/6}} - \frac{\log^5 x_0}{64\pi^2\sqrt{x_0}}\right) \tag{2.20}$$

and

$$\log f(x) \le -\frac{W(x)+2}{\sqrt{x}\log x} + \frac{1}{\sqrt{x}\log^2 x}\left(2+\beta+\frac{4\beta}{\log x_0}+\frac{\sqrt{x_0}\log^2 x_0}{2(x_0-1)}\right), \tag{2.21}$$

which proves (2.16).

Setting $v = -\log f(x)$, it follows from (2.16), (1.19) and (1.3) that

$$v \le \frac{W(x)+2}{\sqrt{x}\log x} \le \frac{2+\beta}{\sqrt{x}\log x} \le v_0 := \frac{2+\beta}{\sqrt{x_0}\log x_0} = 0.00000312\ldots.$$

By Taylor's formula, we have $e^v - 1 \ge v$, which, with (2.16), provides the lower bound of (2.17), and

$$e^v - 1 - v \le \frac{e^{v_0}}{2}v^2 \le \frac{e^{v_0}(2+\beta)^2}{2x\log^2 x} \le \frac{e^{v_0}(2+\beta)^2}{2\sqrt{x_0}\sqrt{x}\log^2 x} = \frac{0.0000662\ldots}{\sqrt{x}\log^2 x},$$

which implies the upper bound in (2.17). ∎

## 3. Bounding $c(n)$

Lemma 3.1. *Let $n$ and $k$ be two integers satisfying $n \ge 2$ and $k \ge 1$. Assume that either the number $j = \omega(n)$ of distinct prime factors of $n$ is equal to $k$, or $N_k \le n < N_{k+1}$. Then*

$$c(n) \le c(N_k). \tag{3.1}$$

*Proof.* It follows from our hypothesis that $n \ge N_k$ and $j \le k$. Let us write $n = q_1^{\alpha_1}\ldots q_j^{\alpha_j}$ (with $q_1 < \cdots < q_j$ as defined in the proof of Lemma 2.4). We have

$$\frac{n}{\varphi(n)} = \prod_{i=1}^{j}\frac{1}{1-1/q_i} \le \prod_{i=1}^{j}\frac{1}{1-1/p_i} \le \prod_{i=1}^{k}\frac{1}{1-1/p_i} = \frac{N_k}{\varphi(N_k)},$$

which yields

$$c(n) \le \left(\frac{N_k}{\varphi(N_k)} - e^\gamma\log\log n\right)\sqrt{\log n} =: h(n) \tag{3.2}$$

and $h(n)$ can be extended to all real $n$. Further,

$$\begin{aligned}\frac{d}{dn}h(n) &= \frac{1}{2n\sqrt{\log n}}\left(\frac{N_k}{\varphi(N_k)} - e^\gamma\log\log n - 2e^\gamma\right)\\ &\le \frac{1}{2n\sqrt{\log n}}\left(\frac{N_k}{\varphi(N_k)} - e^\gamma\log\log N_k - 2e^\gamma\right).\end{aligned}$$

If $k=1$ or 2, it is easy to see that the expression in parentheses is negative, while, if $k\ge 3$, by (1.1), it is smaller than $\frac{2.51}{\log\log N_k}-2e^\gamma$, which is also negative because $\log\log N_k\ge\log\log 30=1.22\ldots$. Therefore, $h(n)\le h(N_k)=c(N_k)$, which, with (3.2), completes the proof of Lemma 3.1. ∎

PROPOSITION 3.1. *Assume that* $x_0=10^9\le p_k\le x<p_{k+1}$. *Under RH, we have*

$$c(N_k)\le e^\gamma(2+W(x))-\frac{0.07}{\log x}\le e^\gamma(2+\beta)-\frac{0.07}{\log x}\tag{3.3}$$

*and*

$$c(N_k)\ge e^\gamma(2+W(x))-\frac{3.7}{\log x}\ge e^\gamma(2-\beta)-\frac{3.7}{\log x}.\tag{3.4}$$

*Proof.* From (1.2) and (1.9), we get

$$c(N_k)=e^\gamma\sqrt{\theta(x)}(\log\theta(x))\left(\frac{1}{f(x)}-1\right).\tag{3.5}$$

By the fundamental theorem of calculus, (1.14) and (1.12), we have

$$\begin{aligned}|\sqrt{\theta(x)}\log\theta(x)-\sqrt{x}\log x| &= \left|\int_x^{\theta(x)}\frac{\log t+2}{2\sqrt{t}}\,dt\right|\le|\theta(x)-x|\frac{\log(4x/5)+2}{2\sqrt{4x/5}}\\ &\le\frac{\sqrt{5}}{4}T(x)\frac{\log x+2}{\sqrt{x}}=\frac{\sqrt{5}}{32\pi}(\log^2 x)(\log x+2),\end{aligned}$$

whence

$$\begin{aligned}\left|\frac{\sqrt{\theta(x)}\log\theta(x)}{\sqrt{x}\log x}-1\right| &\le\frac{\sqrt{5}(\log^2 x)(\log x+2)}{32\pi\sqrt{x}\log x}\\ &\le\frac{\sqrt{5}(\log^2 x_0)(\log x_0+2)}{32\pi\sqrt{x_0}\log x}\le\frac{0.0069}{\log x}.\end{aligned}$$

Therefore, (3.5), (2.17) and (1.19) yield

$$\begin{aligned}c(N_k) &\le e^\gamma\left(2+W(x)-\frac{0.054}{\log x}\right)\left(1+\frac{0.0069}{\log x}\right)\\ &\le e^\gamma(2+W(x))-\frac{e^\gamma}{\log x}(0.054-0.0069(2+\beta)),\end{aligned}$$

which proves (3.3). The proof of (3.4) is similar. ∎

**4. Proof of Theorem 1.1.** It follows from (3.1), (3.3) and (3.4) that

$$\limsup_{n\to\infty}c(n)=e^\gamma\Big(2+\limsup_{x\to\infty}W(x)\Big).$$

As observed in [6, p. 383], by the pigeonhole principle (cf. [3, §2.11] or [4, §11.12]), one can show that $\limsup_{x\to\infty}W(x)=\beta$, which proves (1.4).

To show the other items of Theorem 1.1, we first consider $k_0 = 50847534$, the number of primes up to $x_0 = 10^9$. For all $k \le k_0$, we have calculated $c(N_k)$ in Maple with 30 decimal digits, so that we may think that the first ten are correct.

We have found that for $k_1 = 120568 < k \le k_0$, we have $c(N_k) < e^\gamma(2+\beta)$ (while $c(N_{k_1}) = 3.6444180\ldots > e^\gamma(2+\beta)$) and for $1 \le k \le k_0$, we have $c(N_1) = c(2) \le c(N_k) \le c(N_{66})$.

Further, for $k > k_0$, (3.3) implies $c(N_k) < e^\gamma(2+\beta) < c(N_{66})$, which, together with Lemma 3.1, proves (1.5) and (1.6).

As a challenge, for $k_1 = 120568$, I ask what is the largest number $M$ such that $M < N_{k_1+1}$ and $c(M) \ge e^\gamma(2+\beta)$. Note that $M > N_{k_1}$ since, for $n = N_{k_1-1}p_{k_1+1}$, we have $c(n) = 3.6444178\ldots > e^\gamma(2+\beta)$. Another challenge is to determine all the $n$'s satisfying $n < N_{k_1+1}$ and $c(n) > e^\gamma(2+\beta)$.

Finally, for $k > k_0$, (3.4) implies

$$c(N_k) \ge e^\gamma(2-\beta) - \frac{3.7}{\log(10^9)} = 3.30\ldots > c(2),$$

which completes the proof of (1.7) and of Theorem 1.1. ∎

It is not known if $\liminf_{x\to\infty} W(x) = -\beta$. Let $\rho_1 = 1/2 + it_1$ with $t_1 = 14.13472\ldots$ be the first zero of $\zeta$. By using a theorem of Landau (cf. [3, Th. 6.1 and §2.4]), it is possible to prove that $\liminf_{x\to\infty} W(x) \le -1/(\rho_1(1-\rho_1)) = -0.00499\ldots$. A smaller upper bound is desired.

An interesting question is the following: assume that RH fails. Is it possible to get an upper bound for $k$ such that $k > k_0$ and either $c(N_k) > e^\gamma(2+\beta)$ or $c(N_k) < c(2)$?

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