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Rational periodic points for degree two polynomial morphisms on projective space

by

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1. Introduction and statement of results. Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a morphism of degree two which has a totally ramified fixed point at infinity, in other words, a *polynomial morphism*. We will denote by ϕ^n the nth iterate of ϕ. A point $P \in K$ is called *periodic of period n for ϕ* if there is a positive integer n such that $\phi^n(P) = P$. If n is the smallest such integer, it is called the *primitive period of P for ϕ*. Northcott's theorem [5] tells us that ϕ can have only finitely many rational periodic points defined over a number field and, hence, the primitive periods of rational periodic points must be bounded. For $n = 1$, 2, and 3 there are infinitely many examples of degree two polynomial maps defined over $\mathbb{Q}$ with $\mathbb{Q}$-rational periodic points with primitive period n. Morton [4] showed that there are no such maps with $\mathbb{Q}$-rational primitive 4-periodic points. Flynn, Poonen, and Schaefer [1] showed that there are no such maps with $\mathbb{Q}$-rational primitive 5-periodic points and made the following conjecture:

CONJECTURE 1. For $n \geq 4$ there is no quadratic polynomial $f \in \mathbb{Q}[x]$ with a rational periodic point with primitive period n.

More recently, Stoll [6] has shown conditionally that there are no degree two polynomial maps with $\mathbb{Q}$-rational primitive 6-periodic points. For degree two rational maps, Manes [3, Theorem 4] showed the existence of maps with $\mathbb{Q}$-rational periodic points of primitive period 4 and provided evidence for there being no maps with $\mathbb{Q}$-rational points of primitive period 5 or 6. This article examines the possible primitive period of a $\mathbb{Q}$-rational periodic point for a degree two polynomial morphism on $\mathbb{P}^N$ defined over $\mathbb{Q}$.

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Definition. We define a *polynomial* map $\phi : \mathbb{P}^N \to \mathbb{P}^N$ of degree d with coordinates $[x_0, \dots, x_N]$ as

$$\phi(x_0, \dots, x_N) = [\phi_0(x_0, \dots, x_N), \dots, \phi_{N-1}(x_0, \dots, x_N), x_N^d],$$

where each ϕ_i is a homogeneous form of degree d in the variables $x_0, \dots, x_N$. Such a map is a *morphism* if $\phi_0, \dots, \phi_{N-1}$ have no nontrivial common zeros when $x_N = 0$.

1.1. A first example. Consider a degree two polynomial map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ given by

$$\phi(x, y) = [ax^2 + bxy + cy^2, y^2].$$

We wish to find constants a, b, and c such that $P = [0, 1]$ is a periodic point of primitive period 3 for ϕ. To do so we choose any two distinct other points P_1 and P_2 and solve the three linear equations $\phi(P) = P_1$, $\phi(P_1) = P_2$, and $\phi(P_2) = P$ in the three unknowns a, b, and c to find a suitable map ϕ. Since we will use a similar, albeit more complicated, construction in Theorem 1, we explore this construction in detail for this simple example.

We see that

$$\phi([0, 1]) = [c, 1],$$

so we choose $c \neq 0$, say $c = 1$. Then we have

$$\phi([1, 1]) = [a + b + 1, 1].$$

We now choose b so that

$$\phi([1, 1]) \neq [0, 1], [1, 1],$$

say $b = 1 - a$. Then $\phi([1, 1]) = [2, 1]$ and so

$$\phi([2, 1]) = [2a + 3, 1].$$

Finally, we choose $a = -3/2$ to have $\phi([2, 1]) = [0, 1]$, making the degree two polynomial map

$$\phi([x, y]) = [-3/2x^2 + 5/2x + 1, y^2]$$

have $[0, 1]$ as a primitive 3-periodic point.

Trying to construct a $\mathbb{Q}$-rational primitive 4-periodic point in the same manner, at a minimum, requires more care. The obstruction lies in having to solve four equations in three unknowns. Conjecture 1 states that for a degree two polynomial morphism on $\mathbb{P}^1$ it is never possible to solve these larger systems of equations and the number of coefficients $\binom{1+2}{2} = 3$ is an upper bound on the primitive $\mathbb{Q}$-rational periods.

Theorem 1 demonstrates an infinite family of polynomial maps on $\mathbb{P}^N$ with periodic points with primitive period larger than $\binom{N+2}{2}$. Theorem 2 shows that these families contain infinitely many maps which are in fact morphisms of $\mathbb{P}^N$. Theorem 3 uses Theorems 1 and 2 to show that the primitive period of $\mathbb{Q}$-rational periodic points for polynomial morphisms of $\mathbb{P}^N$ grows faster than $c(k)N^k$ for any k and some constant $c(k)$.

1.2. The main results. In general, we can construct a degree two polynomial map with a $\mathbb{Q}$-rational periodic point with primitive period equal to the number of coefficients of a quadratic form in $N+1$ variables $\binom{N+2}{2} = (N+1)(N+2)/2$, by choosing one coefficient with each successive iterate as we did above for $\mathbb{P}^1$. We show that for $N \geq 2$ we can construct polynomial maps on $\mathbb{P}^N$ with a periodic point with primitive period larger than this value.

THEOREM 1. *Let $N \geq 2$. There is an infinite family of degree two polynomial maps $\phi : \mathbb{P}^N \to \mathbb{P}^N$ with a $\mathbb{Q}$-rational periodic point with primitive period*

$$\leq \begin{cases} 7 = (N+1)(N+2)/2 + 1 & \text{for } N = 2, \\ (N+1)(N+2)/2 + \lfloor (N-1)/2 \rfloor & \text{for } N \geq 3, \end{cases}$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Moreover, the dimension of the family is at least N.

THEOREM 2. *The infinite family of maps constructed in Theorem* 1 *contains infinitely many morphisms.*

THEOREM 3. *For N large enough, there exists a degree two polynomial morphism of $\mathbb{P}^N$ with a $\mathbb{Q}$-rational periodic point with primitive period larger than $c(k)N^k$ for any k and some constant $c(k)$ depending on k.*

In general, the bounds in Theorem 1 are not upper bounds on the primitive period. Several examples of polynomial morphisms with $\mathbb{Q}$-rational points with larger primitive period are included at the end of the article.

2. Proof of Theorem 1. We denote the ith coordinate of a point $P \in \mathbb{P}^N$ as $x_i(P)$, and define the polynomial map $\phi : \mathbb{P}^N \to \mathbb{P}^N$ by

$$x_i(\phi(x_0, \ldots, x_N)) = \begin{cases} \sum_{j=0}^{N-1} \sum_{k=j}^{N} c_i(j,k) x_j x_k & \text{for } i = 0, \ldots, N-1, \\ c_N(N,N) x_N^2 & \text{for } i = N. \end{cases}$$

We denote the nth image of P by ϕ as $\phi^n(P) = P_n$.

The method of construction is to choose appropriate values of the constants $c_i(j,k)$ so that the coordinates of each iterate are linear in at most two of the $c_i(j,k)$. When we have chosen all of the $c_i(j,k)$ with $j \neq k$, we will then be able to choose $c_i(j,j)$ so that $\phi(P)$ is determined and $\phi(\phi(P))$ is linear in one of the $c_i(j,j)$, allowing the primitive period to increase beyond the trivial value $\binom{N+2}{2}$ determined by the number of coefficients. We treat the case of $N = 2$ separately.

In Lemma 1, we choose the initial sequence of images by specifying $c_i(j,k)$, $0 \leq i \leq N$, for one pair (j,k) for each image. Then we proceed with the construction to increase the primitive period beyond the trivial lower bound.

LEMMA 1. *Let $\phi : \mathbb{P}^N \to \mathbb{P}^N$ be a degree two polynomial map. We may choose the first $(N^2+N)/2-1$ images of $[0,\dots,0,1]$ as*

$$\begin{aligned}
&[0,\dots,0,1] \xrightarrow{\phi} [1,0,\dots,0,1] \xrightarrow{\phi} [0,1,0,\dots,0,1] \xrightarrow{\phi} \cdots \xrightarrow{\phi} [0,\dots,0,1,1]\\
&\quad \xrightarrow{\phi} [1,1,0,\dots,0,1] \xrightarrow{\phi} [0,1,1,0,\dots,0,1] \xrightarrow{\phi} \cdots \xrightarrow{\phi} [0,\dots,0,1,1,1]\\
&\quad \xrightarrow{\phi} [1,1,1,0,\dots,0,1] \xrightarrow{\phi} [0,1,1,1,0,\dots,0,1] \xrightarrow{\phi} \cdots \xrightarrow{\phi} [0,\dots,0,1,1,1,1]\\
&\quad \vdots\\
&\quad \xrightarrow{\phi} [1,\dots,1,0,1] \xrightarrow{\phi} [0,1,\dots,1,1] \xrightarrow{\phi} [1,\dots,1]
\end{aligned}$$

by choosing all of the $c_i(j,k)$ except $c_i(0,N-1)$ and $c_i(k,k)$ for each $0 \le k \le N-1$ and each $0 \le i \le N-1$. Furthermore, $x_i(\phi([1,\dots,1]))$ is of the form

$$a_i c_i(0,N-1) + b_i$$

for some constants a_i, b_i for all $0 \le i \le N-1$.

Proof. Let $P = [0,\dots,0,1]$. Then $x_i(\phi(P))$ is linear in $c_i(N,N)$, so we can choose

$$\phi(P) = [1,0,\dots,0,1]$$

by setting

$$c_i(N,N) = \begin{cases} 1 & \text{for } i=0 \text{ and } i=N,\\ 0 & \text{for } 1 \le i \le N-1. \end{cases}$$

Next we choose the sequence of points

$$\begin{aligned}
&[1,0,\dots,0,1] \xrightarrow{\phi} [0,1,0,\dots,0,1] \xrightarrow{\phi} \cdots\\
&\qquad \xrightarrow{\phi} [0,0,\dots,0,1,1] \xrightarrow{\phi} [1,1,0,\dots,0,1],
\end{aligned}$$

where

$$[1,1,0,\dots,0,1] = P_{N+1}.$$

We can do this because $x_i(\phi(P_j))$ is of the form

$$x_i(\phi(P_j)) = \begin{cases} c_i(j-1,j-1) + c_i(j-1,N) + 1 & \text{for } i=0,\\ c_i(j-1,j-1) + c_i(j-1,N) & \text{for } 1 \le i \le N-1,\\ 1 & \text{for } i=N. \end{cases}$$

We choose

$$c_i(j-1,N) = \begin{cases} -1 - c_i(j-1,j-1) & \text{for } i=0,\\ 1 - c_i(j-1,j-1) & \text{for } i=j-1,\\ -c_i(j-1,j-1) & \text{otherwise.} \end{cases}$$

REMARK. With these choices of $c_i(N,N)$ and $c_i(k,N)$ for all $0 \le k \le N-1$, the image $x_k(\phi(x_0,x_1,\ldots,x_{N-1},1))$ contains terms of the form $c_k(i,i)x_i^2 - c_k(i,i)x_i$ for all $0 \le i \le N$ and $0 \le k < N-1$. Consequently, if $x_i = 1$ or $x_i = 0$, then $c_k(i,i)$ does not appear in the kth coordinate of the image.

Next we choose the sequence of images

$$[1,1,0,\ldots,0,0,1] \xrightarrow{\phi} [0,1,1,0,\ldots,0,1] \xrightarrow{\phi} \cdots \xrightarrow{\phi} [0,\ldots,0,1,1,1] \xrightarrow{\phi} [1,1,1,0,\ldots,0,1]$$

until

$$P_{2N-1} = [1,1,1,0,\ldots,0,1].$$

We can do this because we have already chosen $c_i(k,N)$ for all $0 \le k \le N-1$ and $c_i(N,N)$, causing the ith coordinate for all $0 \le i \le N-1$ of each image in this sequence to be linear only in the single coefficient $c_i(k,k+1)$ for some $0 \le k \le N-2$.

Next we choose the sequence of images

$$[1,1,1,0,\ldots,0,1] \xrightarrow{\phi} [0,1,1,1,0,\ldots,0,1] \xrightarrow{\phi} \cdots \xrightarrow{\phi} [0,\ldots,0,1,1,1,1] \xrightarrow{\phi} [1,1,1,1,0,\ldots,0,1].$$

Since we have already chosen $c_i(N,N)$, $c_i(k,N)$ for all $0 \le k \le N-1$, and $c_i(k,k+1)$ for all $0 \le k \le N-2$ and $0 \le i \le N-1$, the ith coordinate of these iterates is linear in $c_i(k,k+2)$ for some $0 \le k \le N-3$.

We repeat this process until we have

$$\begin{aligned} \phi(P_{(N^2+N)/2-3}) &= [1,\ldots,1,0,1,1] && (\text{linear in } c_i(0,N-2)),\\ \phi(P_{(N^2+N)/2-2}) &= [0,1,\ldots,1] && (\text{linear in } c_i(1,N-1)),\\ \phi(P_{(N^2+N)/2-1}) &= [1,\ldots,1] && (\text{linear in } c_i(0,N-1)). \end{aligned}$$

The only coefficients not yet chosen are $c_j(i,i)$ for all $0 \le i \le N-1$ and $c_j(0,N-1)$. We also know that $x_i(\phi(P))$ is linear in $c_i(0,N-1)$ for all $0 \le i \le N-1$. ∎

We are now ready to increase the primitive period beyond the trivial lower bound.

Proof of Theorem 1.

CASE 1: $N=2$. We begin where Lemma 1 finished. We have chosen $c_i(2,2)$ for all $0 \le i \le 2$ and $c_i(0,2)$ and $c_i(1,2)$ for all $0 \le i \le 1$ to have the sequence of points

$$[0,0,1] \xrightarrow{\phi} [1,0,1] \xrightarrow{\phi} [0,1,1] \xrightarrow{\phi} [1,1,1].$$

Now we choose $c_i(0, N-1) = c_i(0,1)$ so that

$$\phi([1,1,1]) = [0, K(0,1), 1]$$

for some constant $K(0,1) \notin \{0,1\}$. Each coordinate $x_i(\phi([0,K(0,1),1]))$ is linear in $c_i(1,1)$. We then choose the $c_i(1,1)$ so that

$$\phi([0,K(0,1),1]) = [0, K(1,1), 1],$$

where $K(1,1)$ is a constant with $K(1,1) \notin \{0,1,K(0,1)\}$. So we have three points chosen of the form $[0,x_1,1]$ whose images are given by

$$[0,0,1] \xrightarrow{\phi} [1,0,1],$$
$$[0,1,1] \xrightarrow{\phi} [1,1,1],$$
$$[0,K(0,1),1] \xrightarrow{\phi} [0,K(1,1),1].$$

Since $x_i(\phi[0,x_1,1])$ is a quadratic polynomial in x_1 for $0 \leq i \leq 1$, the image $\phi([0,K(1,1),1])$ is determined by these three known points and is of the form

$$\phi([0,K(1,1),1]) = [k_0, k_1, 1]$$

for some constants k_0 and k_1. Note that we may need to exclude finitely many choices of $K(0,1)$ and $K(1,1)$ (and hence of $c_1(0,1)$ and $c_1(1,1)$) so that $k_0 \notin \{0,1\}$. Therefore each $x_i(\phi([k_0,k_1,1]))$ is linear in $c_i(0,0)$ and we choose the $c_i(0,0)$ so that

$$\phi(\phi([0,K(1,1),1])) = [0,0,1].$$

This is a primitive 7-periodic point, and the family is dimension 2 since we can choose $c_1(0,1)$ and $c_1(1,1)$ arbitrarily (with finitely many exceptions).

It is easy to modify this construction to get points with primitive periods $1,\ldots,6$ because at each stage we are linear in at most two variables. So we simply choose the constant so that $\phi^n(P) = [0,0,1]$ at the appropriate iterate. The dimension of these families is larger since there are more free coefficients.

Case 2: $N \geq 3$. We begin where Lemma 1 finished and choose the $c_i(0,N-1)$ so that

$$\phi(P_{(N^2+N)/2}) = [0, K(0,N-1), 1, \ldots, 1]$$

where $K(0,N-1) \notin \{0,1\}$ is a constant. Each coordinate $x_i(\phi([0,K(0,N-1), 1,\ldots,1]))$ is linear in $c_i(1,1)$. Choose the $c_i(1,1)$ to have

$$\phi([0,K(0,N-1),1,\ldots,1]) = [K(1,1),1,\ldots,1,0,1],$$

where $K(1,1) \notin \{0,1\}$ is some constant. Now the ith coordinate of the image of $[K(1,1),1,\ldots,1,0,1]$ is linear in $c_i(0,0)$ for all $0 \leq i \leq N-1$. So we choose the $c_i(0,0)$ so that

$$\phi([K(1,1),1,\ldots,1,0,1]) = [0,K(0,0),1,\ldots,1]$$

for some constant $K(0,0) \notin \{0,1,K(0,N-1)\}$. Note that there are three points of the form $[0,x_1,1,\ldots,1]$ whose images are given by

$$[0,1,1,\ldots,1] \xrightarrow{\phi} [1,\ldots,1],$$

$$[0,K(0,N-2),1,\ldots,1] \xrightarrow{\phi} [K(1,1),1,\ldots,1,0,1],$$

$$[0,0,1,\ldots,1] \xrightarrow{\phi} [1,1,\ldots,1,0,1].$$

Since each $x_i(\phi([0,x_1,1,\ldots,1]))$ is a degree two polynomial in x_1, the three known points and their images completely determine the image of any point of the form $[0,x_1,1,\ldots,1]$. The 0th coordinate is a nonconstant function of x_1 and the $(N-1)$st coordinate is a nonconstant function of x_1. The remaining coordinates take on a constant value of 1. Therefore, we have

$$\phi([0,K(0,0),1,\ldots,1]) = [k_0,1,1,\ldots,1,k_{N-1},1]$$

for some constants k_0 and k_{N-1} with $k_{N-1} \notin \{0,1\}$ (again we may need to exclude finitely many choices of $K(1,1)$ and $K(0,N-2)$ so that $k_{N-1} \notin \{0,1\}$). In particular, since the $c_i(0,0)$ are already chosen, each coordinate $x_i(\phi([k_0,1,1,\ldots,1,k_{N-1},1]))$ is linear in $c_i(N-1,N-1)$. From our choice of $[0,K(0,0),1,\ldots,1]$ we have determined $\phi([0,K(0,0),1,\ldots,1])$ and have $\phi(\phi([0,K(0,0),1,\ldots,1]))$ as our next iterate to consider. We have thus increased the primitive period of $[0,\ldots,0,1]$ by two with the choice of the $c_i(0,0)$.

If $N=3$ we are done since we are linear in the $c_i(N-1,N-1)$ and they are the only unchosen coefficients; so we choose the $c_i(N-1,N-1)$ to make the point periodic.

For $N>3$ we repeat the process. Choose the $c_i(N-1,N-1)$ to get

$$[0,0,K(N-1,N-1),1,\ldots,1,0,1]$$

with $K(N-1,N-1) \notin \{0,1\}$. The coordinates of the image are linear in the $c_i(2,2)$ so we choose

$$\phi([0,0,K(N-1,N-1),1,\ldots,1,0,1]) = [0,0,K(2,2),1,\ldots,1,0,1],$$

for $K(2,2) \notin \{0,1,K(N-1,N-1)\}$. Then $\phi([0,0,K(2,2),1,\ldots,1,0,1])$ is completely determined since we have three points of the form $[0,0,x_2,1,\ldots,1,0,1]$ whose images are known. These images are

$$[0,0,K(N-1,N-1),1,\ldots,1,0,1] \xrightarrow{\phi} [0,0,K(2,2),1,\ldots,1,0,1],$$

$$[0,0,1,1,\ldots,1,0,1] \xrightarrow{\phi} [0,0,0,1,\ldots,1,1,1,1],$$

$$[0,0,0,1,\ldots,1,0,1] \xrightarrow{\phi} [0,0,0,0,1,\ldots,1,1,1,1].$$

We have

$$\phi([0,0,K(2,2),1,\ldots,1,0,1]) = [0,0,x,y,1,\ldots,1,z,1]$$

for some constants x, y, and z. Note that we may need to exclude finitely many choices of $K(N-1,N-1)$ and $K(2,2)$ so that $y \notin \{0,1\}$. We have already chosen the $c_i(2,2)$ and the $c_i(N-1,N-1)$, so each coordinate $x_i(\phi(\phi([0,0,K(2,2),1,\ldots,1,0,1])))$ is linear in $c_i(3,3)$, again increasing the primitive period by 2 with the choice of a single set of coefficients $c_i(2,2)$.

Continuing in this manner, we choose the $c_i(k,k)$ to get

$$[0,\ldots,0,K(k,k),1,\ldots,1,0,1],$$

where $K(k,k) \notin \{0,1\}$ and is the $(k+1)$st coordinate. The ith coordinate of the image is linear in $c_i(k+1,k+1)$. We choose

$$\phi([0,\ldots,0,K(k,k),1,\ldots,1,0,1]) = [0,\ldots,0,K(k+1,k+1),1,\ldots,1,0,1],$$

where $K(k+1,k+1)$ is the $(k+1)$st coordinate and $K(k+1,k+1) \notin \{0,1,K(k,k)\}$. The image $\phi([0,\ldots,0,K(k+1,k+1),1,\ldots,1,0,1])$ is completely determined since we have three points of the form $[0,\ldots,0,x_{k+1}, 1,\ldots,1,0,1]$ whose images are known; they are

$$\begin{aligned}
[0,\ldots,0,K(k,k),1,\ldots,1,0,1] &\xrightarrow{\phi} [0,\ldots,0,K(k+1,k+1),1,\ldots,1,0,1],\\
[0,\ldots,0,1,1,\ldots,1,0,1] &\xrightarrow{\phi} [0,\ldots,0,0,1,\ldots,1,1,1,1],\\
[0,\ldots,0,0,1,\ldots,1,0,1] &\xrightarrow{\phi} [0,\ldots,0,0,0,1,\ldots,1,1,1,1].
\end{aligned}$$

We have

$$\phi([0,\ldots,0,K(k+1,k+1),1,\ldots,1,0,1]) = [0,\ldots,0,x,y,1,\ldots,1,z,1]$$

for some constants x, y, and z. Note that we may need to exclude finitely many choices of $K(k,k)$ and $K(k+1,k+1)$ so that $y \notin \{0,1\}$. We have already chosen the $c_i(k+1,k+1)$ and the $c_i(N-1,N-1)$ so each $x_i(\phi(\phi([0,\ldots,0, K(k+1,k+1),1,\ldots,1,0,1])))$ is linear in $c_i(k+2,k+2)$, again increasing the primitive period by 2.

We continue this process until the only unchosen coefficients are either $\{c_i(N-2,N-2), c_i(N-3,N-3)\}$ or $\{c_i(N-2,N-2)\}$. In this first case, we do not have enough unchosen coefficients remaining to increase the primitive period further beyond the trivial value, so we simply choose the $c_i(N-3,N-3)$ to have the point

$$[0,\ldots,0,K(N-3,N-3),1,1]$$

with $K(N-3,N-3) \notin \{0,1\}$. Each $x_i(\phi([0,\ldots,0,K(N-3,N-3),1,1]))$ is linear in $c_i(N-2,N-2)$. We have now reduced to the second case and choose the $c_i(N-2,N-2)$ to have

$$\phi([0,\ldots,0,K(N-3,N-3),1,1]) = [0,0,\ldots,0,1],$$

making the point periodic of primitive period $(N+1)(N+2)/2 + \lfloor (N-1)/2 \rfloor$.

Note that along the way we were able to choose

$$\{c_1(0,N-1), c_1(0,0), c_0(1,1), \ldots, c_{N-2}(N-3,N-3), c_2(N-1,N-1)\}$$

arbitrarily, except for excluding a finite set of values, making this an infinite family of dimension N.

It is easy to modify this construction to get points with periods $< (N+1)(N+2)/2 + \lfloor (N-1)/2 \rfloor$ since at each stage we are linear in at most two variables. So we simply choose the coefficients so that $\phi^n(P) = [0,\ldots,0,1]$ at the appropriate iterate. The dimension of these families is larger since there are more free coefficients. ∎

3. Proof of Theorem 2. We will use the theory of Macaulay resultants to show that we can choose the coefficients of the maps in Theorem 1 so that they are morphisms; in other words, so that $\phi_0,\ldots,\phi_N$ have no nontrivial common zeros. Following [2]: given $N+1$ homogeneous forms $F_0,\ldots,F_N$ of degree d_i in $N+1$ variables $x_0,\ldots,x_N$, construct a matrix denoted $M_d(F_0,\ldots,F_N)$ where $d = 1 + \sum_i (d_i - 1)$. The columns of M_d correspond to the monomials of degree d in the variables $x_0,\ldots,x_N$, and the rows correspond to polynomials of the form rF_i where r is a monomial such that $\deg(rF_i) = d$. The entries of M_d are the coefficients of the column monomials in the row polynomials. The matrix has $\binom{N+d}{d}$ columns and the number of rows corresponding to each F_i is $\binom{N+d-d_i}{d-d_i}$. It is the transpose of the matrix of the linear map

$$(P_0,\ldots,P_N) \mapsto P_0F_0 + \cdots + P_NF_N,$$

where P_i is homogeneous of degree $d - d_i$. Consider the maximal minors of $M_d(F_0,\ldots,F_N)$. The determinants of these minors are polynomials in the coefficients of $F_0,\ldots,F_N$. Let R be the greatest common divisor of these determinants (as polynomials in the coefficients). Then R is called the *resultant* of $F_0,\ldots,F_N$ and (among other properties) it satisfies $R = 0$ if and only if the forms $F_0,\ldots,F_N$ have a common nontrivial zero.

Proof of Theorem 2. We are in the case of $N+1$ homogeneous forms ϕ_i in $N+1$ variables $x_0,\ldots,x_N$. We have each ϕ_i of degree 2 and hence $d = N+2$. We will show that the Macaulay matrix has a maximal minor that has nonzero determinant and hence that the resultant is nonzero for infinitely many maps in the family. In the matrix there are $\binom{2N+2}{N+2}$ columns corresponding to all of the monomials with degree $N+2$ and $(N+1)\binom{2N}{N}$ rows corresponding to the $\binom{2N}{N}$ monomials of degree $d-2$ for each of the $N+1$ forms ϕ_i. We need to extract a $\binom{2N+2}{N+2} \times \binom{2N+2}{N+2}$ minor with nonzero determinant. We first consider the case of largest possible period from Theorem 1.

For $N = 2$ we can write down the matrix (but do not do so here) and explicitly check that it has a maximal minor with nonzero determinant.

For $N \geq 3$ define (with $N-2$ replaced with $N-1$ for $N = 3$)

$$\begin{aligned}
S_N &= \{F \colon \deg(F) = d,\ x_N^2 \mid F\},\\
S_{N-2} &= \{F \colon \deg(F) = d,\ x_N^2 \nmid F,\ \text{ and } x_{N-2}^2 \mid F\},\\
S_{N-1} &= \{F \colon \deg(F) = d,\ x_N^2 \nmid F,\ x_{N-2}^2 \nmid F,\ \text{ and } x_{N-1}^2 \mid F\},\\
S_{N-3} &= \{F \colon \deg(F) = d,\ x_N^2 \nmid F,\ x_{N-1}^2 \nmid F,\ x_{N-2}^2 \nmid F,\ \text{ and } x_{N-3}^2 \mid F\},\\
S_{N-4} &= \{F \colon \deg(F) = d,\ x_N^2 \nmid F, \dots, x_{N-3}^2 \nmid F,\ \text{ and } x_{N-4}^2 \mid F\},\\
&\ \vdots\\
S_0 &= \{F \colon \deg(F) = d,\ x_N^2 \nmid F, \dots, x_1^2 \nmid F,\ \text{ and } x_0^2 \mid F\}.
\end{aligned}$$

Order the columns in reverse lexicographic order, $x_N > x_{N-1} > \cdots > x_0$, with the largest to the left. For the columns corresponding to a monomial in S_N, choose the row with a 1 on the diagonal (the row contains all 0's except one entry which is 1 since $\phi_N(x_0, \dots, x_N) = x_N^2$). For columns corresponding to monomials in S_{2k} with $k \neq 0$, choose the row with $c_{2k}(2k, 2k)$ on the diagonal. For columns corresponding to monomials in S_{2k-1} with $k \neq 1$, choose the row with $c_{2k}(2k-1, 2k-1)$ on the diagonal. For S_1 we choose the row with $c_0(1,1)$ on the diagonal, and for S_0 we choose the row with $c_1(0,0)$ on the diagonal. Finally, for columns corresponding to monomials in S_{N-2} we fix $i > 1$ odd and choose the row with $c_i(N-2, N-2)$ on the diagonal (use S_{N-1} and $c_i(N-1, N-1)$ for $N = 3$).

We have two facts to verify:

(1) These choices contain no duplicate rows.
(2) The resulting minor has nonzero determinant.

The first is clear since S_i and S_j are disjoint for $i \neq j$ and each row associated to an element of S_k has at most one entry containing a $c_i(k,k)$.

For the second, we start by examining the entries in each row. Each row associated to ϕ_i for $i \neq N$ contains a $c_i(N-2, N-2)$ (or $c_i(N-1, N-1)$ for $N = 3$) whose value depends on at least $c_i(N-3, N-3)$ (or $c_i(1,1)$ for $N = 3$) so is not identically 0. In addition, each of these rows contains a corresponding

$$c_i(N-2, N) = \begin{cases} -1 - c_i(N-2, N-2) & \text{for } i = 0,\\ 1 - c_i(N-2, N-2) & \text{for } i = N-2,\\ -c_i(N-2, N-2) & \text{otherwise.} \end{cases}$$

For x_1^2 we have $c_0(1,1)$ and $c_0(1,N) = -1 - c_0(1,1)$. For x_0^2, we have $c_1(0,0)$ and $c_1(0,N) = -c_0(1,1)$. For x_{2k}^2 with $k > 0$, there is a $c_{2k}(2k,2k)$ and a

$c_{2k}(2k,N) = 1 - c_{2k}(2k,2k)$. For $k > 1$ there is a $c_{2k}(2k-1, 2k-1)$ and a $c_{2k}(2k-1,N) = 1 - c_{2k}(2k-1,2k-1)$. The rest of the entries are either constants or depend on $c_1(0, N-1)$. Also note that each row contains each $c_i(k,k)$ at most once (in addition to the corresponding $c_i(k,N)$).

Note that $c_i(k,k)$ and $c_i(k,N)$ are possibly linearly dependent and that our choice of ordering has $c_i(k,k)$ appearing farther right in the matrix than $c_i(k,N)$. For the other entries, we are choosing one coefficient per iteration, so they are either constant, independent, or the next depends on the previous in a quadratic (or higher) fashion (since each ϕ_i is degree 2).

Assume that we have some linear combination of the rows that produces a row identically 0. Each row contains a $c_i(k,k)$ on the diagonal for some i and k and a $c_i(k,N)$ in some other entry. For the linear combination to result in 0, there are three cases to consider.

CASE 1. Assume two rows in the combination contain $c_i(k,k)$ and $c_i(k,N)$ in the same column. By our choice of ordering, the respective $c_i(k,N)$ and $c_i(k,k)$ in those rows would not be in the same column. Hence, we must also include rows in the combination that contain $c_i(k,N)$ and $c_i(k,k)$ in the corresponding columns. Again by our choice of ordering, we need to include at least two rows to do this and then we still have unpaired $c_i(k,k)$ and $c_i(k,N)$ as before. Therefore, we cannot choose any number of rows so that all of the $c_i(k,k)$ and $c_i(k,N)$ are paired by column.

CASE 2. Notice that by our choice of the S_i we have guaranteed that we cannot have $c_j(N-2,N-2)$ and $c_i(k,k)$ in the same column for any $k \neq N-2$. Let j be such that $c_j(N-2,N-2)$ is on the diagonal of the minor. Assume two rows in the combination have $c_i(N-2,N-2)$ and $c_j(N-2,N-2)$ in the same column for $i \neq j$. But with $c_j(N-2,N-2)$ used for S_{N-2}, the row containing $c_i(N-2,N-2)$ must also contain $c_i(k,k)$ for some $k \neq N-2$. As in Case 1, we are unable to find a combination of rows that pairs all of the $c_i(k,k)$ and $c_i(k,N)$.

CASE 3. Assume we have $c_i(k,k)$ and $c_i(k,N)$ paired with constants to get a combination of rows identically 0. However, every row containing a $c_i(k,k)$ with $k \neq N-2$ also contains $c_j(N-2,N-2)$ for some j. These $c_j(N-2,N-2)$ must be paired either with constants or with other $c_t(N-2,N-2)$. However, they cannot be paired with constants since the $c_i(k,k)$ are already paired with constants in a combination that results in 0, and $c_i(k,k)$ and $c_j(N-2,N-2)$ are not related in a linear fashion. Case 2 eliminates the possibility of pairing with another $c_t(N-2,N-2)$ for some t. So we must have $k = N-2$. Then all of the rows in the combination are associated to the same ϕ_j, and hence entries in columns cannot be paired appropriately to result in a combination of 0.

Therefore, no linear combination can have all entries as 0 and the determinant of this minor is not identically 0. Therefore, there are infinitely many choices of the coefficients that produce a map that is a morphism.

For the families with a periodic point with smaller primitive period, the matrix is similar but with more free constants, so similar choices of rows will also produce a minor with nonzero determinant. ∎

4. Proof of Theorem 3

LEMMA 2. *Given $\phi_1 : \mathbb{P}^N \to \mathbb{P}^N$ a polynomial morphism with a point P_1 of primitive period n, and $\phi_2 : \mathbb{P}^M \to \mathbb{P}^M$ a polynomial morphism with P_2 of primitive period m, there exists a polynomial morphism $\psi : \mathbb{P}^{N+M} \to \mathbb{P}^{N+M}$ and a point P with primitive period* $\operatorname{lcm}(n, m)$.

Proof. We restrict ϕ_1 to the affine chart $\mathbb{A}^N$ with $x_N \neq 0$ and ϕ_2 to the affine chart $\mathbb{A}^M$ with $x_M \neq 0$. The restricted points $\widetilde{P_1}$ and $\widetilde{P_2}$ still have period n and m, and the product map $\widetilde{\phi_1} \times \widetilde{\phi_2} : \mathbb{A}^{N+M} \to \mathbb{A}^{N+M}$ has the product of the dehomogenizations $\widetilde{P} = (\widetilde{P_1}, \widetilde{P_2})$ as a periodic point of primitive period $\operatorname{lcm}(n, m)$. This fact is simply the statement that the product of a cyclic group of order n with a cyclic group of order m has order $\operatorname{lcm}(n, m)$.

Now, homogenizing $\widetilde{\phi_1} \times \widetilde{\phi_2}$ to a map $\psi : \mathbb{P}^{N+M} \to \mathbb{P}^{N+M}$ we know that the first N forms and x_{N+M}^2 have no common nontrivial zeros in $x_0, \ldots, x_{N-1}, x_{N+M}$ and the next M forms and x_{N+M}^2 have no common nontrivial zeros in $x_N, \ldots, x_{N+M}$. Since the only variable shared between the two sets of forms is x_{N+M}, the map ψ is also a morphism and the homogenization of $\widetilde{P}$ has primitive period $\operatorname{lcm}(n, m)$ for ψ. ∎

Proof of Theorem 3. From Theorems 1 and 2 we can find morphisms $\phi : \mathbb{P}^N \to \mathbb{P}^N$ with $\mathbb{Q}$-rational periodic points with primitive period $1, 2, \ldots, (N+1)(N+2)/2$. Fix s a positive integer. Let $M = \lfloor N/s \rfloor$. Then $(M+1)(M+2)/2 > (N/s)(N/s)/2 = N^2/2s^2$ and for every prime $p \leq N^2/2s^2$ there is a point with primitive period p for some polynomial morphism of $\mathbb{P}^M$. Fix $\epsilon > 0$ and choose N large enough that the interval $((1-\epsilon)N^2/2s^2, N^2/2s^2)$ has at least s primes $p_1, \ldots, p_s$. We apply Lemma 2 to combine these points and associated morphisms to get a point $P \in \mathbb{P}^{sM} = \mathbb{P}^N$, which has primitive period

$$p_1 \cdots p_s \geq \frac{(1-\epsilon)^s}{2^s s^{2s}} N^{2s}$$

for a polynomial morphism $\psi : \mathbb{P}^N \to \mathbb{P}^N$. ∎

5. Some examples with larger primitive periods. With slightly different choices of coefficients, it is occasionally possible to increase the primitive period by more than 2 with a choice of a single set of coefficients. While a general method to ensure this occurrence was not discovered, in practice it is possible to construct a polynomial map for a specific N with a periodic point with primitive period that exceeds the bound presented in Theorem 1. These can then be combined as in Lemma 2 to produce morphisms of $\mathbb{P}^N$ with $\mathbb{Q}$-rational periodic points of large primitive period. The following examples present such maps for $N = 2$, 3, and 4. For the reader's convenience, the following table shows the trivial lower bound $\binom{N+2}{2}$, the lower bound from Theorem 1, and the primitive period exhibited in the example of a polynomial morphism $\phi : \mathbb{P}^N \to \mathbb{P}^N$. Note that since we are dealing with maps on $\mathbb{P}^N$ outside of the scope of Theorem 2, the maps were verified explicitly to be morphisms, but the details are omitted here.

N	Trivial bound	Theorem 1 bound	Example period
2	6	7	9
3	10	11	24
4	15	16	72

Example 1. The point $[0, 0, 1] \in \mathbb{P}^2$ is a periodic point of primitive period 9 for the morphism

$$\begin{aligned}\phi([x_0, x_1, x_2]) = [&-38/45x_0^2 + (2x_1 - 7/45x_2)x_0 + (-1/2x_1^2 - 1/2x_2x_1 + x_2^2),\\ &-67/90x_0^2 + (2x_1 + 157/90x_2)x_0 - x_2x_1, x_2^2].\end{aligned}$$

Example 2. The point $[0, 0, 0, 1] \in \mathbb{P}^3$ is a periodic point of primitive period 24 for the morphism

$$\begin{aligned}\phi([x_0, x_1, x_2, x_3]) = [&(-x_1 - x_3)x_0 + (-13/30x_1^2 + 13/30x_3x_1 + x_3^2),\\ &-1/2x_0^2 + (-x_1 + 3/2x_3)x_0 + (-1/3x_1^2 + 4/3x_3x_1),\\ &-3/2x_2^2 + 5/2x_2x_3 + x_3^2, x_3^2]\end{aligned}$$

created by combining a periodic point of primitive period 8 in $\mathbb{P}^2$ and a periodic point of primitive period 3 in $\mathbb{P}^1$.

Example 3. The point $[0, 0, 0, 0, 1] \in \mathbb{P}^4$ is a periodic point of primitive period 72 for the morphism

$$\begin{aligned}&\phi([x_0, x_1, x_2, x_3, x_4])\\ &\quad = [-38/45x_0^2 + (2x_1 - 7/45x_4)x_0 + (-1/2x_1^2 - 1/2x_4x_1 + x_4^2),\\ &\qquad -67/90x_0^2 + (2x_1 + 157/90x_4)x_0 - x_4x_1,\\ &\qquad (-x_3 - x_4)x_2 + (-13/30x_3^2 + 13/30x_4x_3 + x_4^2),\\ &\qquad -1/2x_2^2 + (-x_3 + 3/2x_4)x_2 + (-1/3x_3^2 + 4/3x_4x_3), x_4^2]\end{aligned}$$

created by combining periodic points of primitive period 8 and 9 in $\mathbb{P}^2$.

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