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The Fekete–Szegő theorem with splitting conditions: Part II

by

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1. Introduction. A classical theorem of Fekete and Szegő ([5]) says that if $E \subseteq \mathbb{C}$ is a compact set with logarithmic capacity $\gamma(E) \geq 1$, stable under complex conjugation, then every complex neighborhood of E contains infinitely many conjugate sets of algebraic integers. Raphael Robinson [10] strengthened this, showing that if $E \subseteq \mathbb{R}$, then every *real* neighborhood of E contains infinitely many conjugate sets of *totally real* algebraic integers.

In [3], Cantor stated a theorem of Fekete–Szegő–Robinson type for adelic sets in $\mathbb{P}^1$ over a number field K. Write K_v for the completion of K at a place v. Fix algebraic closures $\widetilde{K}$ of K, $\widetilde{K}_v$ of K_v, and let $\mathbb{C}_v$ be the completion of $\widetilde{K}_v$. If v is nonarchimedean, let $\widehat{O}_v$ be the ring of integers of $\mathbb{C}_v$. For each v, let a Galois-stable set $E_v \subset \mathbb{P}^1(\mathbb{C}_v)$ and a neighborhood U_v of E_v in $\mathbb{P}^1(\mathbb{C}_v)$ be given. Cantor developed a theory of capacity for adelic sets $\mathbb{E} = \prod_v E_v \subset \prod_v \mathbb{P}^1(\mathbb{C}_v)$, and defined the capacity $\gamma(\mathbb{E}, \mathfrak{X})$ for such a set with respect to a finite set of global points $\mathfrak{X} \subset \mathbb{P}^1(\widetilde{K})$. Let S be a finite set of places of K such that for each $v \in S$, there is a finite Galois extension L_w/K_v with $E_v \subset \mathbb{P}^1(L_w)$. Cantor's "Fekete–Szegő theorem with splitting conditions" ([3, Theorem 5.1.1, p. 199]) asserts that if $\gamma(\mathbb{E}, \mathfrak{X}) \geq 1$, then there are infinitely many points in $\mathbb{P}^1(\widetilde{K})$ whose conjugates in $\mathbb{C}_v$ belong to U_v for all v, *and* lie in $\mathbb{P}^1(L_w)$, if $v \in S$. We call the latter constraints "splitting conditions". Unfortunately, there were gaps in Cantor's proof. An explicit error occurs in his Lemma 4.2.6, where an adelic polynomial is asserted to have coefficients in a number field. There are also significant difficulties in both the archimedean and nonarchimedean patching processes which his construction did not come to terms with.

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Recently the author [12] established the theorem in the special case where $\mathfrak{X} = \{\infty\}$ and $K = \mathbb{Q}$, with $E_\infty = [-2r, 2r]$, and $E_p = \mathbb{Z}_p$ for finitely many primes p. In this paper, we prove the theorem for $\mathfrak{X} = \{\infty\}$ over an arbitrary number field K, and for general sets E_v. We also strengthen the result, showing that under appropriate hypotheses on the E_v, the algebraic numbers produced exist in all sufficiently large degrees.

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2. Statement of the results. Let K be a number field. For each place v of K, let a set $E_v \subset \mathbb{C}_v$, and a $\mathbb{C}_v$-neighborhood U_v of E_v, be given. Assume that E_v is bounded and stable under the group of continuous automorphisms $\mathrm{Gal}_{\mathrm{c}}(\mathbb{C}_v/K_v)$ for each v, and that $E_v = U_v = \widehat{O}_v$ for all but finitely many v. Write $\log(x) = \ln(x)$. For a compact set E_v, the *local capacity* $\gamma(E_v)$ can be defined by

$$\gamma(E_v) = e^{-V} \quad \text{with} \quad V = \inf_{\nu} \iint\limits_{E_v \times E_v} -\log(|z-w|_v)\, d\nu(z)\, d\nu(w),$$

where the inf is taken over all positive measures ν of total mass 1 supported on E_v. There are several other formulas for the local capacity; these, and methods of computing local capacities for certain classes of sets, will be recalled below. For an arbitrary set E_v, the local capacity is given by

$$\gamma(E_v) = \sup_{\substack{F_v \subset E_v \\ F_v \text{ compact}}} \gamma(F_v).$$

Our $\gamma(E_v)$ is sometimes called the "inner capacity" (see for example [11], [13]). For archimedean compact E_v, the local capacity $\gamma(E_v)$ is just the classical logarithmic capacity, or "transfinite diameter". For nonarchimedean compact E_v, it is the v-adic generalization of the transfinite diameter, first studied by Bertrandias (see [1]).

Cantor's theory defines the capacity $\gamma(\mathbb{E}, \mathfrak{X})$ of an adelic set $\mathbb{E} = \prod_v E_v$ with respect to a finite set of global algebraic points $\mathfrak{X}$. When $\mathfrak{X} = \{\infty\}$, Cantor's capacity reduces to a weighted product of local capacities:

$$\gamma(\mathbb{E}) := \gamma(\mathbb{E}, \{\infty\}) = \prod_v \gamma(E_v)^{D_v}.$$

The weights D_v are the same as the ones in the product formula for K: they (and the local capacities) depend on the normalization of the absolute values $|x|_v$.

In this paper we normalize absolute values as follows. If v is archimedean, we take $|x|_v = |x|$ to be the usual absolute value on $\mathbb{R}$ or $\mathbb{C}$. From an arithmetic point of view it would be preferable to take $|x|_v = |x|^2$ when $K_v \cong \mathbb{C}$, but we use the normalization above to avoid confusion concerning the literature about capacities of sets in $\mathbb{C}$. If v is nonarchimedean, we take $|x|_v$ to be the canonical absolute value on K_v given by the modulus of additive Haar measure (if π_v is a uniformizing element at v, then $|\pi_v|_v = 1/q_v$, where q_v is the order of the residue field of K_v). With this normalization, $D_v = 1$ unless v is archimedean and $K_v \cong \mathbb{C}$, in which case $D_v = 2$.

For nonarchimedean v, let $\log_v(x)$ denote the logarithm to the base $q_v = \#(O_v/\pi_v O_v)$. For archimedean v, write $\log_v(x) = \log(x) = \ln(x)$.

For each v, the absolute value $|x|_v$ on K_v extends uniquely to absolute values on $\widetilde{K}_v$ and $\mathbb{C}_v$, and we denote these extensions by $|x|_v$ as well. The Galois group $\mathrm{Gal}(\widetilde{K}_v/K_v)$ respects $|x|_v$, and its action on $\widetilde{K}_v$ extends to a continuous action on $\mathbb{C}_v$. Conversely, each continuous automorphism of $\mathbb{C}_v/K_v$ arises from an element of $\mathrm{Gal}(\widetilde{K}_v/K_v)$. By $\mathrm{Gal_c}(\mathbb{C}_v/K_v) \cong \mathrm{Gal}(\widetilde{K}_v/K_v)$ we mean the group of continuous automorphisms of $\mathbb{C}_v$, fixing K_v. Given $a_v \in \mathbb{C}_v$, and $r_v > 0$, let the "open" and "closed" balls be

$$B(a_v, r_v)^- = \{z \in \mathbb{C}_v : |z - a_v|_v < r_v\},$$
$$B(a_v, r_v) = \{z \in \mathbb{C}_v : |z - a_v|_v \le r_v\}.$$

By an *adelic neighborhood* $\mathbb{U}$ of $\mathbb{E}$, we mean a set $\mathbb{U} = \prod_v U_v$, where each U_v is an open set in $\mathbb{C}_v$ containing E_v.

Our main theorem is as follows:

THEOREM 2.1 (Fekete–Szegő theorem with splitting conditions). *Let K be a number field, and let $\mathbb{E} = \prod_v E_v$ be an adelic set over K such that each E_v is bounded and stable under $\mathrm{Gal_c}(\mathbb{C}_v/K_v)$, with $E_v = \widehat{O}_v$ for all but finitely many v. Suppose that $\prod_v \gamma(E_v)^{D_v} > 1$, or that $\prod_v \gamma(E_v)^{D_v} = 1$ and E_v is compact for at least one archimedean v. Let S be a finite set of places of K, such that for each $v \in S$ there is a finite Galois extension L_w/K_v with $E_v \subset L_w$. Then for each adelic neighborhood $\mathbb{U} = \prod_v U_v$ of $\mathbb{E}$, there are infinitely many numbers $\alpha \in \widetilde{K}$ with the following properties*:

(1) *for each v, all the conjugates of α in $\mathbb{C}_v$ belong to U_v*;

(2) *for each $v \in S$, these conjugates belong to $L_w \cap U_v$.*

Under appropriate hypotheses, we can require that the numbers α produced by Theorem 2.1 belong to the sets E_v:

THEOREM 2.2. *Let K be a number field, and let $\mathbb{E} = \prod_v E_v$ be an adelic set over K such that each E_v is bounded and stable under $\mathrm{Gal_c}(\mathbb{C}_v/K_v)$, and $E_v = \widehat{O}_v$ for all but finitely many v. Assume that $\prod_v \gamma(E_v)^{D_v} > 1$ and that*

(1) *for each archimedean v, either*

  (A) *E_v is the closure of its interior, and ∂E_v is a finite union of arcs*; *or*

  (B) *$K_v \cong \mathbb{R}$, and $E_v \subset \mathbb{R}$ is a finite union of closed intervals*;

(2) *for each nonarchimedean* v, *either*

(A) E_v *is a finite union of open and/or closed balls in* $\mathbb{C}_v$; *or*

(B) $E_v = \bigcup_{i=1}^{M}(a_i + \pi_w^{k_i} O_w) \subset L_w$ *is a finite union of cosets of the ring of integers of a finite Galois extension* L_w/K_v.

Then there are infinitely many $\alpha \in \widetilde{K}$ *such that for each* v, *all the conjugates of* α *in* $\mathbb{C}_v$ *belong to* E_v.

Theorem 2.2 follows from Theorem 2.1 by shrinking the archimedean E_v slightly, and taking the U_v to be the interiors of the original sets.

In a different direction, we can ask whether for sufficiently large n, numbers α as in Theorems 2.1 and 2.2 can be found with $[K(\alpha) : K] = n$. In general this is not possible; for example if $K = \mathbb{Q}$ and the archimedean set E_∞ and neighborhood U_∞ are stable under complex conjugation but do not meet $\mathbb{R}$, then the numbers α produced must be totally complex, so $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ must be even. Similar constraints arise at finite places. However, we can prove:

Theorem 2.3. *Let* K *be a number field, and let* $\mathbb{E}$ *and* $\mathbb{U}$ *satisfy the hypotheses of Theorem* 2.1 (*resp. Theorem* 2.2). *Suppose in addition that each* E_v *contains a point of* K_v, *and that* $L_w = K_v$ *for each* $v \in S$. *Then for all sufficiently large* n, *there exist algebraic numbers* α *satisfying the conclusions of Theorem* 2.1 (*resp. Theorem* 2.2) *with* $[K(\alpha) : K] = n$. *In addition, if* S_1' *and* S_2' *are finite sets of nonarchimedean places of* K *where* $E_v = \widehat{O}_v$, *disjoint from* S *and from each other, then for each* $v \in S_1'$ *we can require that* v *is inert in* $K(\alpha)/K$, *and for each* $v \in S_2'$ *we can require that* v *is totally ramified.*

It follows from Theorem 2.3 that there is a number Q, depending only on $\mathbb{E}$ and $\mathbb{U}$, such that the numbers α in Theorems 2.1 and 2.2 can be found with $[K(\alpha) : K] = n$ for all sufficiently large n divisible by Q.

The following concrete example generalizes the main theorem of [12]:

Corollary 2.4. *Let* $\mathbb{E} = \prod_v E_v$ *be an adelic set over* K *such that*

(1) *for each archimedean* v *with* $K_v \cong \mathbb{C}$, $E_v = B(a_v, r_v)$ *is a closed ball*;

(2) *for each archimedean* v *with* $K_v \cong \mathbb{R}$, $E_v = [a_v, b_v]$ *is a closed interval*;

(3) *for each nonarchimedean* v *in a finite set of places* S, $E_v = O_v$, *the ring of integers of* K_v;

(4) *for the remaining nonarchimedean* v, $E_v = \widehat{O}_v$.

Assume

$$\prod_{K_v \cong \mathbb{C}} r_v^2 \cdot \prod_{K_v \cong \mathbb{R}} \frac{b_v - a_v}{4} \cdot \prod_{v \in S} q_v^{-1/(q_v - 1)} > 1.$$

Then for all sufficiently large n, *there exist numbers* α *with* $[K(\alpha) : K] = n$ *whose conjugates all belong to* E_v, *for each* v.

Algebraic numbers satisfying a variety of congruence conditions, with controlled archimedean conjugates, can be constructed by imposing appropriate geometric conditions on the sets E_v. As a whimsical example, taking $K = \mathbb{Q}$ in Theorem 2.2, with $E_\infty = B(0, 2+\varepsilon)$, $E_2 = \mathbb{Z}_2$, $E_3 = \mathbb{Z}_3^{\mathrm{nr}}$ (the ring of integers of the maximal unramified extension of $\mathbb{Q}_3$), $E_5 = B_5(1,1)^-$, $E_7 = B_7(0,1)^-$, and $E_p = \widehat{O}_p$ for all remaining p, we have

COROLLARY 2.5. *For any $\varepsilon > 0$, there are infinitely many algebraic integers α such that*

(1) *the archimedean conjugates of α satisfy $|\sigma(\alpha)| < 2+\varepsilon$;*
(2) *the prime 2 splits completely in $\mathbb{Q}(\alpha)$;*
(3) *the prime 3 is unramified in $\mathbb{Q}(\alpha)$;*
(4) *at all places v of $\mathbb{Q}(\alpha)$ above 5, we have $\alpha \equiv 1 \pmod{\wp_v}$;*
(5) *at all places v of $\mathbb{Q}(\alpha)$ above 7, we have $\alpha \in \wp_v$.*

Here, $\gamma(B(0, 2+\varepsilon)) = 2+\varepsilon$, and $\gamma(\mathbb{Z}_2) = 2^{-1/(2-1)} = 1/2$. The condition "3 is unramified" is cost-free, since

$$\gamma(\mathbb{Z}_3^{\mathrm{nr}}) = \sup_{\substack{L_w/\mathbb{Q}_3 \\ \text{finite, unramified}}} \gamma(O_w) = \lim_{m\to\infty} 3^{-1/(3^m-1)} = 1.$$

Likewise, the "open" conditions at the primes 5 and 7 do not impose any cost, since the capacities of the sets $B_5(1,1)^-$ and $B_7(0,1)^-$ are both 1: the capacity of an open or closed ball is simply its radius, r. For all other primes, $\gamma(\widehat{O}_p) = 1$. Thus, $\gamma(\mathbb{E}) = 1 + \varepsilon/2 > 1$.

We should also mention a converse to the Fekete–Szegő theorem, which shows that the hypothesis $\gamma(\mathbb{E}) \geq 1$ is sharp. By a *PL-domain* in $\mathbb{C}_v$ ("Polynomial Lemniscate Domain") we mean a set of the form

$$U_v = \{z \in \mathbb{C}_v : |f_v(x)|_v \leq R_v\}$$

where $f_v(z) \in \mathbb{C}_v[z]$ is a nonconstant polynomial, and R_v belongs to the value group of $\mathbb{C}_v^\times$. For a set $E_v \subset \mathbb{C}_v$, we say that E_v is *algebraically capacitable* if

$$\gamma(E_v) = \inf_{\substack{U_v \supset E_v \\ U_v = \text{PL-domain}}} \gamma(U_v).$$

For archimedean v, Hilbert's Lemniscate Theorem shows that compact sets are algebraically capacitable (see [11, Proposition 3.3.3]). For nonarchimedean v, it is shown in [11, §4.4] that compact sets, PL-domains, and any finite combination of unions or intersections of such sets, are algebraically capacitable; in addition, finite unions of open balls or closed balls are algebraically capacitable. The following is a special case of [3, Theorem 5.1.2, p. 199]; see [11, Theorem 6.3.1] for a generalization to curves:

THEOREM 2.6 (Fekete's theorem). *Let* $\mathbb{E} = \prod_v E_v$ *be an affine adelic set over* K *such that each* E_v *is bounded*, *stable under* $\mathrm{Gal_c}(\mathbb{C}_v/K_v)$, *and algebraically capacitable*, *such that*

$$\prod_v \gamma(E_v)^{D_v} < 1.$$

Then there exists an adelic neighborhood $\mathbb{U} = \prod_v U_v$ *of* $\mathbb{E}$, *which contains only finitely many algebraic numbers* α *whose conjugates in* $\mathbb{C}_v$ *belong to* U_v *for all* v.

3. Local capacities. In this section we will recall some of the properties of the capacity, and explain how to compute local capacities for certain types of sets.

3.1. *The archimedean case.* If v is archimedean, fix an isomorphism of $\widetilde{K}_v$ with $\mathbb{C}$, and regard E_v as a subset of $\mathbb{C}$. Assume E_v is compact; then $\gamma(E_v)$ is the classical logarithmic capacity. Three well-known formulas for the logarithmic capacity are as follows (see for example [13, pp. 71–75]):

$$\begin{aligned}\gamma(E_v) &= \lim_{n\to\infty} \sup_{\{x_1,\ldots,x_n\}\subseteq E_v} \Big(\prod_{i\neq j} |x_i - x_j|\Big)^{1/(n^2-n)} \\ &= \lim_{n\to\infty} \inf_{\substack{f(x)\in\mathbb{C}[x] \\ \text{monic}, \deg(f)=n}} (\|f(x)\|_{E_v})^{1/n} \\ &= e^{-V(E_v)}\end{aligned}$$

where $\|f(x)\|_{E_v}$ is the sup norm of $f(x)$ on E_v under $|x|_v$, and

$$V(E_v) = \inf_{\nu} \iint_{E_v\times E_v} -\log(|z-w|)\, d\nu(z)\, d\nu(w)$$

where ν runs over probability measures (positive measures of total mass 1) supported on E_v.

The first formula, $\gamma(E_v)$ as the "transfinite diameter", was the definition originally used by Fekete. It shows that $\gamma(E_v)$ is translation-invariant and homogeneous under scaling. The second formula, $\gamma(E_v)$ as the "Chebyshev constant", provides a connection with polynomials. Let $\partial E_v^{\mathrm{out}}$ be the outer boundary of E_v, the part of ∂E_v meeting the closure of the unbounded component of $\mathbb{C}\setminus E_v$. By the maximum principle, $\gamma(E_v) = \gamma(\partial E_v) = \gamma(\partial E_v^{\mathrm{out}})$.

In the third formula, the number $V(E_v)$ is called the "Robin constant" of E_v. If $\gamma(E_v) > 0$ (equivalently, if $V(E_v) < \infty$), there is a unique probability measure μ, called the equilibrium distribution of E_v, for which the inf is achieved. This holds, for example, if E_v contains an arc. When $\gamma(E_v) > 0$, we define the "Green function of E_v relative to the point ∞" by

$$G(z,\infty;E_v) = V(E_v) + \int_{E_v} \log(|z-w|)\,d\mu(w).$$

The Green function is characterized by the following properties: it is continuous in the complement of E_v and upper semicontinuous everywhere; it is 0 on E_v except possibly on a set of capacity 0 contained in $\partial E_v^{\mathrm{out}}$, and it is harmonic in $\mathbb{C}\setminus E_v$ with a logarithmic pole at ∞ (that is, $G(z,\infty;E_v)-\log(|z|)$ remains bounded as $z\to\infty$). The restriction of $G(z,\infty;E_v)$ to the unbounded component of $\mathbb{C}\setminus E_v$ is the classical Green function $g(z,\infty)$ of that domain. Clearly

$$V(E_v) = \lim_{z\to\infty} \left(G(z,\infty;E_v) - \log(|z|)\right). \tag{3.1}$$

If ∂E_v is a union of arcs, then $G(z,\infty;E_v)$ is continuous everywhere, and $G(z,\infty;E_v)=0$ on E_v: the exceptional set in ∂E_v is empty.

Formula (3.1) and the characterization of the Green function make it possible to compute the capacities of many sets: if $G(z,\infty;E_v)$ can be found, then the capacity can be read off. In particular, for a closed ball $B(a,r)$,

$$G(z,\infty;B(a,r)) = \begin{cases} \log(|(z-a)/r|) & \text{if } z\notin B(a,r),\\ 0 & \text{if } z\in B(a,r),\end{cases}$$
$$\gamma(B(a,r)) = r.$$

If an explicit conformal mapping $f(z)$ can be found which takes $\mathbb{C}\setminus E_v$ to $\mathbb{C}\setminus B(0,1)$, then $G(z,\infty;E_v)=\log(|f(z)|)$ for $z\notin E_v$. In this way, capacities can be computed for straight line segments, ellipses, triangles, rectangles, regular n-gons, bent segments, arcs of circles, and so on; a fairly extensive table of capacities for archimedean sets is given in [11, pp. 348–351]. For a segment,

$$\gamma([a,b]) = (b-a)/4.$$

For an ellipse $E_v = \{x+iy\in\mathbb{C} : x^2/a^2+y^2/b^2\le 1\}$,

$$\gamma(E_v) = (a+b)/2. \tag{3.2}$$

More generally, if $h(z)$ is a monic polynomial of degree n, and $F_v = h^{-1}(E_v)$, then $G(z,\infty;F_v) = (1/n)G(h(z),\infty;E_v)$, giving the pullback formula

$$\gamma(h^{-1}(E_v)) = \gamma(E_v)^{1/n}. \tag{3.3}$$

Two consequences of the pullback formula we will need are:

COROLLARY 3.1. *Let $h(z)\in\mathbb{C}[z]$ be monic of degree n.*

(A) *If $E_v = \{z\in\mathbb{C} : |h(z)|\le R_v^n\}\subset\mathbb{C}$, then $\gamma(E_v)=R_v$.*

(B) *If $E_v=\bigcup_{i=1}^M [a_i,b_i]\subset\mathbb{R}$ is a finite union of segments, and if $h(z)\in\mathbb{R}[z]$ "oscillates n times between $\pm 2R_v^n$ on E_v" (so that $E_v=\{z\in\mathbb{C} : |h(z)|\le 2R_v^n\}$), then $\gamma(E_v)=R_v$.*

3.2. *The nonarchimedean case*: *PL-domains.* If K_v is nonarchimedean let $h(z) \in \mathbb{C}_v[z]$ be a monic polynomial of degree n, and let R_v belong to the value group of $\mathbb{C}_v^\times$. Following Cantor, we call a set of the form

$$E_v = \{z \in \mathbb{C}_v : |h(z)|_v \le R_v^n\}$$

a *Polynomial Lemniscate Domain* ("PL-domain"). Motivated by the pull-back formula in the archimedean case, Cantor ([3, §3.2]) defined the capacity, Robin constant, and Green function of E_v by

$$(3.4)\qquad \begin{aligned} \gamma(E_v) &= R_v, \\ V(E_v) &= -\log_v(R_v), \\ G(z,\infty;E_v) &= \begin{cases} (1/n)\log_v(|h(z)|_v/R_v^n) & \text{if } z \notin E_v, \\ 0 & \text{if } z \in E_v, \end{cases} \end{aligned}$$

and showed that these are independent of the choice of $h(z)$.

Note that if $E_v = \widehat{O}_v$, then taking $f(z) = z$ and $R_v = 1$ in the formulas above yields $\gamma(\widehat{O}_v) = 1$. In our definition of an adelic set $\mathbb{E}$, we required that $E_v = \widehat{O}_v$ for all but finitely many v. Thus, the global capacity

$$\gamma(\mathbb{E}) = \prod_v \gamma(E_v)^{D_v}$$

is in fact a finite product.

An important part of Cantor's theory was the identification of PL-domains with "finite unions of balls", sets of the form

$$E_v = \bigcup_{i=1}^{M} B(a_i, r_i),$$

where each $B(a_i, r_i) = \{z \in \mathbb{C}_v : |z - a_i|_v \le r_i\}$ is a closed ball, with radius r_i in the value group of $\mathbb{C}_v^\times$. By the ultrametric inequality, we can assume the union is disjoint, which means that $|a_i - a_j|_v > \max(r_i, r_j)$ for each $i \ne j$. Cantor showed, by induction on the number of zeros of $h(z)$, that each PL-domain was a finite union of balls ([3, Theorem 3.12, p. 180]).

Conversely, given a finite union of balls as above, he showed that there was a polynomial realizing it as a PL-domain. The construction is as follows. Define the $(M+1) \times (M+1)$ symmetric real matrix

$$(3.5)\qquad \Theta_v = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & \log_v(r_1) & \log_v(|a_1 - a_2|_v) & \cdots & \log_v(|a_1 - a_M|_v) \\ 1 & \log_v(|a_2 - a_1|_v) & \log_v(r_2) & \cdots & \log_v(|a_2 - a_M|_v) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \log_v(|a_M - a_1|_v) & \log_v(|a_M - a_2|_v) & \cdots & \log_v(r_M) \end{bmatrix}.$$

Then Θ_v is nonsingular, and, in the unique solution to the system of equations

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \Theta_v \begin{bmatrix} V(E_v) \\ w_1 \\ \vdots \\ w_M \end{bmatrix}, \tag{3.6}$$

the numbers w_i are positive. (For a proof, see Appendix II below.) Since Θ_v has rational entries, $V(E_v)$ and the w_i are rational as well. Let n be a common denominator for the w_i. By Lemma 12.1 of Appendix II, a polynomial realizing E_v as a PL-domain is given by

$$h(z) = \prod_{i=1}^{M} (z - a_i)^{nw_i}, \tag{3.7}$$

with $R_v^n = q_v^{-nV(E_v)}$. It follows that

$$\gamma(E_v) = q_v^{-V(E_v)}.$$

Thus, the problem of computing the capacity for finite unions of balls reduces to linear algebra.

3.3. *The nonarchimedean case*: *compact sets.* If v is nonarchimedean and E_v is compact, we define the Robin constant

$$V(E_v) = \inf_{\nu} \iint_{E_v \times E_v} - \log_v(|z - w|_v)\, d\nu(z)\, d\nu(w) \tag{3.8}$$

where the inf is taken over all probability measures ν supported on E_v. If $V(E_v)$ is finite, there is a unique probability measure μ, the equilibrium distribution, for which the inf is achieved (see [11, §4.1]). By analogy with the archimedean case, we define the capacity and Green function of E_v by

$$\gamma(E_v) = q_v^{-V(E_v)},$$
$$G(z, \infty; E_v) = V(E_v) + \int_{E_v} \log_v(|z - w|_v)\, d\mu(w).$$

It is often convenient to work with the potential function

$$\omega_\mu(z; E_v) = \int_{E_v} - \log_v(|z - w|_v)\, d\mu(w) = V(E_v) - G(z, \infty; E_v)$$

rather than the Green function. The equilibrium distribution μ is characterized by the property that $\omega_\mu(z; E_v)$ takes a constant value a.e. on E_v (here a.e. means except on a set of capacity 0); clearly this value is $V(E_v)$.

For example, take $E_v = a + bO_v$. Since E_v is translation invariant under bO_v, the equilibrium distribution μ is translation invariant as well. Thus

μ must be additive Haar measure restricted to E_v, normalized so that $\mu(E_v) = 1$. Put $r = |b|_v$. It is not hard to compute that

$$\omega_\mu(z; E_v) = \begin{cases} \dfrac{1}{q_v - 1} - \log_v(r) & \text{for } z \in E_v, \\ -\log_v(|z-a|_v) & \text{for } |z-a|_v > r, \end{cases}$$

so that $\gamma(E_v) = r \cdot q_v^{-1/(q_v-1)}$. In particular, when $K_v = \mathbb{Q}_p$ and $E_v = \mathbb{Z}_p$,

$$\gamma(\mathbb{Z}_p) = p^{-1/(p-1)},$$

a formula used implicitly in [12].

More generally, suppose $E_v \subset K_v$ is a disjoint union of cosets

$$\bigcup_{i=1}^{M} (a_i + b_i O_v).$$

Put $r_i = |b_i|_v$ for each i; the disjointness means that $\max(r_i, r_j) > |a_i - a_j|_v$ for all $i \neq j$. Using the ultrametric inequality and the characterization of μ, one sees that μ must be a weighted sum of the equilibrium distributions μ_i of the cosets $E_{v,i} = a_i + b_i O_v$:

$$\mu = \sum_{i=1}^{M} w_i \mu_i.$$

We claim that $V(E_v)$ and the w_i satisfy a system of equations much like the ones that determine the polynomial $h(z)$ for a PL-domain. Indeed, let Θ_v be defined as in (3.5) for $\bigcup_{i=1}^{M} B(a_i, r_i)$, and I_0 be the $(M+1) \times (M+1)$ matrix

$$I_0 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

The fact that $\omega_\mu(z; E_v)$ must take the same value on each $E_{v,i}$ yields the system of equations

$$\begin{bmatrix} 1 \\ \vec{0} \end{bmatrix} = \left(\Theta_v - \frac{1}{q_v - 1} I_0 \right) \begin{bmatrix} V(E_v) \\ \vec{w} \end{bmatrix}. \tag{3.9}$$

More explicitly, since $-\log_v(|a_i - a_j|_v) = \mathrm{ord}_v(a_i - a_j)$ and $-\log_v(r_i) =$

$\mathrm{ord}_v(b_i)$, the system is

$$(3.10)\quad \begin{cases} \displaystyle\sum_{i=1}^{M} w_i = 1, \quad \text{and for } j = 1, \ldots, M, \\ \displaystyle\sum_{i=1,\, i \neq j}^{M} w_i(\mathrm{ord}_v(a_i - a_j)) + w_j\left(\mathrm{ord}_v(b_j) + \frac{1}{q_v - 1}\right) = V(E_v). \end{cases}$$

By the existence and uniqueness of μ, this system has a unique solution, in which $w_i = \mu(E_{v,i})$. Since the matrices have rational entries, $V(E_v)$ and the w_i are rational. Clearly each $w_i \geq 0$; in fact, each $w_i > 0$ since if $w_j = 0$ for some j then we would have $G(z, \infty; E_v) > 0$ on $E_{v,j}$. However, this is impossible since $G(z, \infty; E_v) = 0$ a.e. on E_v (see [11, Theorem 4.1.11, p. 195]). As before,

$$\gamma(E_v) = q_v^{-V(E_v)}.$$

Thus, again, computation of capacities reduces to linear algebra. The equations (3.10) will play a key role in the proof of Theorem 2.3 (see Lemmas 8.7 and 8.8).

The definitions of the capacity for PL-domains and compact sets are consistent. In [11, §4.3] it is shown that for a PL-domain U_v,

$$\gamma(U_v) = \sup_{\substack{E_v \subset U_v \\ E_v = \text{compact}}} \gamma(E_v),$$

while for a compact set E_v,

$$\gamma(E_v) = \inf_{\substack{U_v \supset E_v \\ U_v = \text{ PL-domain}}} \gamma(U_v).$$

4. Reductions. In this section, we will derive Theorem 2.2 from Theorem 2.1, and Theorem 2.1 from Theorem 2.3. We will also outline the strategy for the proof of Theorem 2.3.

Preliminary reductions. Before proving Theorems 2.1 and 2.3, we will make some adjustments to the sets $\mathbb{E} = \prod_v E_v$ and $\mathbb{U} = \prod_v U_v$. By assumption, each E_v is stable under $\mathrm{Gal}_\mathrm{c}(\mathbb{C}_v/K_v)$ and each U_v is open. We will arrange that in addition each E_v is closed and has a simple form, and that each U_v is stable under $\mathrm{Gal}_\mathrm{c}(\mathbb{C}_v/K_v)$. Since the theorems concern the existence of points $\alpha \in \widetilde{K}$ with conjugates in $\mathbb{U}$, the conclusions remain valid as long as we do not enlarge the sets U_v.

Reduction 1. We claim that we can assume that $\gamma(\mathbb{E}) > 1$. The original hypothesis of Theorem 2.1 was that either $\gamma(\mathbb{E}) > 1$, or that $\gamma(\mathbb{E}) = 1$ and at least one archimedean E_v is compact. In the latter case, choose an archimedean place where E_v is compact. If $K_v \cong \mathbb{C}$, let Z_v be a closed ball in U_v which belongs to the unbounded component of $\mathbb{C}\backslash E_v$. If $K_v \cong \mathbb{R}$, let Z_v be a closed interval in $\mathbb{R}\cap U_v$ which belongs to $\mathbb{C}\backslash E_v$. Then $\gamma(E_v \cup Z_v) > \gamma(E_v)$, so by replacing E_v with $E_v \cup Z_v$ we can make $\gamma(\mathbb{E}) > 1$.

REDUCTION 2. Let S be the distinguished set of places where $E_v \subset \mathbb{P}^1(L_w)$ for a finite galois L_w/K_v; we can assume S contains all archimedean v. We claim that we can assume that for each $v \in S$, E_v is compact; and for each nonarchimedean $v \in S$, E_v has the form

$$E_v = \bigcup_{i=1}^{M} a_i + \pi_w^{n_i} O_w, \tag{4.1}$$

where O_w is the ring of integers of L_w. Furthermore, we claim we can assume that for each $v \in S$, U_v is stable under $\mathrm{Gal_c}(\mathbb{C}_v/K_v)$, and for each nonarchimedean $v \in S$, $E_v = L_w \cap U_v$.

Let $\widehat{S}$ be the finite set of places of K consisting of all the archimedean places and all the nonarchimedean places where $E_v \neq \widehat{O}_v$; clearly $S \subset \widehat{S}$. Since

$$\prod_{v \in \widehat{S}} \gamma(E_v)^{D_v} = \gamma(\mathbb{E}) > 1,$$

and since

$$\gamma(E_v) = \sup_{\substack{F_v \subset E_v \\ F_v \text{ compact}}} \gamma(F_v),$$

for each $v \in \widehat{S}$ we can choose a compact $F_v \subset E_v$ such that

$$\prod_{v \in \widehat{S}} \gamma(F_v)^{D_v} > 1.$$

However, a priori F_v may not be stable under $\mathrm{Gal_c}(\mathbb{C}_v/K_v)$.

In the archimedean case, if $K_v \cong \mathbb{C}$, replace E_v by F_v; if $K_v \cong \mathbb{R}$, let σ be complex conjugation, and replace E_v by $F_v \cup \sigma(F_v)$ and U_v by $U_v \cap \sigma(U_v)$.

In the nonarchimedean case, for v in S, since U_v is open, for each $x \in E_v$ there is a ball $B(x, r_x) \subset U_v$. Since $E_v \subset L_w$ is stable under $\mathrm{Gal}(L_w/K_v)$, and $\mathrm{Gal}(L_w/K_v)$ is finite, we can assume r_x is small enough that $B(\sigma x, r_x) \subset U_v$ for all $\sigma \in \mathrm{Gal}(L_w/K_v)$; after further reducing r_x, we can assume that r_x belongs to the value group of $L_w^\times$. F_v is compact, so we can cover it by a finite number of balls $B(x, r_x)$. Replacing U_v by the union of these balls and their conjugates under $\mathrm{Gal}(L_w/K_v)$, and then replacing E_v by $L_w \cap U_v$ (which contains F_v), yields all the claims.

REDUCTION 3. We claim that for each nonarchimedean $v \in \widehat{S}$, but not in S, we can assume that E_v is a PL-domain of the form

$$E_v = \bigcup_{i=1}^{M} B(a_i, r_i), \tag{4.2}$$

and that $U_v = E_v$; in particular U_v is stable under $\mathrm{Gal_c}(\mathbb{C}_v/K_v)$.

For $v \notin S$ the set F_v constructed above need not be contained in a finite extension L_w/K_v, and may have infinitely many conjugates under $\mathrm{Gal_c}(\mathbb{C}_v/K_v)$. However, we will show that F_v is contained in a Galois-stable set of the form (4.2), contained in U_v. Fixing $x \in F_v$, and using the fact that $x \in F_v \subset E_v$, the Galois-stability of E_v means that $\sigma x \in E_v$ for each $\sigma \in \mathrm{Gal_c}(\mathbb{C}_v/K_v)$. Since U_v is open, for each σ there is an $r > 0$ such that $B(\sigma x, r)^- \subset U_v$. Put

$$r_\sigma = \sup_{B(\sigma x, r)^- \subset U_v} r,$$

so that $B(\sigma x, r_\sigma)^- \subset U_v$. The map

$$\varphi_x : \mathrm{Gal_c}(\mathbb{C}_v/K_v) \to \mathbb{R}, \qquad \varphi_x(\sigma) = r_\sigma$$

is continuous and locally constant, since for a given $\sigma_0 \in \mathrm{Gal_c}(\mathbb{C}_v/K_v)$, if $\sigma x \in B(\sigma_0 x, r_{\sigma_0})^-$, the ultrametric inequality shows that $r_\sigma = r_{\sigma_0}$. Since $\mathrm{Gal_c}(\mathbb{C}_v/K_v)$ is compact, there is an $r > 0$ such that $B(\sigma x, r) \subset U_v$ for all $\sigma \in \mathrm{Gal_c}(\mathbb{C}_v/K_v)$. Without loss of generality, take r in the value group of $\mathbb{C}_v^\times$. Fix $\alpha \in \widetilde{K}_v \cap B(x, r)$. The fact that $\mathrm{Gal_c}(\mathbb{C}_v/K_v)$ preserves $|x|_v$ means that $B(\sigma x, r) = B(\sigma\alpha, r)$ for all σ. Since α has only finitely many conjugates, there are only finitely many distinct balls $B(\sigma x, r)$.

Since F_v is compact, we can cover it a finite number of balls $B(x, r)$. Replacing E_v by the union of these balls and their conjugates, we obtain (4.2). This set is open and contained in U_v, so we can replace U_v with it as well; by construction, it is stable under $\mathrm{Gal_c}(\mathbb{C}_v/K_v)$.

REDUCTION 4. Finally, we claim that for each $v \notin \widehat{S}$, we can assume that

$$E_v = U_v = \widehat{O}_v.$$

Since $v \notin \widehat{S}$, by hypothesis we have $E_v = \widehat{O}_v$; after shrinking U_v, we can assume that $U_v = \widehat{O}_v$ as well.

Proof of Theorem 2.2, assuming Theorem 2.1. Let $\mathbb{E} = \prod_v E_v$ satisfy the hypotheses of Theorem 2.2. We will construct sets $E_v' \subset U_v' \subset E_v$ such that $\mathbb{E}' = \prod_v E_v'$ and $\mathbb{U}' = \prod_v U_v'$ satisfy the hypotheses of Theorem 2.1. By hypothesis, for each nonarchimedean v, either E_v is a union of balls and hence is open in the v-adic topology, so we can take $E_v' = U_v' = E_v$; or $E_v = \bigcup_{i=1}^{M}(a_i + b_i O_w)$, so we can take $v \in S$, and put $E_v' = E_v$, $U_v' =$

$\bigcup_{i=1}^M B(a_i, |b_i|_v)$. For each archimedean v with $K_v \cong \mathbb{R}$ and $E_v \subset \mathbb{R}$, E_v is a finite union of closed intervals; let $E'_v = U'_v$ be the (real) interior of E_v. Since U'_v is obtained from E_v by removing a finite set of points, which is a set of capacity 0, Theorem III.18 of [13] shows that $\gamma(E'_v) = \gamma(E_v)$. For each archimedean v with $K_v \cong \mathbb{C}$ or with $K_v \cong \mathbb{R}$ but $E_v \not\subseteq \mathbb{R}$, let $E'_v = U'_v$ be the (complex) interior of E_v. Since E_v is the closure of U'_v, Lemma 11.1 of Appendix I shows that $\gamma(E'_v) = \gamma(E_v)$. Each E'_v is Galois-stable since E_v is Galois-stable; each U'_v is open; and

$$\prod_v \gamma(E'_v)^{D_v} = \prod_v \gamma(E_v)^{D_v} > 1.$$

Thus, we can apply Theorem 2.1 to $\mathbb{E}'$ and $\mathbb{U}'$. ■

Proof of Theorem 2.1, assuming Theorem 2.3. Suppose the hypotheses of Theorem 2.1 hold, and let S, $\widehat{S}$ be as above.

The global capacity $\gamma(\mathbb{E})$ has good functoriality under base change: by [11, Theorem 5.1.13, p. 333], for any finite extension L/K, we can pull back $\mathbb{E} = \mathbb{E}_K$ to a set $\mathbb{E}_L$ for which

$$\gamma(\mathbb{E}_L) = \gamma(\mathbb{E}_K)^{[L:K]}. \tag{4.3}$$

$\mathbb{E}_L$ is defined as follows: given a place w of L, let v be the place of K below w. Fix an isomorphism of $\mathbb{C}_w$ with $\mathbb{C}_v$, and use it to define a set $E_w \cong E_v$. Since E_v is stable under $\mathrm{Gal_c}(\mathbb{C}_v/K_v)$, E_w is independent of the choice of isomorphism, and is stable under $\mathrm{Gal_c}(\mathbb{C}_w/L_w)$. If we put $\mathbb{E}_L = \prod_w E_w$, then (4.3) holds.

After our preliminary reductions, we can assume that each U_v is stable under $\mathrm{Gal_c}(\mathbb{C}_v/K_v)$, so we can also pull back $\mathbb{U} = \mathbb{U}_K$ to a well-defined adelic neighborhood $\mathbb{U}_L$ of $\mathbb{E}_L$.

We claim that by replacing K with an appropriate finite Galois extension L/K, we can assume that $E_v \cap K_v$ is nonempty for each v, and that $K_v = L_w$ for each $v \in S$. We will now construct L. For each $v \in S$ we are given a finite Galois extension L_w/K_v with $E_v \subset L_w$. For each $v \in \widehat{S} \setminus S$, choose a finite Galois L_w/K_v such that $E_v \cap L_w \neq \emptyset$. Let D be the least common multiple of the $[L_w : K_v]$, and put $m_v = D/[L_w : K_v]$. For each v, choose m_v distinct, nonconjugate primitive elements $\alpha_{v,j}$ for L_w/K_v. Let $f_{v,j}(z) \in K_v[z]$ be the minimal polynomial for $\alpha_{v,j}$, and put

$$f_v(z) = \prod_{j=1}^{m_v} f_{v,j}(z).$$

By the weak approximation theorem, we can find a monic $f(x) \in K[x]$ which approximates each $f_v(x)$ so closely that each root of $f(x)$ in $\mathbb{C}_v$ is again a primitive element for L_w/K_v (continuity of the roots, plus Krasner's lemma). We can also arrange that $f(x)$ be irreducible, by requiring it to be an Eisenstein polynomial at some fixed $v_0 \notin \widehat{S}$. Let L be the splitting field of $f(x)$ over K. Then L/K is a finite Galois extension, with the prescribed completion L_w at each place $w \mid v$, for each $v \in \widehat{S}$.

We can now apply Theorem 2.3 to $\mathbb{E}_L$ and $\mathbb{U}_L$. If $\alpha \in \widetilde{L} = \widetilde{K}$ is a number whose conjugates over L (in $\mathbb{C}_w$) belong to U_w for each w, then because each U_v is stable under $\mathrm{Gal_c}(\mathbb{C}_v/K_v)$ and $U_w = U_v$ for each $w \mid v$, the conjugates of α over K (in $\mathbb{C}_v$) belong to U_v, for each v. Thus, without loss of generality, we can replace K by L, and Theorem 2.1 follows from Theorem 2.3. ∎

Outline of the proof of Theorem 2.3. The strategy is to show that under the hypotheses of Theorem 2.3, for each sufficiently large integer n it is possible to find an irreducible monic polynomial $u(z) \in K[z]$ whose roots, for each $v \in \widehat{S}$, belong to U_v; whose coefficients are integral at all $v \notin \widehat{S}$; and which has the desired ramification at the places in $S' = S_1' \cup S_2'$. We can assume that the preliminary reductions for Theorem 2.1 have been carried out, so that $\gamma(\mathbb{E}) > 1$ and the sets E_v are of restricted types. The roots of the $u(z)$ will be the algebraic numbers required by the theorem.

The polynomials $u(z)$ will be constructed by "patching together" local polynomials $u_v^{(0)}(z) \in K_v[z]$, for each $v \in \widehat{S} \cup S'$. The patching process has both a local and a global aspect.

For each v, the local part of the patching process consists of choosing a monic polynomial $u_v^{(0)}(z) \in K_v[z]$ of degree n which has all its roots in U_v (and in K_v, if $v \in S$), and successively replacing it by polynomials $u_v^{(1)}(z), u_v^{(2)}(z), \ldots, u_v^{(n)}(z)$, where $u_v^{(k)}(z)$ has the same $k-1$ highest-order coefficients as $u_v^{(k-1)}(z)$, and whose kth coefficient differs from that of $u_v^{(k-1)}(z)$ by a specified quantity $\Delta_v^{(k)} \in K_v$; its lower-order coefficients may be changed in arbitrary ways. (We will call this "patching with numbers $\Delta_v^{(k)}$".) Thus, formally the kth step of the patching process consists of replacing $u_v^{(k-1)}(z)$ by $u_v^{(k)}(z) = u_v^{(k-1)}(z) + \Delta_v^{(k)} w_v^{(k)}(z)$, where $w_v^{(k)}(z) \in K_v[z]$ is monic of degree $n-k$. In the following sections, we will show that if $u_v^{(0)}(z)$ is chosen carefully and the $\Delta_v^{(k)}$ are subject to appropriate constraints, namely that for certain constants B_v, h_v and L

$$|\Delta_v^{(k)}|_v \le \begin{cases} B_v & \text{for } k \le L, \\ h_v^k & \text{for } k > L, \end{cases} \tag{4.4}$$

then the $w_v^{(k)}(z)$ can be chosen so that each $u_v^{(k)}(z)$ has all of its roots in U_v (and in K_v, if $v \in S$).

From a global point of view, the patching process consists of choosing the numbers $\Delta_v^{(k)}$ in such a way that in the kth step, the coefficient $c_{v,k}^{(k)}$

of z^{n-k} in each $u_v^{(k)}(z)$ becomes a global $\widehat{S}$-integer $c_k \in K$ independent of v. In practice, we first choose the target coefficient c_k, and if $c_{v,k}^{(k-1)}$ is the coefficient of z^{n-k} in $u_v^{(k-1)}(z)$ we then take

$$\Delta_v^{(k)} = c_k - c_{v,k}^{(k-1)}.$$

The burden of the global part of the argument is to show that numbers c_k can be found so that the local conditions (4.4) are satisfied. This will be a consequence of the Strong Approximation Theorem. At the end of the patching process, all the $u_v^{(n)}(z)$ will be equal to a single global polynomial $u(z) \in K[z]$. This polynomial $u(z)$ will therefore have all its roots in U_v (and in K_v, if $v \in S$) for all $v \in \widehat{S}$, and will satisfy the desired ramification properties, for $v \in S'$. Since its coefficients are $\widehat{S}$-integers, its roots also belong to $U_v = \widehat{O}_v$ for all $v \notin \widehat{S}$.

We will provide the details of the proof after presenting the local patching constructions. However, to give some perspective on the argument, we indicate how the various parameters are related.

By Reduction 1, $\prod_{v \in \widehat{S} \cup S'} \gamma(E_v)^{D_v} = \gamma(\mathbb{E}) > 1$. For each $v \in \widehat{S} \cup S'$, fix numbers $0 < h_v < r_v < \gamma(E_v)$ subject to the condition $\prod_{v \in \widehat{S} \cup S'} h_v^{D_v} > 1$. The numbers h_v and r_v control the freedom in the local patching processes.

In the construction, we will choose a constant L, the number of coefficients to be deemed "high-order". L is subject to a finite number of local constraints depending only on E_v, U_v, h_v and r_v, for each v, and also to a global constraint arising from the Strong Approximation Theorem; each of these constraints is satisfied for all large L.

After L has been fixed, we consider the numbers B_v. For nonarchimedean v, the B_v are determined by the local patching process and depend on L. However, for archimedean v, the B_v can be chosen as large as we wish, provided n is big enough. If we make $\prod_{v \in \widehat{S} \cup S'} B_v^{D_v}$ sufficiently large, the global patching process can be carried out. In this way the construction succeeds for all sufficiently large n. ∎

Sections 5 through 9 contain the local patching constructions. Section 10 gives the global patching argument.

5. Local patching for sets in $\mathbb{C}$. When $K_v \simeq \mathbb{C}$, or when $K_v \simeq \mathbb{R}$ and E_v is stable under complex conjugation, our patching process is an extension of a method going back to Fekete and Szegő [5].

PROPOSITION 5.1. *Suppose that K_v is archimedean, and that $E_v \subset U_v \subset \mathbb{C}$, where E_v is compact and U_v is open. There is a number M_v, depending only on E_v and U_v, with the following property. Let $0 < h_v < r_v < \gamma(E_v)$ be* *given, and let L be an integer satisfying*

$$\left(\frac{h_v}{r_v}\right)^L \frac{M_v}{1-h_v/r_v} < \frac{1}{8}. \tag{5.1}$$

If $K_v \simeq \mathbb{C}$ (resp. if $K_v \simeq \mathbb{R}$, E_v is stable under complex conjugation, and $E_v \cap \mathbb{R}$ is nonempty) then, for any $B_v > 0$, for each sufficiently large n there is a monic polynomial $u_v^{(0)}(z) \in \mathbb{C}[z]$ (resp. $\mathbb{R}[z]$) of degree n, whose roots belong to U_v and which can be patched with arbitrary $\Delta_v^{(k)} \in \mathbb{C}$ (resp. $\mathbb{R}$) satisfying

$$|\Delta_v^{(k)}| \le \begin{cases} B_v & \text{for } k \le L, \\ h_v^k & \text{for } k > L, \end{cases}$$

in such a way that its roots remain in U_v.

Proof. We will give the construction when $K_v \simeq \mathbb{C}$, noting minor changes in the argument when $K_v \simeq \mathbb{R}$.

We first reduce to the case where E_v is a finite union of arcs, and $\mathbb{C} \setminus E_v$ is connected. To achieve this, first replace E_v with a finite number of closed balls contained in U_v, whose interiors cover E_v; this only increases $\gamma(E_v)$. Let $\partial E_v^{\text{out}}$ be the outer boundary of E_v; as noted earlier, $\gamma(E_v) = \gamma(\partial E_v^{\text{out}})$. Replacing E_v by $\partial E_v^{\text{out}}$, we can assume that E_v consists of a finite union of arcs $\bigcup_{i=1}^M F_i$. For each i, choose a point $a_i \in F_i$, and let $B(a_i, r)^-$ be the open ball of radius r centered at a_i. For each i, put $e_i(r) = E_v \cap B(a_i, r)^-$, and put

$$E_v(r) := E_v \setminus \Big(\bigcup_{i=1}^M B(a_i, r)^-\Big),$$

so that $E_v = E_v(r) \cup (\bigcup_{i=1}^M e_i(r))$. Then $\mathbb{C} \setminus E_v(r)$ is connected, and if r is small enough, no component of $E_v(r)$ is reduced to a point. Fix R large enough that $E_v \subset B(0, R)$. By [13, Theorem III.17, p. 63] and the homogeneity of $\gamma(E)$,

$$\frac{1}{\log(2R/\gamma(E_v))} \le \frac{1}{\log(2R/\gamma(E_v(r)))} + \sum_{i=1}^M \frac{1}{\log(2R/\gamma(e_i(r)))}.$$

Furthermore, $\gamma(e_i(r)) \le \gamma(B(a_i, r)) = r$. Since $r_v > 0$ is fixed and $\gamma(E_v) > r_v$, if r is small enough the inequality above implies that $\gamma(E_v(r)) > r_v$. Replacing E_v by $E_v(r)$ for a suitably small r yields a set with the properties we need. If $K_v \simeq \mathbb{R}$, the construction can be carried out in a way stable under complex conjugation.

After these reductions the Green function $G(z, \infty; E_v)$ is continuous, with $G(z, \infty; E_v) = 0$ for all $z \in E_v$, and $G(z, \infty; E_v) > 0$ for all $z \notin E_v$. By continuity, there is an $\varepsilon > 0$ such that

$$\{z \in \mathbb{C} : G(z, \infty; E_v) \le \varepsilon\} \subset U_v. \tag{5.2}$$

Put

$$\begin{aligned} \Omega_\varepsilon &= \{z \in \mathbb{C} : G(z, \infty; E_v) < \varepsilon\}, \\ \partial\Omega_\varepsilon &= \{z \in \mathbb{C} : G(z, \infty; E_v) = \varepsilon\}. \end{aligned} \tag{5.3}$$

Then Ω_ε is open, and $E_v \subset \Omega_\varepsilon \subset U_v$.

We can approximate the equilibrium distribution μ of E_v by a discrete measure $\sum_{i=1}^{n_v}(1/n_v)\delta_{\alpha_i}(z)$ sufficiently well that the monic polynomial

$$g(z) = \prod_{i=1}^{n_v}(z - \alpha_i) \in \mathbb{C}[z]$$

has all its roots in E_v, and for all $z \notin \Omega_\varepsilon$ it satisfies

$$\left| \frac{1}{n_v} \log_v(|g(z)|) - (G(z, \infty; E_v) - V(E_v)) \right| < \frac{\varepsilon}{2} \tag{5.4}$$

(see [6, pp. 294–295], or [11, Lemma 3.3.5, p. 169]). When $K_v \simeq \mathbb{R}$, μ is stable under complex conjugation, and $g(z)$ can be chosen in $\mathbb{R}[z]$. This polynomial $g(z)$, and its degree n_v, will be fixed for the rest of the construction. Put $R_v = \gamma(E_v)$.

Fix a point $a \in E_v$ (if $K_v \simeq \mathbb{R}$, choose $a \in E_v \cap \mathbb{R}$), and put

$$\begin{aligned} C_v^- &= \min_{0 \le h < n_v} \left(\left(\min_{z \in \partial\Omega_\varepsilon} |(z-a)^h| \right) / (R_v e^{2\varepsilon})^h \right), \\ C_v^+ &= \max_{0 \le h < n_v} \left(\left(\max_{z \in \partial\Omega_\varepsilon} |(z-a)^h| \right) / R_v^h \right). \end{aligned} \tag{5.5}$$

Finally, put

$$M_v = C_v^+ / C_v^- . \tag{5.6}$$

This is the constant in the proposition.

Construction of the initial patching polynomials. The condition imposed by v on the globally chosen constant L will be that

$$\left(\frac{h_v}{r_v} \right)^L \frac{M_v}{1 - h_v/r_v} < \frac{1}{8}. \tag{5.7}$$

This holds for all sufficiently large L. Suppose L has been fixed, and choose an integer N_v large enough that $n_v N_v > L$. Set

$$\widetilde{g}(z) = g(z)^{N_v}.$$

By continuity there is a $\delta > 0$ such that for any monic $\widehat{g}(z) \in \mathbb{C}[z]$ (resp. $\mathbb{R}[z]$) of degree $n_v N_v$, whose coefficients differ from those of $\widetilde{g}(z)$ by less than δ in absolute value, all the roots of $\widehat{g}(z)$ belong to Ω_ε, and for all $z \in \partial\Omega_\varepsilon$

$$\left| \frac{1}{n_v N_v} \log_v(|\widehat{g}(z)|) - \frac{1}{n_v} \log_v(|g(z)|) \right| < \min\left(\frac{\varepsilon}{2}, \frac{\log_v(2)}{n_v N_v} \right). \tag{5.8}$$

If $\widehat{g}(z)$ is such a polynomial, then $\frac{1}{n_v N_v}\log_v(|\widehat{g}(z)|) - \frac{1}{n_v}\log_v(|g(z)|)$ is harmonic in the complement of Ω_ε and remains bounded as $z \to \infty$. By (5.8) and the maximum modulus principle for harmonic functions, (5.8) holds for all $z \notin \Omega_\varepsilon$. From (5.4) and (5.8), and from $R_v = e^{-V(E_v)}$, it follows that

$$|\widehat{g}(z)| > R_v^{n_v N_v} \quad \text{for all } z \notin \Omega_\varepsilon. \tag{5.9}$$

Given n, to construct the initial polynomial $u_v^{(0)}(z)$, first write $n = s + m \cdot n_v N_v$, where s, m are integers and $0 \le s < n_v N_v$. Decompose $s = s_1 + s_2 \cdot n_v$ where $0 \le s_1 < n_v$ and $0 \le s_2 \le N_v$, and put

$$f_s(z) = (z-a)^{s_1} g(z)^{s_2}.$$

Then, put

$$u_v^{(0)}(z) = f_s(z)\widetilde{g}(z)^m.$$

Note that $u_v^{(0)}(z)$ has all its zeros in Ω_ε. For the rest of the construction, we hold s fixed, and view n as a function of m. It suffices to prove the proposition for each fixed s, and all sufficiently large m.

Phase 1. *Patching the high-order coefficients* ($k \le L$). We will patch the high-order coefficients of $u_v^{(0)}(z)$ by sequentially modifying the coefficients of $\widetilde{g}(z)$, taking advantage of a phenomenon of "magnification". This idea is new, and is what ultimately enables archimedean and nonarchimedean polynomials to be patched together.

We can assume m is large enough that $m\delta > B_v$, where B_v is the number specified in the proposition.

Put $\widehat{g}^{(0)}(z) = \widetilde{g}(z)$. For $k = 1, \dots, L$, we will inductively construct a polynomial $\widehat{g}^{(k)}(z)$ by adding a number $\delta_k \in \mathbb{C}$ (resp. $\delta_k \in \mathbb{R}$) with $|\delta_k| < \delta$, to the kth coefficient of $\widehat{g}^{(k-1)}(z)$; we will then put

$$u_v^{(k)}(z) = f_s(z)\widehat{g}^{(k)}(z)^m.$$

Since $|\delta_k| < \delta$, the discussion above applies to each $\widehat{g}^{(k)}(z)$. Fixing k, write

$$f_s(z) = z^s + \sum_{h=1}^{s} a_h z^{s-h}, \qquad \widehat{g}^{(k-1)}(z) = z^{n_v N_v} + \sum_{j=1}^{n_v N_v} b_j z^{d-j},$$

and expand

$$u_v^{(k-1)}(z) := f_s(z)\widehat{g}^{(k-1)}(z)^m = z^n + \sum_{j=1}^{n} c_{v,j} z^{n-j}.$$

By the multinomial theorem (since $n_v N_v > L$ and $k \le L$), $c_{v,k}$ has the form

$$c_{v,k} = \sum_{h, l_0, l_1, \dots, l_k} \binom{m}{l_0\, l_1\, \dots\, l_k} a_h b_1^{l_1} \dots b_k^{l_k} \tag{5.10}$$

where the sum is over all integers $h, l_0, l_1, \ldots, l_k \geq 0$ satisfying

$$h + l_1 + 2l_2 + \ldots + kl_k = k, \quad l_0 + l_1 + \ldots + l_k = m, \quad 0 \leq h \leq s.$$

Note that there are only a bounded number of possibilities for $h, l_1, \ldots, l_k$, independent of m; and if $m > L$ and a choice of $h, l_1, \ldots, l_k$ is given we can take $l_0 = m - l_1 - \ldots - l_k$. Moreover, there is only one term in (5.10) in which b_k appears, namely the one with $h = l_1 = \ldots = l_{k-1} = 0$, $l_k = 1$. For that term, the multinomial coefficient is m, and $a_0 = 1$. Thus, for sufficiently large m, $c_{v,k}$ is a polynomial in m and $b_1, \ldots, b_k$ of the form

$$c_{v,k} = mb_k + Q_{k,s}(m, b_1, \ldots, b_{k-1}),$$

which depends linearly on b_k.

If we define $\widehat{g}^{(k)}(z)$ by adding δ_k to the coefficient b_k in $\widehat{g}^{(k-1)}(z)$, the coefficients $c_{v,l}$ in $u_v^{(k-1)}(z)$ with $l < k$ remain unchanged, $c_{v,k}$ is changed to $c_{v,k} + m\delta_k$, and the coefficients $c_{v,l}$ with $l > k$ are modified in ways that are unimportant. Thus, since $m\delta > B_v$, for $k = 1, \ldots, L$ we can vary the kth coefficient of $u^{(k-1)}(z)$ by a quantity $\Delta_{v,k}$ up to B_v in magnitude, by adjusting the kth coefficient of $\widehat{g}^{(k-1)}(z)$ by $\delta_k = \Delta_{v,k}/m$, a quantity at most δ in magnitude.

The $\Delta_{v,k}$ will be chosen on the basis of global considerations; for the remainder of the proof we assume that $\Delta_{v,1}, \ldots, \Delta_{v,L}$ have been fixed, and write $\widehat{g}(z)$ for the corresponding polynomial $\widehat{g}^{(L)}(z)$. In the discussion below, we use only that $\widehat{g}(z)$ satisfies (5.8) (hence also (5.9)), and that it is monic with degree $n_v N_v$.

Phase 2. *Patching the low-order coefficients* ($L < k \leq n$). We begin this phase of the patching process with

$$u_v^{(L)}(z) = f_s(z)\widehat{g}(z)^m, \tag{5.11}$$

where $\widehat{g}(z) = \widehat{g}^{(L)}(z)$. Since $f_s(z)$ and $\widehat{g}(z)$ have all their zeros in Ω_ε, the same is true for $u_v^{(L)}(z)$. Moreover, by (5.5) and (5.9),

$$|u_v^{(L)}(z)| > C_v^- R_v^n \quad \text{for all } z \notin \Omega_\varepsilon. \tag{5.12}$$

In the kth patching step, for $k > L$, let $w_v^{(k)}(z)$ be the monic polynomial of degree $n - k$ defined by

$$w_v^{(k)}(z) = (z-a)^{h_1} g(z)^{h_2} \widehat{g}(z)^{h_3}, \tag{5.13}$$

where the integers h_1, h_2, h_3 are determined by

$$n - k = h_1 + h_2 n_v + h_3 n_v N_v,$$
$$0 \leq h_1 < n_v, \quad 0 \leq h_2 < N_v, \quad 0 \leq h_3 < m.$$

We define $u_v^{(k)}(z)$ by setting

$$u_v^{(k)}(z) = u_v^{(k-1)}(z) + \Delta_{v,k} w_v^{(k)}(z).$$

This has the effect of changing the kth coefficient $c_{v,k}$ of $u_v^{(k-1)}(z)$ to $c_{v,k} + \Delta_{v,k}$ in $u_v^{(k)}(z)$, and leaving the coefficients $c_{v,l}$ with $l < k$ unchanged. The $\Delta_{v,k}$ will be chosen on the basis of global considerations, subject to $|\Delta_{v,k}| \le h_v^k$.

We need bounds for $|w_v^{(k)}(z)|$ and $|u_v^{(L)}(z)|$ in terms of $|\widehat{g}(z)|$. First, we bound $|z-a|^h$, for $h = 0, \ldots, n_v - 1$. By (5.5), on $\partial\Omega_\varepsilon$,

$$C_v^-(R_v e^{2\varepsilon})^h \le |z-a|^h \le C_v^+ R_v^h.$$

On the other hand by (5.4) and (5.8), on $\partial\Omega_\varepsilon$ also $R_v \le |\widehat{g}(z)|^{1/(n_v N_v)} \le R_v e^{2\varepsilon}$. Thus

$$C_v^-|\widehat{g}(z)|^{h/(n_v N_v)} \le |z-a|^h \le C_v^+|\widehat{g}(z)|^{h/(n_v N_v)}. \tag{5.14}$$

By the maximum modulus principle for harmonic functions, applied to $\frac{h}{n_v N_v}\log_v(|\widehat{g}(z)|) - h\log(|z-a|)$ on $\mathbb{P}^1(\mathbb{C}) \setminus \Omega_\varepsilon$, these inequalities hold for all $z \notin \Omega_\varepsilon$. Second, we must bound $|g(z)|^h$, for $h = 0, \ldots, N_v - 1$. By (5.8), on $\partial\Omega_\varepsilon$ we have

$$\frac{1}{2}|\widehat{g}(z)|^{h/N_v} \le |g(z)|^h \le 2|\widehat{g}(z)|^{h/N_v} \tag{5.15}$$

and again by the maximum modulus principle, these inequalities hold for all $z \notin \Omega_\varepsilon$. Combining (5.14) and (5.15, we find that for all $z \notin \Omega_\varepsilon$,

$$|w_v^{(k)}(z)| = |(z-a)^{h_1} g_v(z)^{h_2} \widehat{g}_v(z)^{h_3}| \le 2C_v^+|\widehat{g}(z)|^{(n-k)/(n_v N_v)}, \tag{5.16}$$

$$|u_v^{(L)}(z)| = |(z-a)^{s_1} g(z)^{s_2} \widehat{g}(z)^m| \ge \frac{1}{2}C_v^-|\widehat{g}(z)|^{n/(n_v N_v)}. \tag{5.17}$$

At the end of the construction we have

$$u_v^{(n)}(z) = u_v^{(L)}(z) + \sum_{k=L+1}^{n} \Delta_{v,k} w_v^{(k)}(z). \tag{5.18}$$

By (5.16), (5.17), and (5.7), together with $M_v = C_v^+/C_v^-$ and the inequalities $|\Delta_{v,k}| \le h_v^k$ and $|\widehat{g}(z)|^{1/(n_v N_v)} > R_v > r_v$, for all $z \notin \Omega_\varepsilon$ we have

$$\begin{aligned} \left|\frac{u_v^{(n)}(z) - u_v^{(L)}(z)}{u_v^{(L)}(z)}\right| &\le 4M_v \sum_{k=L+1}^{n} (h_v)^k |\widehat{g}(z)|^{-k/(n_v N_v)} \\ &< 4M_v \sum_{k=L}^{\infty} \left(\frac{h_v}{r_v}\right)^k \le \frac{1}{2}. \end{aligned} \tag{5.19}$$

Finally from (5.19) and (5.12) we see that for all $z \notin \Omega_\varepsilon$,

$$|u_v^{(n)}(z)| \ge \tfrac{1}{2}|u_v^{(L)}(z)| > \tfrac{1}{2}C_v^- R_v^n > 0. \tag{5.20}$$

This means that $u_v^{(n)}(z)$ has all its zeros in Ω_ε, and hence in U_v. ∎

6. Local patching for sets in $\mathbb{R}$. When $K_v \cong \mathbb{R}$ and $E_v \subset \mathbb{R}$, our patching process uses Chebyshev polynomials, extending a method due to Robinson [10]. Replacing U_v by $U_v \cap \mathbb{R}$, we can assume U_v is a real neighborhood of E_v.

PROPOSITION 6.1. *Suppose $K_v \cong \mathbb{R}$, and that $E_v \subset U_v \subset \mathbb{R}$, where E_v is compact and U_v is open. There is a number M_v, depending only on E_v and U_v, with the following property. Let $0 < h_v < r_v < \gamma(E_v)$ be given, and let L be an integer such that*

$$\left(\frac{h_v}{r_v}\right)^L \frac{M_v}{1 - h_v/r_v} < \frac{1}{16}. \tag{6.1}$$

Given $B_v > 0$, then for each sufficiently large n there is a monic polynomial $u_v^{(0)}(z) \in \mathbb{R}[z]$ of degree n, whose roots belong to U_v and which can be patched with arbitrary $\Delta_v^{(k)} \in \mathbb{R}$ satisfying

$$|\Delta_v^{(k)}|_v \le \begin{cases} B_v & \text{for } k \le L, \\ h_v^k & \text{for } k > L, \end{cases}$$

in such a way that its roots remain in U_v.

Proof. We begin with some preliminary reductions. After shrinking U_v, we can assume that it has compact closure, and after enlarging E_v within U_v, we can assume that it is a finite union of closed intervals:

$$E_v = \bigcup_{i=1}^{M} [a_i, b_i], \quad \text{where} \quad a_1 < b_1 < a_2 < b_2 < \ldots < a_M < b_M.$$

Put $R_v = \gamma(E_v)$. By definition, the *Chebyshev polynomial* of degree d for E_v is a monic polynomial $T_d(z; E_v) \in \mathbb{R}[z]$ of degree d, with minimal sup norm on E_v. It is known that such polynomials exist, and that

$$\|T_d(z; E_v)\|_{E_v} \ge 2R_v^d.$$

Furthermore, since E_v is a finite union of closed intervals, $T_d(z; E_v)$ is unique, its roots are simple and belong to $\mathbb{R}$, and between each pair of roots, it achieves its maximum magnitude at a point of E_v. Its roots are real and lie in $[a_1, b_M]$, and there is at most one root in each "gap" (b_i, a_{i+1}). For these facts, see e.g. [10].

Chebyshev polynomials for an interval $[-2r, 2r]$ are known explicitly, and are discussed for example in [9]. We will write $T_{d,r}(z)$ for $T_d(z; [-2r, 2r])$; it is the polynomial defined by

$$T_{d,r}(2r\cos(\theta)) = 2r^d \cos(d\theta).$$

$T_{d,r}(z)$ can be expanded as

$$T_{d,r}(z) = z^d + \sum_{k=1}^{\lfloor d/2 \rfloor} (-1)^k \frac{d}{k} \binom{d-k-1}{k-1} r^{2k} z^{d-2k}$$

(see [10]). The graph of $T_{d,r}(z)$ oscillates d times between $\pm 2r^d$ on $[-2r, 2r]$, and $T_{d,r}(z)$ takes each value between $\pm 2r^d$ exactly d times on $[-2r, 2r]$ (counting multiplicities), so that

$$[-2r, 2r] = \{z \in \mathbb{C} : T_{d,r}(z) \in [-2r^n, 2r^n]\}. \tag{6.2}$$

Based on (6.2) one might hope that for a general set E_v, the polynomial $T_d(z; E_v)$ oscillates d times between $\pm 2R_v^d$ on E_v, so that

$$E_v = \{z \in \mathbb{C} : T_d(z; E_v) \in [-2R_v^d, 2R_v^d]\}. \tag{6.3}$$

Unfortunately, in general (6.3) is false: it fails, for example, if $T_d(z; E_v)$ has a zero in one of the gaps (b_i, a_{i+1}). However, a result of Raphael Robinson ([10, p. 422]) asserts that after enlarging the intervals $[a_i, b_i]$ slightly within U_v and choosing d appropriately, we can assume that $T_d(z; E_v)$ and E_v do satisfy (6.3). Robinson proved this by carefully analyzing the properties of certain Schwarz–Christoffel mappings. Robinson's theorem is the key to our construction, and the absence of a purely potential-theoretic proof of it is the chief obstruction to extending the Fekete–Szegő theorem with splitting conditions to a more general setting.

In general, if $f(z) \in \mathbb{R}[z]$ is a polynomial of degree d, then for any $R > 0$ we will say that "$f(z)$ oscillates d times between $\pm R$ on a set $E \subset \mathbb{R}$" if $\{z \in \mathbb{C} : f(z) \in [-R, R]\} = E$, and we will say that "$f(z)$ oscillates d times with magnitude at least R on E" if $\{z \in \mathbb{C} : f(z) \in [-R, R]\} \subseteq E$.

In the following, we will assume that E_v and d have been chosen so that (6.3) holds. Since our reductions have only enlarged E_v, the hypothesis $0 < h_v < r_v < \gamma(E_v)$ continues to hold. Put

$$g(z) = T_d(z; E_v), \qquad n_v = d = \deg(g(z)),$$

so that $g(z) \in \mathbb{R}[z]$ is monic of degree n_v and

$$E_v = g^{-1}([-2R_v^{n_v}, 2R_v^{n_v}]). \tag{6.4}$$

An important consequence of (6.2) and (6.4) is that by composing $g(z)$ with the polynomials $T_{m,R_v^{n_v}}(z)$, we can obtain Chebyshev polynomials for E_v of higher degree:

$$T_{m,n_v}(z; E_v) = T_{m,R_v^{n_v}}(g(z)).$$

This polynomial $g(z)$ and its degree n_v will be fixed for the rest of the construction. The constant in the proposition will be

$$M_v = \max_{0 \le h < n_v} (\|z^h\|_{U_v} / R_v^h). \tag{6.5}$$

Construction of the initial patching polynomials. The condition imposed by the place v on the globally chosen constant L will be that

$$\left(\frac{h_v}{r_v}\right)^L \frac{M_v}{1-h_v/r_v} < \frac{1}{16}. \tag{6.6}$$

Clearly this holds for all sufficiently large L. Assuming L has been fixed, choose an integer N_v large enough that

$$n_v N_v > L \quad \text{and} \quad R_v^{n_v N_v} > 2r_v^{n_v N_v} \tag{6.7}$$

and put

$$\widetilde{g}(z) = T_{N_v, R_v^{n_v}}(g(z)).$$

Then $\widetilde{g}(z)$ is monic of degree $n_v N_v$, and by (6.4)

$$E_v = \widetilde{g}^{-1}([-2R_v^{n_v N_v}, 2R_v^{n_v N_v}]). \tag{6.8}$$

By continuity, there is a number $\delta > 0$ such that for each monic $\widehat{g}(z)$ in $\mathbb{R}[z]$ of degree $n_v N_v$, whose coefficients differ from those of $\widetilde{g}(z)$ by at most δ in magnitude, we have

$$|\widehat{g}(z) - \widetilde{g}(z)| < R_v^{n_v N_v} \tag{6.9}$$

for all $z \in E_v$. Put

$$\widehat{R}_v = 2^{-1/(n_v N_v)} R_v, \tag{6.10}$$

noting that by (6.7),

$$r_v < \widehat{R}_v < R_v. \tag{6.11}$$

Finally, choose an interval $E_{v,0} = [c, d]$ contained in $U_v \setminus E_v$. For each integer s in the range $0 \leq s < n_v N_v$, let $f_s(z)$ be the Chebyshev polynomial of degree s for $E_{v,0}$. Put

$$\varrho := \gamma(E_{v,0}) = (d-c)/4,$$

so that $f_s(z)$ oscillates s times between $\pm 2\varrho^s$ on $E_{v,0}$, and

$$|f_s(z)| > 2\varrho^s \quad \text{on } \mathbb{R} \setminus E_{v,0}. \tag{6.12}$$

In the following, we will write $\widehat{T}_m(z)$ for $T_{m, \widehat{R}_v^{n_v N_v}}(z)$. Given n, to define the initial polynomial $u_v^{(0)}(z)$, decompose $n = s + m \cdot n_v N_v$, where s, m are integers and $0 \leq s < n_v N_v$, and put

$$u_v^{(0)}(z) = f_s(z) \widehat{T}_m(\widetilde{g}(z)).$$

For the rest of the construction, we hold s fixed, and view n as a function of m. Since there are only finitely many progressions $n = s + m \cdot n_v N_v$, it suffices to prove the proposition for each s, and all sufficiently large m.

Phase 1. *Patching the high-order coefficients* ($k \leq L$). Just as in the case $K_v \cong \mathbb{C}$, we patch the high-order coefficients of $u_v^{(0)}(z)$ by modifying the coefficients of $\widetilde{g}(z)$, taking advantage of "magnification". Let $B_v > 0$ be the number specified in the proposition.

Assume m is large enough that $m > L$ and $m\delta > B_v$. Write $\widehat{g}^{(0)}(z) = \widetilde{g}(z)$, so that $u_v^{(0)}(z) = f_s(z)\widehat{T}_m(\widehat{g}^{(0)}(z))$. For each $k = 1, \ldots, L$, we will define $\widehat{g}^{(k)}(z)$ by adding a number $\delta_k \in \mathbb{R}$ with $|\delta_k| < \delta$ to the kth coefficient of $\widehat{g}^{(k-1)}(z)$, and patch $u^{(k-1)}(z)$ by taking $u^{(k)}(z) = f_s(z)\widehat{T}_m(\widehat{g}^{(k)}(z))$. The choice of δ_k will be explained below. Since $|\delta_k| < \delta$, each $\widehat{g}^{(k)}(z)$ satisfies (6.9). Write

$$f_s(z) = z^s + \sum_{h=1}^{s} a_h z^{s-h}, \qquad \widehat{g}^{(k-1)}(z) = z^{n_v N_v} + \sum_{j=1}^{n_v N_v} b_j z^{n_v N_v - j},$$

and expand

$$\begin{aligned} (6.13) \qquad u_v^{(k-1)}(z) &= f_s(z)\widehat{T}_m(\widehat{g}^{(k-1)}(z)) \\ &= f_s(z)\widehat{g}^{(k-1)}(z)^m \\ &\quad + \sum_{j=1}^{\lfloor m/2 \rfloor} (-1)^j \frac{m}{j} \binom{m-j-1}{j-1} \widehat{R}_v^{2j} f_s(z)(\widehat{g}^{(k-1)}(z))^{m-2j} \\ &= z^n + \sum_{l=1}^{n} c_{v,l} z^{n-l}. \end{aligned}$$

Each term $f_s(z)\widehat{g}^{(k-1)}(z)^{m-2j}$ in the sum has degree $s + (m-2j)n_v N_v < n - L$, so the L high-order coefficients of $u_v^{(k-1)}(z)$ are the same as those of $f_s(z)\widehat{g}^{(k-1)}(z)^m$.

Thus, just as when $K_v \cong \mathbb{C}$ (see (5.10)), the kth coefficient $c_{v,k}$ is a polynomial in m and $b_1, \ldots, b_k$ of the form

$$c_{v,k} = mb_k + Q_{s,k}(m, b_1, \ldots, b_{k-1}),$$

which depends linearly on b_k. If δ_k is added to the coefficient b_k in $\widehat{g}^{(k-1)}(z)$, then $c_{v,k}$ is changed to $c_{v,k} + m\delta_k$ while the coefficients of $u_v^{(k-1)}(z)$ with $j < k$ remain unchanged. Hence, since $m\delta > B_v$, we can sequentially vary the first L coefficients of $u_v^{(0)}(z)$ by quantities $\Delta_{v,k}$ up to B_v in magnitude, by sequentially varying the first L coefficients of $\widetilde{g}(z)$ by quantities $\delta_k = \Delta_{v,k}/m$ at most δ in magnitude.

The choice of the numbers $\Delta_{v,k}$ will be made on the basis of global considerations; for the remainder of this section we assume that $\Delta_{v,1}, \ldots, \Delta_{v,L} \in \mathbb{R}$ have been fixed, and write $\widehat{g}(z)$ for $\widehat{g}^{(L)}(z)$. In the discussion below, we use only the fact that $\widehat{g}(z) \in \mathbb{R}[z]$ satisfies (6.9) and that it is monic with degree $n_v N_v > L$.

Phase 2. *Patching the middle coefficients* ($L < k \le n - s$). By (6.8), $\widetilde{g}(z)$ oscillates $n_v N_v$ times between $\pm 2R_v^{n_v N_v}$ on E_v. At any point z where $|\widetilde{g}(z)| = 2R_v^{n_v N_v}$, it follows from (6.9) that $|\widehat{g}(z)| > R_v^{n_v N_v}$, so $\widehat{g}(z)$ oscillates

$n_v N_v$ times with magnitude at least $R_v^{n_v N_v} = 2\widehat{R}_v^{n_v N_v}$ on E_v. The set

$$\widehat{E}_v := \{z \in \mathbb{C} : \widehat{g}(z) \in [-2\widehat{R}_v^{n_v N_v}, 2\widehat{R}_v^{n_v N_v}]\}$$

satisfies $\widehat{E}_v \subset E_v$, and $\widehat{g}(z)$ oscillates $n_v N_v$ times between $\pm 2\widehat{R}_v^{n_v N_v}$ on $\widehat{E}_v$. We have

$$u_v^{(L)}(z) = f_s(z)\widehat{T}_m(\widehat{g}(z)) \quad \text{and} \quad n = s + m \cdot n_v N_v = \deg(u_v^{(L)}(z)).$$

Since $\widehat{T}_m(z) := T_{m,\widehat{R}_v^{n_v N_v}}(z)$ is the Chebyshev polynomial of degree m for $[-2\widehat{R}_v^{n_v N_v}, 2\widehat{R}_v^{n_v N_v}]$, it follows that $\widehat{T}_m(\widehat{g}(z))$ oscillates $n_v N_v m$ times between $\pm 2\widehat{R}_v^{n_v N_v m}$ on $\widehat{E}_v$.

Our goal is to patch $u_v^{(L)}(z)$ in such a way that $u_v^{(n)}(z)$ oscillates with large magnitude on $\widehat{E}_v$. Write $T_h(z) = T_{h,R_v^{n_v}}(z)$ for the Chebyshev polynomial of the interval $[-2R_v^{n_v}, 2R_v^{n_v}]$, and let $w_v^{(k)}(z)$ be the monic polynomial of degree $n - k - s$ given by

$$w_v^{(k)}(z) = z^{h_1} T_{h_2}(g(z))\widehat{T}_{h_3}(\widehat{g}(z)), \tag{6.14}$$

where the integers h_1, h_2, h_3 are determined by the conditions

$$n - k - s = h_1 + h_2 n_v + h_3 n_v N_v,$$
$$0 \le h_1 < n_v, \quad 0 \le h_2 < N_v, \quad 0 \le h_3 < m.$$

At the kth step of the patching process, $L < k \le n - s$, we will put

$$u_v^{(k)}(z) = u_v^{(k-1)}(z) + \Delta_{v,k} f_s(z) w_{v,k}(z),$$

where $\Delta_{v,k} \in \mathbb{R}$ satisfies $|\Delta_{v,k}| \le h_v^k$.

We now seek to bound $|w_{v,k}(z)|$ on $\widehat{E}_v$. From (6.5), together with (6.10) and $0 \le h_1 < n_v$, it follows that

$$\|z^{h_1}\|_{\widehat{E}_v} \le M_v R_v^{h_1} \le M_v \cdot 2\widehat{R}_v^{h_1}.$$

By the properties of Chebyshev polynomials and the fact that $h_2 < N_v$,

$$\|T_{h_2}(g(z))\|_{\widehat{E}_v} \le \|T_{h_2}(g(z))\|_{E_v} = 2R_v^{h_2 n_v} \le 2 \cdot 2\widehat{R}_v^{h_2 n_v}.$$

Likewise, by the properties of Chebyshev polynomials,

$$\|\widehat{T}_{h_3}(\widehat{g}(z))\|_{\widehat{E}_v} = 2\widehat{R}_v^{h_3 n_v N_v}.$$

Thus,

$$\|w_{v,k}(z)\|_{\widehat{E}_v} \le 16 M_v \widehat{R}_v^{n-k}.$$

At the end of this phase of the patching process, we will have

$$\begin{aligned} u_v^{(n-s)}(z) &= u_v^{(L)}(z) + \sum_{k=L+1}^{n-s} \Delta_{v,k} f_s(z) w_v^{(k)}(z) \\ &= f_s(z)\Big(\widehat{T}_m(\widehat{g}(z)) + \sum_{k=L+1}^{n-s} \Delta_{v,k} w_v^{(k)}(z)\Big). \end{aligned}$$

Put $q(z) = \widehat{T}_m(\widehat{g}(z)) + \sum_{k=L+1}^{n-s} \Delta_{v,k} w_v^{(k)}(z)$. By the inequalities above, and (6.6), for $z \in \widehat{E}_v$,

$$|q(z) - \widehat{T}_m(\widehat{g}(z))| \le 16 M_v \sum_{k=L+1}^{n} h_v^k \widehat{R}_v^{n-k} \le 16 M_v \sum_{k=L}^{\infty} \left(\frac{h_v}{r_v}\right)^k \widehat{R}_v^n \le \widehat{R}_v^n. \tag{6.15}$$

Since $\widehat{T}_m(\widehat{g}(z))$ oscillates $n_v N_v m$ times between $\pm 2\widehat{R}_v^{n_v N_v m}$ on $\widehat{E}_v$, it follows that $q(z)$ oscillates $n_v N_v m$ times with magnitude at least $\widehat{R}_v^{n_v N_v m}$ on $\widehat{E}_v$, and also that

$$\|q(z)\|_{\widehat{E}_v} \le 3\widehat{R}_v^{n_v N_v m}. \tag{6.16}$$

Before proceeding further, we will need a lemma.

LEMMA 6.2. *Let $E \subset \mathbb{R}$ be a compact set of positive capacity with connected complement, and put $R = \gamma(E)$. Then for any compact set $F \subset \mathbb{C} \backslash E$ there is a constant $C(E, F)$ with the following property: if $f(z)$ is a monic polynomial of degree d with all its roots in E, and if $\|f\|_E \le BR^d$ for some number B, then for all $z \in F$,*

$$\left| G(z, \infty; E) + \log(R) - \frac{1}{d}\log(|f(z)|) \right| \le \frac{C(E, F)\log(B)}{d}.$$

Proof. Let B_0 be such that $\|f\|_E = B_0 R^d$, and put

$$h(z) = G(z, \infty; E) + \frac{1}{d}\log(B_0) + \log(R) - \frac{1}{d}\log(|f(z)|).$$

Then $h(z)$ is harmonic in $\mathbb{C} \setminus E$, and $\lim_{z\to\infty} h(z) = \frac{1}{d}\log(B_0)$, so $h(z)$ extends to a function harmonic in $\mathbb{P}^1(\mathbb{C}) \setminus E$ with $h(\infty) = \frac{1}{d}\log(B_0)$. For each $x \in \partial(\mathbb{P}^1(\mathbb{C}) \setminus E)$,

$$\liminf_{\substack{z\to x \\ z\in\mathbb{P}^1(\mathbb{C})\setminus E}} h(z) \ge 0.$$

By the minimum principle for harmonic functions, we have $h(z) \ge 0$ for all $z \in \mathbb{P}^1(\mathbb{C}) \setminus E$. Hence, by Harnack's theorem, there is a constant C_0 such that for all $z \in F$,

$$0 \le h(z) \le C_0 h(\infty) = C_0 \frac{1}{d}\log(B_0).$$

Thus, we can take $C(E, F) = \max(1, C_0 - 1)$. ∎

PHASE 3. *Patching the low-order coefficients* $(n - s < k \le n)$. To motivate our procedure for patching the low order coefficients, we need lower bounds for $|f_s(z)|$ on $\widehat{E}_v$ and for $|q(z)|$ on $E_{v,0}$.

Since $f_s(z)$ is the Chebyshev polynomial for $E_{v,0}$, and $\widehat{E}_v \subset E_v \subset \mathbb{R}$ is disjoint from $E_{v,0}$, it follows from (6.12) that $|f_s(z)| \ge 2\varrho^s$ on $\widehat{E}_v$. Thus, at each point of $\widehat{E}_v$ where $|q(z)| \geq \widehat{R}_v^{n_v N_v m}$, we have

$$|u_v^{(n-s)}(z)| = |f_s(z)q(z)| \geq 2\varrho^s \widehat{R}_v^{n_v N_v m} = 2(\varrho/\widehat{R}_v)^s \widehat{R}_v^n.$$

On the other hand, there is a positive lower bound σ for $G(z,\infty;\widehat{E}_v)$ on $E_{v,0}$. Applying Lemma 6.2, with $E = \widehat{E}_v$ and $F = E_{v,0}$, taking $f(z) = q(z)$ and using (6.16), we see that if m is sufficiently large, then for all $z \in E_{v,0}$,

$$\text{(6.17)} \qquad \frac{1}{n_v N_v m} \log(|q(z)|) \geq \sigma + \log(\widehat{R}_v) - \frac{C(\widehat{E}_v; E_{v,0}) \log(3)}{n_v N_v m} \geq \log(\widehat{R}_v),$$

and so at each point of $E_{v,0}$ where $|f_s(z)| = 2\varrho^s$ we also have

$$|u_v^{(n-s)}(z)| = |f_s(z)q(z)| \geq 2\varrho^s \widehat{R}_v^{n_v N_v m} = 2(\varrho/\widehat{R}_v)^s \widehat{R}_v^n.$$

Put

$$A_1 = \min_{0 \leq s < n_v N_v} (2(\varrho/\widehat{R}_v)^s), \qquad A_2 = \max_{0 \leq s < n_v N_v} (\|z^s\|_{U_v}/h_v^s).$$

Since $\widehat{R}_v = 2^{-1/(n_v N_v)} R_v$, these constants depend only on E_v, U_v, and our choice of n_v and N_v. As $f_s(z)$ has all its zeros in $E_{v,0}$, it has constant sign on each subinterval of $\widehat{E}_v$. Similarly, since $q(z)$ has all its zeros in $\widehat{E}_v$, it has constant sign on $E_{v,0}$. Since $q(z)$ oscillates $n_v N_v m$ times with magnitude at least $\widehat{R}_v^{n_v N_v m}$ on $\widehat{E}_v$, and $f_s(z)$ oscillates s times between $\pm 2\varrho^s$ on $E_{v,0}$, it follows that $u_v^{(n-s)}(z)$ oscillates n times, with magnitude at least $A_1 \widehat{R}_v^n$, on $\widehat{E}_v \cup E_{v,0}$.

To patch the coefficients $c_{v,k}$ for $n - s < k \leq n$, set

$$u_v^{(k)}(z) = u_v^{(k-1)}(z) + \Delta_{v,k} z^{n-k}.$$

On the set $\widehat{E}_v \cup E_{v,0}$, we have $|\Delta_{v,k} z^{n-k}| \leq h_v^k A_2 h_v^{n-k} = A_2 h_v^n$, and the total patching correction in passing from $u_v^{(n-s)}(z)$ to $u_v^{(n)}(z)$ has magnitude at most $n_v N_v \cdot A_2 h_v^n$. Recall that by (6.11), $h_v < r_v < \widehat{R}_v$. If m (hence n) is sufficiently large then we will have

$$\text{(6.18)} \qquad n_v N_v A_2 h_v^n \leq \frac{1}{2} A_1 \widehat{R}_v^n.$$

Consequently, if n is large enough that (6.17) and (6.18) hold, then $u_v^{(n)}(z)$ oscillates n times, with magnitude at least $\frac{1}{2} A_1 \widehat{R}_v^n$, on $\widehat{E}_v \cup E_{v,0}$. It follows that $u_v^{(n)}(z)$ has all its roots in $\widehat{E}_v \cup E_{v,0}$, and hence in U_v. ■

7. Local patching for nonarchimedean PL-domains. When K_v is nonarchimedean, and E_v is a PL-domain, our patching procedure generalizes a method introduced by Cantor [3].

PROPOSITION 7.1. *Let K_v be nonarchimedean, and suppose*

$$E_v = \bigcup_{i=1}^{M} B(a_i, r_i)$$

is a finite union of balls with radii in the value group of $\mathbb{C}_v^\times$, stable under $\mathrm{Gal}_c(\mathbb{C}_v/K_v)$, and such that $E_v \cap K_v$ is nonempty. There is a constant M_v, depending only on E_v, with the following property. Let $0 < h_v < r_v < \gamma(E_v)$ be given. For each integer L large enough that

$$\left(\frac{h_v}{r_v}\right)^L M_v \le \frac{1}{2}, \tag{7.1}$$

there is a constant B_v (depending on L), such that for each sufficiently large n there is a monic polynomial $u_v^{(0)}(z) \in K_v[z]$ of degree n whose roots belong to E_v, and which can be patched with any numbers $\Delta_v^{(k)} \in K_v$ satisfying

$$|\Delta_v^{(k)}|_v \le \begin{cases} B_v & \text{for } k \le L, \\ h_v^k & \text{for } k > L, \end{cases}$$

in such a way that its roots remain in E_v.

Proof. We begin with some reductions. In the decomposition $E_v = \bigcup_{i=1}^M B(a_i, r_i)$, we can assume that the balls $B(a_i, r_i)$ are pairwise disjoint and that the a_i belong to $\widetilde{K}_v$. Furthermore, since by hypothesis $E_v \cap K_v$ is nonempty, some $B(a_i, r_i)$ contains a point of K_v; we can assume that $i = 1$ and that $a_1 \in K_v$. Lemma 12.1 of Appendix II shows there is a polynomial $g_0(z) \in \widetilde{K}_v[z]$ of the form $g_0(z) = c_0 \prod_{i=1}^M (z - a_i)^{d_i}$, with each $d_i > 0$, such that

$$E_v = \{z \in \mathbb{C}_v : |g_0(z)|_v \le 1\}. \tag{7.2}$$

If $\sigma \in \mathrm{Gal}_c(\mathbb{C}_v/K_v)$, then since E_v is Galois-stable,

$$|g_0(z)|_v \le 1 \iff z \in E_v \iff |(\sigma g_0)(z)|_v \le 1.$$

Thus taking the product of the conjugates of $g_0(z)$ over K_v, and scaling the product so it is monic, we obtain a polynomial $g(z) \in K_v[z]$ of degree n_v, say, and a number R_v in the value group of $\mathbb{C}_v^\times$ for which

$$E_v = \{z \in \mathbb{C}_v : |g(z)|_v \le R_v^{n_v}\}. \tag{7.3}$$

By (3.4), $R_v = \gamma(E_v)$. The polynomial $g(z)$ will be fixed for the rest of the construction.

We will need the nonarchimedean Maximum Modulus Principle in the following form (see [3, Theorem 3.1.1, p. 180] or [11, Theorem 1.4.2, p. 51]).

LEMMA 7.2 (Maximum Principle with Distinguished Boundary). *Let $f(z) \in \mathbb{C}_v(z)$ be a nonconstant rational function, and let R belong to the value* *group of* $\mathbb{C}_v^\times$. *Put*

$$D = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |f(z)|_v \le R\},$$
$$\partial D = \partial D(f) = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |f(z)|_v = R\}.$$

Then any rational function $F(z) \in \mathbb{C}_v(z)$ *with no poles in* D *achieves its maximum absolute value on* D *at a point of* ∂D. *In particular, for any* M, *if* $|F(z)|_v \le M$ *on* ∂D, *then* $|F(z)|_v \le M$ *on all of* D.

Put

$$D_v = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |g(z)|_v \ge R_v^{n_v}\} = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |1/g(z)|_v \le 1/R_v^{n_v}\} \tag{7.4}$$

noting that

$$\partial E_v(g) = \partial D_v(1/g) = \{z \in \mathbb{C}_v : |g(z)|_v = R_v^{n_v}\}. \tag{7.5}$$

We will write $\partial D_v = \partial E_v$ for this common boundary. Write $a = a_1$, and note that a is a root of $g(z)$. By the maximum modulus principle, applied to $F(z) = 1/(z-a)$ on D_v relative to ∂D_v, the number $\min_{z\in\partial E_v}(|z-a|_v)$ exists and is positive. Put

$$C_v^- = \min_{0\le h<n_v}((\min_{z\in\partial E_v} |(z-a)^h|_v)/R_v^h), \tag{7.6}$$

$$C_v^+ = \max_{0\le h<n_v}((\max_{z\in\partial E_v} |(z-a)^h|_v)/R_v^h). \tag{7.7}$$

The constant M_v in the proposition will be

$$M_v = C_v^+/C_v^-. \tag{7.8}$$

The condition imposed on the globally chosen constant L by the place v is

$$M_v\left(\frac{h_v}{r_v}\right)^L \le \frac{1}{2}, \tag{7.9}$$

and the constant B_v in the proposition will be

$$B_v = \frac{1}{2}\min_{1\le k\le L}(R_v^k/M_v). \tag{7.10}$$

The patching construction. If $n = s + mn_v$, where s, m are integers with $0 \le s < n_v$, we take the initial patching polynomial to be

$$u_v^{(0)}(z) = (z-a)^s g(z)^m \in K_v[z].$$

By (7.5) and (7.6), for all $z \in \partial E_v$,

$$|u_v^{(0)}(z)|_v \ge C_v^- R_v^n. \tag{7.11}$$

The Maximum Modulus Principle, applied to $1/u_v^{(0)}(z)$ on ∂D_v, shows this holds for all $z \notin E_v$.

Let $w_v^{(k)}(z)$ be the monic polynomial of degree $n-k$

$$w_v^{(k)}(z) = (z-a)^{h_1} g(z)^{h_2}, \tag{7.12}$$

where the integers h_1, h_2, are determined by

$$n-k = h_1 + h_2 n_v, \quad 0 \le h_1 < n_v,\ 0 \le h_2 < m. \tag{7.13}$$

By (7.8) and (7.3), for all $z \in E_v$,

$$|w_v^{(k)}(z)|_v \le C_v^+ R_v^{h_1} R_v^{h_2 n_v} = C_v^+ R_v^{n-k}. \tag{7.14}$$

At the kth step of the patching process, $1 \le k \le n$, we will put

$$u_v^{(k)}(z) = u_v^{(k-1)}(z) + \Delta_{v,k} w_{v,k}(z)$$

where $\Delta_{v,k} \in K_v$ is chosen on the basis of global considerations, and satisfies the conditions in the proposition. At the end of the construction,

$$u_v^{(n)}(z) = u_v^{(0)}(z) + \sum_{k=1}^{n} \Delta_v^{(k)} w_v^{(k)}(z). \tag{7.15}$$

We claim that the conditions on $|\Delta_v^{(k)}|_v$ imply that for all $z \in E_v$,

$$\Big|\sum_{k=1}^{n} \Delta_v^{(k)} w_v^{(k)}(z)\Big|_v \le \frac{1}{2} C_v^- R_v^n. \tag{7.16}$$

By the ultrametric inequality, it suffices to check this for each k. When $k \le L$ we have $|\Delta_v^{(k)}|_v \le B_v \le \frac{1}{2} R_v^k/M_v$ by (7.10), while if $k > L$, then $|\Delta_v^{(k)}|_v \le h_v^k \le \frac{1}{2} r_v^k/M_v \le \frac{1}{2} R_v^k/M_v$ by (7.9). Thus, in either case, since $M_v = C_v^+/C_v^-$, by (7.14),

$$|\Delta_v^{(k)} w_v^{(k)}(z)|_v \le \left(\frac{1}{2} R_v^k/M_v\right)(C_v^+ R_v^{n-k}) = \frac{1}{2} C_v^- R_v^n.$$

We now apply the Maximum Modulus Principle to

$$F(z) = (u_v^{(n)}(z) - u_v^{(0)}(z))/u_v^{(0)}(z) = \Big(\sum_{k=1}^{n} \Delta_v^{(k)} w_v^{(k)}(z)\Big)/u_v^{(0)}(z)$$

on the domain D_v, relative to the boundary $\partial D_v = \partial E_v$.

Here $u_v^{(n)}(z)$ and $u_v^{(0)}(z)$ are monic polynomials of degree n, and the zeros of $u_v^{(0)}(z) = (z-a)^s g(z)^m$ belong to $\mathbb{P}^1(\mathbb{C}_v) \setminus D_v$, so $F(z)$ has no poles in D_v. On $\partial D_v = \partial E_v$, (7.16) and (7.11) show that

$$|F(z)|_v \le 1/2.$$

Hence $|F(z)|_v \le 1/2$ throughout D_v. This means that for all $z \notin E_v$,

$$|u_v^{(n)}(z) - u_v^{(0)}(z)|_v \le \frac{1}{2} |u_v^{(0)}(z)|_v,$$

and so by the ultrametric inequality, for all $z \notin E_v$,

$$|u_v^{(n)}(z)|_v = |u_v^{(0)}(z)|_v \ge C_v^- R_v^n > 0. \tag{7.17}$$

In particular, $u_v^{(n)}(z)$ has all its roots in E_v. ∎

8. Local patching for nonarchimedean compact sets. Let K_v be nonarchimedean, and suppose $E_v \subset K_v$ is compact. By our preliminary reductions, we can assume E_v is a finite disjoint union of cosets of O_v,

$$E_v = \bigcup_{i=1}^{M} (a_i + b_i O_v),$$

where the $a_i, b_i \in K_v$ and the $b_i \neq 0$. Let R be such that $E_v \subset B(0,R)$.

PROPOSITION 8.1. *Suppose K_v is nonarchimedean and*

$$E_v = \bigcup_{i=1}^{M} (a_i + b_i O_v) \subset K_v.$$

There is a number M_v, depending only on E_v, with the following property. Let $0 < h_v < r_v < \gamma(E_v)$ be given. If $L \geq 1$ is large enough that

$$L\left(\frac{h_v}{r_v}\right)^L M_v \leq 1, \tag{8.1}$$

then there is a constant B_v (depending on L) such that for each sufficiently large n, there is a monic polynomial $u_v^{(0)}(z) \in K_v[z]$ of degree n, which has all its roots in E_v, and can be patched with arbitrary $\Delta_v^{(k)} \in K_v$ satisfying

$$|\Delta_v^{(k)}|_v \leq \begin{cases} B_v & \text{for } k \leq L, \\ h_v^k & \text{for } k > L, \end{cases}$$

in such a way that its roots remain in E_v.

Before giving the proof we have several tasks. First, we need to construct the "basic well-distributed sequence for E_v" and define the notion of a "regular sequence in E_v". Second, we need to introduce generalized Stirling polynomials, which play a role analogous to those of Chebyshev polynomials in the case $K_v \cong \mathbb{R}$ (this idea goes back to Cantor [3]). Finally, we must prove several lemmas which govern the patching process.

The basic well-distributed sequence. First consider the case where $E_v = O_v$. Write $q = q_v = \#(O_v/\pi_v O_v)$, and let $\beta_v(k)$, for $k = 0, \ldots, q-1$, be a set of representatives for $O_v/\pi_v O_v$, with $\beta_v(0) = 0$. For $k \geq q$, expand k as

$$k = \sum_{i=0}^{N} d_i(k) q^i$$

where $N = \lfloor \log_q(k) \rfloor$ and $0 \leq d_i(k) \leq q-1$ are the base q digits of k; put

$$\beta_v(k) = \sum_{i=0}^{N} \beta_v(d_i(k)) \pi_v^i.$$

In this way, the sequence $\{\beta_v(k)\}_{0\le k<\infty}$ extends $\{\beta_v(k)\}_{0\le k<q}$, and uniformly fills out the cosets of $O_v/\pi_v^e O_v$ for each $e \ge 0$. Define $\mathrm{val}_q(k)$ to be the smallest i such that $d_i(k) \ne 0$. Then it is easy to see that for each k,

$$\mathrm{ord}_v(\beta_v(k)) = \mathrm{val}_q(k), \tag{8.2}$$

and that for all $k \ne l$,

$$\mathrm{ord}_v(\beta_v(k) - \beta_v(l)) = \mathrm{val}_q(|k-l|). \tag{8.3}$$

We call $\{\beta_v(k)\}$ the *basic well-distributed sequence* in O_v. Note that for each $n > 0$, if $k, l < n$ and $k \ne l$, then

$$\mathrm{ord}_v(\beta_v(k) - \beta_v(l)) \le \lfloor \log_v(n) \rfloor, \tag{8.4}$$

while for each $z \in O_v$, there is a $k < n$ such that

$$\mathrm{ord}_v(z - \beta_v(k)) \ge \lfloor \log_v(n) \rfloor. \tag{8.5}$$

Now consider an arbitrary set of the form $E_v = \bigcup_{i=1}^M E_{v,i}$, where $E_{v,i} = a_i + b_i O_v$ with $a_i, b_i \in K_v$, $b_i \ne 0$, and where the cosets are disjoint. Let μ be the equilibrium distribution of E_v, and define weights

$$w_i = \mu(E_{v,i}), \qquad i = 1, \ldots, M.$$

Then each $w_i > 0$ as was noted earlier, and $\sum_{i=1}^M w_i = 1$. We want the basic well-distributed sequence in E_v to assign elements to each $E_{v,i}$ in proportion to its weight w_i. That this can be done, with small error, follows from a combinatorial lemma due to Balinski and Young ([2, Theorem 3, p. 714]):

Lemma 8.2. *Let $w_1, \ldots, w_M > 0$ be such that $\sum_i w_i = 1$. Then there is a* 1-1 *correspondence $\Phi : \mathbb{N} \to \mathbb{N}^M$, $\Phi(n) = (\Phi_1(n), \ldots, \Phi_M(n))$, such that*:

(A) *For each i, $\Phi_i(0) = 0$ and $\Phi_i(n)$ is nondecreasing with n*;

(B) *For each $n \ge 0$, $\sum_{i=1}^M \Phi_i(n) = n$*;

(C) *For each n and i, $\lfloor w_i n \rfloor \le \Phi_i(n) \le \lceil w_i n \rceil$. In particular, if $w_i n \in \mathbb{N}$, then $\Phi_i(n) = w_i n$.*

Remark. This lemma has an unusual source. It was originally established in the context of the "Alabama Paradox" concerning the apportionment of members in the U.S. House of Representatives. The question was, is there a rule for allotting representatives to the various states in proportion to population, in such a way that no state loses representatives when the size of the House is increased? The author thanks Peter Rice for pointing the lemma out to him.

Clearly for each n there is exactly one index i for which $\Phi_i(n+1) > \Phi_i(n)$. We will write $i(n)$ for this i.

Definition 8.1. The *basic well-distributed sequence* in E_v is the sequence $\{\lambda_v(n)\}_{n\geq 0}$ defined by

$$\lambda_v(n) = a_{i(n)} + b_{i(n)} \cdot \beta_v(\Phi_{i(n)}(n)).$$

That is, the nth element of the sequence is assigned to $E_{v,i(n)}$, and the elements assigned to $E_{v,i} = a_i + b_i O_v$ fill out $E_{v,i}$ like the basic well-distributed sequence in O_v. Among $\lambda_v(0), \ldots, \lambda_v(n-1)$, precisely $\Phi_i(n)$ elements belong to $E_{v,i}$. If $\lambda_v(k)$ and $\lambda_v(l)$ belong to distinct cosets $E_{v,i}$, $E_{v,j}$, then

$$|\lambda_v(k) - \lambda_v(l)|_v = |a_i - a_j|_v, \tag{8.6}$$

while if they belong to the same coset $E_{v,i}$, then, writing $r_i = |b_i|_v$,

$$|\lambda_v(k) - \lambda_v(l)|_v = r_i |\pi_v|_v^{\mathrm{val}_q(\Phi_i(k) - \Phi_i(l))}. \tag{8.7}$$

Note that the basic well-distributed sequence in E_v depends on the representation of E_v as a union of sets $a_i + b_i O_v$, and also on the choice of the function Φ (which in general is not unique). For the remainder of the construction we will assume these are fixed.

Stirling polynomials and regular sequences. The classical Stirling polynomial of degree n is $S_n(z) = \prod_{i=0}^{n-1}(z - i)$. Pólya introduced Stirling polynomials for the rings of integers O_v, putting

$$S_n(z; O_v) = \prod_{k=0}^{n-1}(z - \beta_v(k))$$

(cf. [8]). For general E_v, we put $S_0(z; E_v) = 1$ and define the *Stirling polynomial* of degree $n \geq 1$ for E_v to be

$$S_n(z; E_v) = \prod_{k=0}^{n-1}(z - \lambda_v(k)). \tag{8.8}$$

However, for our needs it is not enough to consider Stirling polynomials alone, but also polynomials with roots which behave sufficiently like them.

Lemma 8.3. *The basic well-distributed sequence in E_v has the following property. Put*

$$A_0 = \max(0, \max_i(\mathrm{ord}_v(b_i))).$$

Then for each $n > 0$ and each k, l with $0 \leq k, l < n$ and $k \neq l$,

$$\mathrm{ord}_v(\lambda_v(k) - \lambda_v(l)) < A_0 + \log_v(n).$$

Proof. Let n, k and l be as above. If $i(k) \neq i(l)$, then $\mathrm{ord}_v(a_{i(k)} - a_{i(l)}) < \mathrm{ord}_v(b_{i(k)}) \leq A_0$, so

$$\mathrm{ord}_v(\lambda_v(k) - \lambda_v(l)) = \mathrm{ord}_v(a_{i(k)} - a_{i(l)}) < A_0 + \log_v(n).$$

On the other hand, if $i(k) = i(l) = i$, then

$$\mathrm{ord}_v(\lambda_v(k) - \lambda_v(l)) = \mathrm{ord}_v(b_i) + \mathrm{val}_q(|\Phi_i(k) - \Phi_i(l)|) < A_0 + \log_v(n). \qquad \blacksquare$$

REMARK 8.1. A_0 has the following properties:

(1) $A_0 \geq 0$;
(2) $A_0 \geq -\log_v(R)$ for any R with $E_v \subset B(0,R)$;
(3) $A_0 > \mathrm{ord}_v(a_i - a_j)$ for each $i \neq j$; and
(4) if $\alpha \in K_v$ satisfies $\mathrm{ord}_v(\alpha - x) \geq A_0$ for some $x \in E_v$, then $\alpha \in E_v$.

DEFINITION 8.2. A *regular sequence* of length n in E_v is a sequence $\{\alpha_0, \ldots, \alpha_{n-1}\} \subset E_v$ such that

$$\mathrm{ord}_v(\alpha_k - \lambda_v(k)) \geq A_0 + \log_v(n)$$

for each $k = 0, \ldots, n-1$.

Thus, in a regular sequence of length n, if $0 \leq k, l < n$ and $k \neq l$ then

$$\mathrm{ord}_v(\alpha_k - \alpha_l) = \mathrm{ord}_v(\lambda_v(k) - \lambda_v(l)) < A_0 + \log_v(n). \tag{8.9}$$

We now establish some simple properties of regular sequences.

LEMMA 8.4. *There is a constant $A_1 \geq 0$ such that for any n, any regular sequence $\{\alpha_0, \ldots, \alpha_{n-1}\}$ of length n in E_v, and any $z \in E_v$, there is an index J for which* $\mathrm{ord}_v(z - \alpha_J) \geq \log_v(n) - A_1$.

Proof. Suppose $z \in E_{v,i} = a_i + b_i O_v$, and let n_i be the number of elements of the sequence which belong to $E_{v,i}$. Let J be an index for which $\mathrm{ord}_v(z - \alpha_J)$ is maximal. If $n_i > 0$, then $\alpha_J \in E_{v,i}$, and by (8.5),

$$\mathrm{ord}_v(z - \alpha_J) \geq \mathrm{ord}_v(b_i) + \lfloor \log_v(n_i) \rfloor \geq \log_v(n) + \mathrm{ord}_v(b_i) + \log_v(w_i) - 1.$$

On the other hand if $n_i = 0$ then necessarily $w_i n < 1$ and so

$$\mathrm{ord}_v(z - \alpha_J) \geq -\log_v(R) > \log_v(n) - \log_v(R) + \log_v(w_i).$$

Since there are only finitely many $E_{v,i}$ the result follows. ∎

LEMMA 8.5. *Let $\{\alpha_0, \ldots, \alpha_{n-1}\}$ be a regular sequence of length n in E_v. Given $z \in E_v$, let $0 \leq J < n$ be an index for which $\mathrm{ord}_v(z - \alpha_J)$ is maximal. Then for any $i \neq J$,*

$$\mathrm{ord}_v(z - \alpha_i) \leq \mathrm{ord}_v(\alpha_J - \alpha_i).$$

Moreover, if n is large enough that each $E_{v,h}$ contains points of the regular sequence, then for any $i \neq J$,

$$\mathrm{ord}_v(z - \alpha_i) = \mathrm{ord}_v(\alpha_J - \alpha_i).$$

Proof. To prove the inequality, first suppose i is such that $\mathrm{ord}_v(z - \alpha_i) < \mathrm{ord}_v(z - \alpha_J)$. Then

$$\mathrm{ord}_v(\alpha_J - \alpha_i) = \min(\mathrm{ord}_v(z - \alpha_J), \mathrm{ord}_v(z - \alpha_i)) = \mathrm{ord}_v(z - \alpha_i).$$

Next suppose that i is an index for which $\mathrm{ord}_v(z - \alpha_i) = \mathrm{ord}_v(z - \alpha_J)$. If $\mathrm{ord}_v(z - \alpha_i) > \mathrm{ord}_v(\alpha_J - \alpha_i)$, then

$$\mathrm{ord}_v(z - \alpha_J) = \min(\mathrm{ord}_v(z - \alpha_i), \mathrm{ord}_v(\alpha_J - \alpha_i)) = \mathrm{ord}_v(\alpha_J - \alpha_i) < \mathrm{ord}_v(z - \alpha_i),$$

a contradiction. Hence $\mathrm{ord}_v(z - \alpha_i) \leq \mathrm{ord}_v(\alpha_J - \alpha_i)$.

If n is large enough that each $E_{v,h}$ contains points of the regular sequence, then necessarily α_J belongs to the same coset $E_{v,h} = a_h + b_h O_v$ as z. If α_i belongs to another coset $E_{v,l} = a_l + b_l O_v$, then

$$\mathrm{ord}_v(z - \alpha_i) = \mathrm{ord}_v(a_h - a_l) = \mathrm{ord}_v(\alpha_J - \alpha_i).$$

If α_i belongs to $E_{v,h}$, let n_h be the number of elements of the regular sequence in $E_{v,h}$. By the construction of the basic well-distributed sequence, since $\mathrm{ord}_v(z - \alpha_J)$ is maximal, by (8.5) we must have

$$\mathrm{ord}_v(z - \alpha_J) \geq \mathrm{ord}_v(b_h) + \lfloor \log_v(n_h) \rfloor, \tag{8.10}$$

while by (8.4), since $\alpha_i \neq \alpha_J$,

$$\mathrm{ord}_v(\alpha_J - \alpha_i) \leq \mathrm{ord}_v(b_h) + \lfloor \log_v(n_h) \rfloor. \tag{8.11}$$

Hence $\mathrm{ord}_v(z-\alpha_i) \geq \mathrm{ord}_v(\alpha_J-\alpha_i)$ by the ultrametric inequality. Combined with our earlier inequality, this gives $\mathrm{ord}_v(z - \alpha_i) = \mathrm{ord}_v(\alpha_J - \alpha_i)$. ∎

The following lemma shows that each regular sequence of length n contains "many" disjoint regular subsequences of shorter length.

LEMMA 8.6. (A) *For any $C_1 > 0$, there is a $C_2 > 0$ such that for all sufficiently large n, any regular sequence of length n in E_v contains at least $\lceil C_2\sqrt{n}\,\rceil$ pairwise disjoint subsets, each of which is a regular sequence in E_v of length $\lceil C_1\sqrt{n}\,\rceil$.*

(B) *For any $C_3 > 0$, there is a $C_4 > 0$ such that for all sufficiently large n, if $S_1, \ldots, S_Z$ are regular sequences in E_v, where each S_l has length $n_l \leq \lceil C_3\sqrt{n}\,\rceil$ and $Z \leq \lceil C_4\sqrt{n}\,\rceil$, then any regular sequence $\{\alpha_i\}_{0\leq i<n}$ in E_v contains a regular subsequence $Q = \{\alpha_{i_0}, \ldots, \alpha_{i_{W-1}}\}$ of length $W = \lceil C_3\sqrt{n}\,\rceil$ such that for each $\alpha_{i_k} \in Q$ and each $\theta \in \bigcup_{l=1}^{Z} S_l$,*

$$\mathrm{ord}_v(\alpha_{i_k} - \theta) < A_0 + \log_v(n).$$

Proof. For (A), recall that for any n, each $E_{v,i} = a_i + b_i O_v$ receives $\Phi_i(n) \geq \lfloor w_i n \rfloor$ elements from a regular sequence of length n. Write $B_i = \mathrm{ord}_v(b_i)$. Then for each integer $B \geq B_i$, these $\Phi_i(n)$ elements are distributed among the cosets $d + \pi_v^B O_v$ in $E_{v,i}$ in such a way that each coset receives at least $\lfloor \lfloor w_i n \rfloor / q^{B-B_i} \rfloor$ elements. Now take $B = \lceil A_0 + \log_v(\lceil C_1\sqrt{n}\,\rceil) \rceil$. Then each coset $d + \pi_v^B O_v$ contained in $E_{v,i}$ receives at least

$$\begin{aligned}\lfloor \lfloor w_i n \rfloor / q^{B-B_i} \rfloor &\geq w_i n / q^{A_0 + \log_v(\sqrt{n}) + \log_v(C_1) + 2 - B_i} - 2 \\ &= (w_i / q^{A_0 - B_i + \log_v(C_1) + 2})\sqrt{n} - 2\end{aligned}$$

elements from a regular sequence with length n, but at most one element from a regular sequence of length $\lceil C_1\sqrt{n}\,\rceil$ (by the definition of a regular sequence).

Let $C_2 > 0$ be less than the minimum of $w_i / q^{A_0 - B_i + \log_v(C_1) + 2}$ for $i = 1, \ldots, M$. For sufficiently large n, each regular sequence of length n in E_v

contains regular subsequences of length $\lceil C_1\sqrt{n}\,\rceil$ (the initial elements form such a subsequence). And, if B is as above, then for sufficiently large n each coset $d+\pi_v^B O_v$ in E_v receives at least $\lceil C_2\sqrt{n}\,\rceil$ elements from the regular sequence, so there are at least that many disjoint regular subsequences of length $\lceil C_1\sqrt{n}\,\rceil$ in a regular sequence of length n.

The proof of (B) is similar. ∎

Products like $\prod_{j=0}^{n-1}(\beta_v(n)-\beta_v(j))$ are analogous to factorials: by (8.3),

$$\operatorname{ord}_v\Big(\prod_{j=0}^{n-1}(\beta_v(n)-\beta_v(j))\Big)=\sum_{j=1}^{n}\operatorname{val}_q(j). \tag{8.12}$$

We claim that

$$\sum_{j=1}^{n}\operatorname{val}_q(j)=\frac{n}{q-1}-\frac{1}{q-1}\sum_{i\ge 0}d_i(n). \tag{8.13}$$

This generalizes the familiar formula for $\operatorname{ord}_p(n!)$. For each $k\ge 1$, there are exactly $\lfloor n/q^k\rfloor$ numbers j in the range $1\le j\le n$ for which $\operatorname{val}_q(j)\ge k$. Hence, writing $d_i=d_i(n)$, we have

$$\begin{aligned}\sum_{j=1}^{n}\operatorname{val}_q(j)&=\sum_{k\ge 1}\left\lfloor\frac{n}{q^k}\right\rfloor\\ &=(d_1+d_2q+d_3q^2+\ldots)+(d_2+d_3q+\ldots)+\ldots\\ &=d_1\cdot\frac{q-1}{q-1}+d_2\cdot\frac{q^2-1}{q-1}+d_3\cdot\frac{q^3-1}{q-1}+\ldots\\ &=\frac{d_0+d_1q+d_2q^2+\ldots}{q-1}-\frac{d_0+d_1+d_2+\ldots}{q-1}\\ &=\frac{n}{q-1}-\frac{1}{q-1}\sum_{i\ge 0}d_i(n).\end{aligned}$$

For the next lemma, it is crucial that the weights w_i defining a regular sequence arise from the equilibrium distribution of E_v.

LEMMA 8.7. *There is a constant $A_2\ge 0$ such that for all n, if $\{\alpha_i\}_{0\le i<n}$ is a regular sequence of length n in E_v and $f(z)=\prod_{i=0}^{n-1}(z-\alpha_i)$, then:*

(A) *For each J in the range $0\le J<n$, we have*

$$nV(E_v)-2\log_v(n)-A_2\le\operatorname{ord}_v\Big(\prod_{\substack{i=0\\ i\ne J}}^{n-1}(\alpha_J-\alpha_i)\Big)\le nV(E_v)+A_2. \tag{8.14}$$

(B) *For each $z\in E_v$, if $0\le J<n$ is such that $\operatorname{ord}_v(z-\alpha_J)$ is maximal, then*

$$nV(E_v)-\log_v(n)-A_2\le\operatorname{ord}_v(f(z))\le nV(E_v)+A_2+\operatorname{ord}_v(z-\alpha_J). \tag{8.15}$$

In particular, $\|f\|_{E_v}\le q^{-nV(E_v)+\log_v(n)+A_2}$.

Proof. First consider the special case $E_v = O_v$ with $\{\lambda_v(k)\} = \{\beta_v(k)\}$. Then $V(O_v) = 1/(q-1)$ and $A_0 = 0$.

To prove (A), fix J and note that if $i \neq J$, then $\mathrm{ord}_v(\alpha_J - \alpha_i) = \mathrm{ord}_v(\beta_v(J) - \beta_v(i)) = \mathrm{val}_q(|J-i|)$. Hence

$$\mathrm{ord}_v\Big(\prod_{\substack{i=0\\ i\neq J}}^{n-1}(\alpha_J - \alpha_i)\Big) = \sum_{l=1}^{J}\mathrm{val}_q(l) + \sum_{l=1}^{n-J-1}\mathrm{val}_q(l)$$
$$= \frac{n}{q-1} - \frac{\sum d_i(J) + \sum d_i(n-J-1) + 1}{q-1}.$$

It is easy to see that

$$0 \leq \sum d_i(J) + \sum d_i(n-J-1) + 1 \leq 2(q-1)\lceil \log_v(n)\rceil$$

and hence that

$$\frac{n}{q-1} - 2\log_v(n) - 2 \leq \mathrm{ord}_v\Big(\prod_{\substack{i=0\\ i\neq J}}^{n-1}(\alpha_J - \alpha_i)\Big) \leq \frac{n}{q-1}. \tag{8.16}$$

This yields assertion (A).

To obtain (B), let J be an index for which $\mathrm{ord}_v(z-\alpha_J)$ is maximal. Then

$$\mathrm{ord}_v(z-\alpha_J) \geq \lfloor \log_v(n)\rfloor \geq \log_v(n) - 1$$

and by Lemma 8.5, using the fact that $E_v = O_v$ consists of only one coset, for each $i \neq J$,

$$\mathrm{ord}_v(z-\alpha_i) = \mathrm{ord}_v(\alpha_J - \alpha_i).$$

Hence, adding the term $\mathrm{ord}_v(z-\alpha_J)$ to (8.16), we find that

$$\frac{n}{q-1} - \log_v(n) - 3 \leq \mathrm{ord}_v(f(z)) \leq \frac{n}{q-1} + \mathrm{ord}_v(z-\alpha_J). \tag{8.17}$$

Now let E_v be arbitrary: $E_v = \bigcup_{l=1}^{M} E_{v,l}$, where $E_{v,l} = a_l + b_l O_v$. We can assume n is large enough that any regular sequence of length n in E_v contains elements from each $E_{v,l}$.

To prove (A), fix J, and let h be the index for which $\alpha_J \in E_{v,h}$. If $l \neq h$ and $\alpha_i \in E_{v,l}$, then $|\alpha_J - \alpha_i|_v = |a_l - a_h|_v$, so

$$\mathrm{ord}_v\Big(\prod_{\alpha_i \in E_{v,l}}(\alpha_J - \alpha_i)\Big) = \Phi_l(n)\,\mathrm{ord}_v(a_l - a_h). \tag{8.18}$$

On the other hand, the $\alpha_i \in E_{v,h}$ form a regular sequence of length $\Phi_h(n)$ in $E_{v,h}$, and are an affine transformation of a regular sequence in O_v. By (8.16),

$$\Phi_h(n)\Big(\frac{1}{q-1} + \mathrm{ord}_v(b_h)\Big) - 2\log_v(\Phi_h(n)) - 2 - \mathrm{ord}_v(b_h) \tag{8.19}$$
$$\leq \mathrm{ord}_v\Big(\prod_{\substack{\alpha_i \in E_{v,h}\\ i\neq J}}(\alpha_J - \alpha_i)\Big) \leq \Phi_h(n)\Big(\frac{1}{q-1} + \mathrm{ord}_v(b_h)\Big) - \mathrm{ord}_v(b_h).$$

For each l, we have $\lfloor w_l n \rfloor \le \Phi_l(n) \le \lceil w_l n \rceil$, while by (3.10),

$$\sum_{l \neq h} w_l \operatorname{ord}_v(a_l - a_h) + w_h\left(\frac{1}{q-1} + \operatorname{ord}_v(b_h)\right) = V(E_v). \tag{8.20}$$

Summing the inequalities (8.18) and (8.19) and using (8.20), we find that there is a constant A_2 (which can be taken independent of h) such that

$$nV(E_v) - 2\log_v(n) - A_2 \le \operatorname{ord}_v\Big(\prod_{\substack{i=0\\ i\neq J}}^{n-1}(\alpha_J - \alpha_i)\Big) \le nV(E_v) + A_2.$$

This yields (A). We obtain (B) as before, possibly after increasing A_2. ∎

We can now state the basic lemma governing the patching process.

LEMMA 8.8. *Let $\{\alpha_i\}_{0\le i<m}$ be a regular sequence of length m in E_v, and put $f(z) = \prod_{i=0}^{m-1}(z-\alpha_i)$. Then for any $M \ge A_0 + \log_v(m)$, and any $\Delta \in K_v$ satisfying*

$$\operatorname{ord}_v(\Delta) \ge mV(E_v) + A_2 + M,$$

the roots of $f^*(z) := f(z) + \Delta$ again form a regular sequence of length m in E_v, and if we write $f^*(z) = \prod_{i=0}^{m-1}(z-\alpha_i^*)$, the roots α_i^* can be uniquely labeled in such a way that

$$\operatorname{ord}_v(\alpha_i^* - \alpha_i) \ge M$$

for each i. More generally, if $E_v \subset B(0,R)$, let $\Delta(z) \in K_v[z]$ be any polynomial of degree $< m$ such that

$$\operatorname{ord}_v(\Delta(z)) \ge mV(E_v) + A_2 + M$$

for all $z \in B(0,R)$. Then the same assertions hold for $f^*(z) := f(z) + \Delta(z)$.

Proof. Fix a root α_J of $f(z)$, and expand $f(z)$, $f^*(z)$ and $\Delta(z)$ about α_J, writing

$$\begin{aligned} f(z) &= b_1(z-\alpha_J) + b_2(z-\alpha_J)^2 + \ldots + b_{m-1}(z-\alpha_J)^{m-1} + (z-\alpha_J)^m,\\ f^*(z) &= c_0 + c_1(z-\alpha_J) + c_2(z-\alpha_J)^2 + \ldots + (z-\alpha_J)^m,\\ \Delta(z) &= \Delta_0 + \Delta_1(z-\alpha_J) + \ldots + \Delta_{m-1}(z-\alpha_J)^{m-1}. \end{aligned}$$

Here $b_1 = \pm\prod_{i\neq J}(\alpha_J - \alpha_i)$, so by Lemma 8.7(A), $\operatorname{ord}_v(b_1) \le mV(E_v) + A_2$. For each $k \ge 2$,

$$b_k = \pm b_1 \sum_{\substack{i_1<\ldots<i_{k-1}\\ \text{each } i_j\neq J}} \frac{1}{(\alpha_J - \alpha_{i_1})\ldots(\alpha_J - \alpha_{i_{k-1}})}$$

and so by (8.9),

$$\operatorname{ord}_v(b_k) > \operatorname{ord}_v(b_1) - (k-1)(A_0 + \log_v(m)).$$

On the other hand, by the hypothesis on $\Delta(z)$ and an easy estimate using the maximum principle, for each $k \geq 0$,

$$\mathrm{ord}_v(\Delta_k) \geq (mV(E_v) + A_2 + M) + k\log_v(R).$$

Here $A_0 \geq -\log_v(R)$ by Remark 8.1, and $M \geq A_0 + \log_v(m) > A_0$ (assuming $m \geq 2$, which is permissible since the lemma is trivial when $m = 1$), so for each $k \geq 1$,

$$\mathrm{ord}_v(\Delta_k) > \mathrm{ord}_v(b_1) - (k-1)(A_0 + \log_v(m)).$$

Now consider the Newton polygon of $f^*(z)$, expanded about α_J. By the estimates above, we have

$$\begin{aligned}
&\mathrm{ord}_v(c_0) = \mathrm{ord}_v(\Delta_0) \geq \mathrm{ord}_v(b_1) + M,\\
&\mathrm{ord}_v(c_1) = \mathrm{ord}_v(b_1),\\
&\mathrm{ord}_v(c_k) > \mathrm{ord}_v(b_1) - (k-1)(A_0 + \log_v(m)) \quad \text{for all } k \geq 2.
\end{aligned}$$

Thus the Newton polygon has a break at the point $(1, \mathrm{ord}_v(c_1))$, and $f^*(z)$ has a unique root α_J^* for which

$$\mathrm{ord}_v(\alpha_J^* - \alpha_J) \geq M.$$

By its uniqueness, α_J^* belongs to K_v, and by Remark 8.1 it belongs to E_v, since $M \geq A_0$.

Since this is true for each J, it follows that $\{\alpha_J^*\}_{0 \leq J < m}$ is a regular sequence of length m in E_v. ∎

Recall that numbers h_v and r_v were fixed at the beginning of the proof, satisfying

$$0 < h_v < r_v < \gamma(E_v) = q^{-V(E_v)}.$$

We are constructing a constant L subject to various local conditions; for this place v the condition will be that

$$L \cdot (-\log_v(h_v/r_v)) \geq A_2 + A_0 + \log_v(L) + 2. \tag{8.21}$$

The line on the left side of (8.21) meets the logarithmic curve on the right side at most twice. Since $(A_2 + A_0 + 2)\log(q_v) \geq 2\log(2) > 1$, if there are two intersections, one occurs for $L < 1$. Hence there is a unique least positive integer L for which (8.21) holds; for all $l \geq L$,

$$l \cdot (-\log_v(h_v/r_v)) \geq A_2 + A_0 + \lceil \log_v(l) \rceil. \tag{8.22}$$

Taking $M_v = q_v^{A_2 + A_0 + 2}$, (8.21) is equivalent to

$$L\left(\frac{h_v}{r_v}\right)^L M_v \leq 1,$$

which is the condition stated in Proposition 8.1. The constant B_v in Proposition 8.1 will be defined by

$$-\log_v(B_v) = \max_{1\le k\le L}(\lceil kV(E_v) + A_2 + A_0 + \lceil\log_v(k)\rceil\rceil). \tag{8.23}$$

Proof of Proposition 8.1. Given n, choose a regular sequence $\{\alpha_l\}_{0\le l<n}$ of length n in E_v, and take the initial patching polynomial to be

$$u_v^{(0)}(z) = \prod_{l=0}^{n-1}(z-\alpha_l).$$

In the construction, if n is sufficiently large, we will successively modify $u_v^{(0)}(z)$ to polynomials $u_v^{(1)}(z), u_v^{(2)}(z), \ldots, u_v^{(n)}(z)$, where

$$u_v^{(k)}(z) = z^n + \sum_{i=1}^{n} c_{v,i}^{(k)} z^{n-i} = \prod_{l=0}^{n-1}(z-\alpha_l^{(k)}).$$

At each step, all the roots of $u_v^{(k)}(z)$ will belong to E_v. There will be a well-defined correspondence between the roots of $u_v^{(k-1)}(z)$ and those of $u_v^{(k)}(z)$, so that a given root $\alpha_l^{(0)} = \alpha_l$ of $u_v^{(0)}(z)$ is successively modified to $\alpha_l^{(1)}, \alpha_l^{(2)}, \ldots, \alpha_l^{(n)}$. Because of this, if S is a subset of the original regular sequence $\{\alpha_0, \ldots, \alpha_{n-1}\}$, it makes sense to speak of the set

$$\{\alpha_l^{(k)} : \alpha_l \in S\}.$$

The $k-1$ high-order coefficients of $u_v^{(k)}(z)$ will be the same as those of $u_v^{(k-1)}(z)$, while $c_{v,k}^{(k)} = c_{v,k}^{(k-1)} + \Delta_v^{(k)}$. In applications, the numbers $\Delta_v^{(k)}$ will be chosen on the basis of global considerations.

The construction has five phases. The first two phases involve "building up strength" regarding the amount of movement which a patching correction $c_{v,k}^{(k)} = c_{v,k}^{(k-1)} + \Delta_v^{(k)}$ causes in the roots of $u_v^{(k-1)}(z)$. These initial steps will cause large movement in some of the roots, destroying the property that the roots form a regular sequence of length n. The last three phases compensate for this.

The basic idea in the first two phases is as follows. To pass from $u_v^{(k-1)}(z)$ to $u_v^{(k)}(z)$, we take a subset $S = \{\alpha_{i_0}^{(k-1)}, \ldots, \alpha_{i_{k-1}}^{(k-1)}\}$ of the roots $u_v^{(k-1)}(z)$ which forms a regular sequence of length k. Put

$$f_k(z) = \prod_{\alpha_l^{(k-1)}\in S}(z-\alpha_l^{(k-1)}), \tag{8.24}$$

$$w_v^{(k)}(z) = \prod_{\alpha_l^{(k-1)}\notin S}(z-\alpha_l^{(k-1)}) = u_v^{(k-1)}(z)/f_k(z), \tag{8.25}$$

and set

$$u_v^{(k)}(z) = u_v^{(k-1)}(z) + \Delta_v^{(k)} w_v^{(k)}(z) = (f_k(z) + \Delta_v^{(k)}) w_v^{(k)}(z). \tag{8.26}$$

We then apply Lemma 8.8 to $f_k(z)$ and $\Delta_v^{(k)}$, checking that the condition on $\Delta_v^{(k)}$ assures that the roots of $f_k^*(z) = f_k(z) + \Delta_v^{(k)}$ belong to E. Since $w_v^{(k)}(z)$ is monic of degree $n-k$, the step (8.26) changes $c_{v,k}^{(k-1)}$ to $c_{v,k}^{(k)} = c_{v,k}^{(k-1)} + \Delta_v^{(k)}$ and accomplishes the desired patching correction in $u_v^{(k-1)}(z)$.

Write $\Omega = \{\alpha_l\}_{0 \le l < n}$ for the initial regular sequence of length n, and let T be the least positive integer such that

$$T \cdot (-\log_v(h_v/r_v)) \ge A_2 + A_0 + \log_v(n) + 2. \tag{8.27}$$

Just as in (8.21), for all $k \ge T$ we have

$$k \cdot (-\log_v(h_v/r_v)) \ge A_2 + A_0 + \lceil \log_v(n) \rceil. \tag{8.28}$$

There is a constant A_3 such that for all sufficiently large n,

$$T \le A_3 \log_v(n).$$

The construction will require us to choose T pairwise disjoint regular subsequences from Ω, one of each length $1, 2, \ldots, T$. By Lemma 8.6(A) this can be done if n is large enough, which we henceforth assume. Let $S_1, \ldots, S_T$ be these subsequences. Only roots $\alpha_l \in S_k$ are moved in the kth step of the construction, for $k = 1, \ldots, T$.

Phase 1. *Patching the high-order coefficients*, $k = 1, \ldots, L$. Inductively suppose $u_v^{(k-1)}(z)$ has been determined, and that its roots, apart from the α_l in $S_1, \ldots, S_{k-1}$, are the same as those of $u_v^{(0)}(z)$. Take $S = S_k$, and define $f_k(z)$, $w_v^{(k)}(z)$, and $u_v^{(k)}(z)$ as in (8.24)–(8.26). Since $|\Delta_v^{(k)}|_v \le B_v$ it follows from (8.23) that

$$\mathrm{ord}_v(\Delta_v^{(k)}) \ge kV(E_v) + A_2 + A_0 + \lceil \log_v(k) \rceil$$

and so by Lemma 8.8,

$$f_k^*(z) = f_k(z) + \Delta_v^{(k)} = \prod_{\alpha_l \in S_k} (z - \alpha_l^*)$$

is a polynomial whose roots belong to E_v; moreover Lemma 8.8 gives a unique correspondence between the roots α_l and the α_l^* such that

$$\mathrm{ord}_v(\alpha_l^* - \alpha_l) \ge A_0 + \lceil \log_v(k) \rceil.$$

In passing from $u_v^{(k-1)}(z)$ to $u_v^{(k)}(z)$, the roots $\alpha_l \in S_k$ are replaced by the α_l^*, and the remaining roots are unchanged; $S_k^* = \{\alpha_l^*\}_{\alpha_l \in S_k}$ is a regular sequence of length k in E_v.

Phase 2. *Patching the coefficients for* $k = L+1, \ldots, T$. In this phase, the construction is exactly the same as in Phase 1, except that now $|\Delta_v^{(k)}|_v$

$\le h_v^k$. By (8.22),

$$\mathrm{ord}_v(\Delta_v^{(k)}) \ge kV(E_v) + A_0 + A_2 + \lceil \log_v(k) \rceil$$

and so Lemma 8.8 applies as before. In passing from $u_v^{(k-1)}(z)$ to $u_v^{(k)}(z)$, the roots in S_k are replaced by the roots in $S_k^* = \{\alpha_l^*\}_{\alpha_l \in S_k}$ forming a new regular sequence of length k, and the remaining roots are unchanged.

PHASE 3. *Moving the roots apart.* Put $\Omega_0 = \Omega \setminus (S_1 \cup \ldots \cup S_T)$. At this point, we have a polynomial $u_v^{(T)}(z)$ whose roots compose the set $\Omega_0 \cup S_1^* \cup \ldots \cup S_T^*$. Although the roots within Ω_0 and within each S_i^* are well-separated from each other in the sense that each pair $\alpha_k^{(T)} \ne \alpha_l^{(T)}$ of such roots satisfies $\mathrm{ord}_v(\alpha_k^{(T)} - \alpha_l^{(T)}) < A_0 + \log_v(n)$, some roots in S_i^* may be very near to roots in Ω_0 or roots in other S_j^*. Therefore, we pause to move them apart. Put

$$\begin{aligned} B = {} & \lceil (T-1)(V(E_v) + \log_v(R)) \rceil \\ & + \lceil \max(0, V(E_v)) + A_2 + A_0 + \lceil \log_v(n) \rceil \rceil. \end{aligned} \tag{8.29}$$

Since $B \ge A_0 + \lceil \log_v(n) \rceil$, each coset of the form $d + \pi_v^B O_v$ in E_v contains at most $T+1$ elements from $\Omega_0 \cup S_1^* \cup \ldots \cup S_T^*$. Let $D \ge 0$ be the smallest integer for which

$$q^D \ge T + 1.$$

Then it is possible to move the roots α_l^* in $S_1^* \cup \ldots \cup S_T^*$ to numbers $\alpha_l^\# = \alpha_l^* + \varepsilon_l$, by adding on quantities $\varepsilon_l \in K_v$ with $\mathrm{ord}_v(\varepsilon_l) \ge B$, in such a way that each $\alpha_l^\#$ satisfies

$$\begin{cases} \mathrm{ord}_v(\alpha_l^\# - \alpha_j^\#) \le B + D & \text{for all } \alpha_j^\# \ne \alpha_l^\#, \\ \mathrm{ord}_v(\alpha_l^\# - \alpha_j) \le B + D & \text{for all } \alpha_j \in \Omega_0. \end{cases}$$

Let $S_k^\#$ be the set of roots $\alpha_l^\#$ obtained from S_k^*; each $S_k^\#$ is a regular sequence of length k in E_v. Since $T \le A_3 \log_v(n)$, there is a constant A_4 such that for all sufficiently large n,

$$B + D < A_4 \log_v(n).$$

Put

$$\widetilde{u}_v^{(T)}(z) = \prod_{k=1}^{T} \Big(\prod_{\alpha_l^\# \in S_k^\#} (z - \alpha_l^\#) \Big) \cdot \prod_{\alpha_l \in \Omega_0} (z - \alpha_l).$$

The roots of $\widetilde{u}_v^{(T)}(z)$ are "moderately well-separated" in the sense that (for sufficiently large n) any two of its roots $\widetilde{\alpha}_l \ne \widetilde{\alpha}_j$ satisfy $\mathrm{ord}_v(\widetilde{\alpha}_l - \widetilde{\alpha}_j) \le A_4 \log_v(n)$. However, in general it will not have the same T high-order coefficients as $u_v^{(T)}(z)$. Write

$$\widetilde{u}_v^{(T)}(z) = z^n + \sum_{l=1}^{n} \widetilde{c}_{v,l}^{(T)} z^{n-l}$$

and for each $l \le T$, put

$$\delta_l = c_{v,l}^{(T)} - \widetilde{c}_{v,l}^{(T)}.$$

Considering the expansions of the $c_{v,l}^{(T)}$ and $\widetilde{c}_{v,l}^{(T)}$ as symmetric functions in the roots of $u_v^{(T)}(z)$ and $\widetilde{u}_v^{(T)}(z)$, we see that

$$\operatorname{ord}_v(\delta_l) \ge B - (l-1)\log_v(R). \tag{8.30}$$

To change the T high-order coefficients of $\widetilde{u}_v^{(T)}(z)$ back to those of $u_v^{(T)}(z)$, we move a second set of roots so as to compensate for the first.

For this purpose, choose a regular subsequence Q_T of length T from Ω_0, which has the property that for each $\alpha_i \in Q_T$, and each $\alpha_j^{\#} \in S_1^{\#} \cup \ldots \cup S_T^{\#}$,

$$\operatorname{ord}_v(\alpha_i - \alpha_j^{\#}) < A_0 + \log_v(n). \tag{8.31}$$

That is, Q_T is a regular sequence in E_v of length T, contained in the original regular sequence Ω of length n, whose elements are "well-separated" from elements in $S_1, \ldots, S_T$ and $S_1^{\#}, \ldots, S_T^{\#}$. If n is sufficiently large, the existence of such a subsequence follows from Lemma 8.6(B).

We now need a lemma.

Lemma 8.9. *Let $h(z) \in K_v[z]$ be a monic polynomial of degree N whose roots belong to $B(0,R)$. Then for any $m \ge 0$, there is a monic polynomial $g_m(z) \in K_v[z]$ of degree m, with $\|g_m(z)\|_{B(0,R)} \le R^m$, such that $g_m(z)h(z)$ has the form*

$$g_m(z)h(z) = z^{N+m} + \sum_{k=1}^{N} d_k z^{N-k}.$$

Proof. Write $h(z) = z^N + \sum_{i=1}^{N} h_i z^{N-i}$; by the hypothesis on the roots of $h(z)$, we have $|h_i|_v \le R^i$ for all i. Similarly, expand $g_m(z) = z^m + \sum_{j=1}^{m} \gamma_j z^{N-j}$ with undetermined coefficients. For $g_m(z)h(z)$ to have the desired form, we must have

$$\begin{cases} \gamma_1 = -h_1, \\ \gamma_2 = -h_1\gamma_1 - h_2, \\ \vdots \\ \gamma_m = -h_1\gamma_{m-1} - h_2\gamma_{m-2} - \ldots - h_m, \end{cases}$$

where we take $h_k = 0$ if $k > N$. These equations recursively determine $\gamma_1, \ldots, \gamma_m$; moreover one sees inductively that $|\gamma_j|_v \le R^j$ for each j. This in turn yields $\|g_m(z)\|_{B(0,R)} \le R^m$. ∎

To apply this, take $N = n - T$ and put

$$h(z) = \widetilde{u}_v^{(T)}(z) \,/ \prod_{\alpha_l \in Q_T} (z - \alpha_l), \qquad \widetilde{f}_T(z) = \prod_{\alpha_l \in Q_T} (z - \alpha_l).$$

With the $g_j(z)$ from the lemma, put

$$\Delta(z) = \sum_{l=1}^{T} \delta_l \, g_{T-l}(z).$$

By (8.29), (8.30) and the bounds $\|g_{T-l}(z)\|_{B(0,R)} \le R^{T-l}$,

$$\begin{aligned} \operatorname{ord}_v(\Delta(z)) &\ge B - (T-1)\log_v(R) \\ &\ge TV(E_v) + A_2 + A_0 + \lceil \log_v(n) \rceil \end{aligned} \tag{8.32}$$

for all $z \in B(0,R)$.

Now define

$$\overline{u}_v^{(T)}(z) = \widetilde{u}_v^{(T)}(z) + \Delta(z)h(z) = (\widetilde{f}_T(z) + \Delta(z))h(z).$$

By Lemma 8.9 the high-order coefficients of $\overline{u}_v^{(T)}(z)$ are $c_{v,l}^{(T)}$ for $l = 1, \ldots, T$. On the other hand, $\widetilde{f}_T(z)$ and $\Delta(z)$ satisfy the conditions of Lemma 8.8, so by (8.32),

$$\widetilde{f}_T^*(z) = \widetilde{f}_T(z) + \Delta(z) = \prod_{\alpha_l \in Q_T} (z - \alpha_l^*)$$

is a polynomial whose roots belong to E_v and can uniquely be put in correspondence with the $\alpha_l \in Q_T$ in such a way that

$$\operatorname{ord}_v(\alpha_l^* - \alpha_l) \ge A_0 + \lceil \log_v(n) \rceil$$

for all i. By (8.31) these α_l^* have not moved nearer to any of the $\alpha_j^{\#}$. Put

$$S_0 = (\Omega_0 \setminus Q_T) \cup \{\alpha_l^* : \alpha_l \in Q_T\}.$$

Then S_0 is a subset of a regular sequence of length n in E_v (indeed, $S_0 \cup S_1 \cup \ldots \cup S_T$ is such a sequence), and the roots of $\overline{u}_v^{(T)}(z)$ form the set $S_0 \cup S_1^{\#} \cup \ldots \cup S_T^{\#}$, where each $S_k^{\#}$ is a regular sequence of length k in E_v. Furthermore each pair of distinct roots $\overline{\alpha}_i$, $\overline{\alpha}_j$ of $\overline{u}_v^{(T)}(z)$ satisfies $\operatorname{ord}_v(\overline{\alpha}_i - \overline{\alpha}_j) < A_4 \log_v(n)$.

Replace $u_v^{(T)}(z)$ by $\overline{u}_v^{(T)}(z)$.

Phase 4. *Carry on.* By (8.28), at the end of Phase 2 the construction reached a point that further patching moves roots only by quantities ε with $\operatorname{ord}_v(\varepsilon) \ge A_0 + \lceil \log_v(n) \rceil$, preserving their position in a regular sequence of length n. In Phase 3, we arranged that $\operatorname{ord}_v(\alpha_i^{(T)} - \alpha_j^{(T)}) < A_4 \log_v(n)$ for all $i \ne j$. The purpose of Phase 4 is to carry on the construction, using the basic patching lemma, until further patching moves roots by quantities ε with $\mathrm{ord}_v(\varepsilon) \geq \lceil A_4 \log_v(n) \rceil$, meaning that the "delicate" roots in $S_1^\# \cup \ldots \cup S_T^\#$ can again be included in the patching process.

To motivate the duration of Phase 4, we need the following generalization of the basic patching lemma, which will be used in Phase 5.

LEMMA 8.10. *Let $\Theta = \{\theta_i\}_{0 \leq i < n} \subset E_v$ be a sequence obtained by perturbing a regular sequence $\Omega = \{\alpha_i\}_{0 \leq i < n}$ of length n; assume Θ can be partitioned into disjoint subsets $\Theta_0 \cup \Theta_1 \cup \ldots \cup \Theta_Z$ such that for each $l = 1, \ldots, Z$, Θ_l is a regular sequence of length n_l in E_v. Suppose n is large enough that*

$$\log_v(n) \geq A_0 + A_1 + \max_{1 \leq l \leq Z}(\log_v(n_l)).$$

Furthermore, suppose that

$$\mathrm{ord}_v(\theta_i - \alpha_i) \geq \begin{cases} A_0 + \log_v(n) & \text{for each } \theta_i \in \Theta_0, \\ A_0 + \log_v(n_l) & \text{for each } \theta_i \in \Theta_l \text{ with } l \geq 1. \end{cases} \tag{8.33}$$

Finally, suppose that for some $M \geq A_0 + \log_v(n)$, all $\theta_i \neq \theta_j$ in Θ satisfy $\mathrm{ord}_v(\theta_i - \theta_j) < M$. Given $k \leq n$, put $f_k(z) = \prod_{i=0}^{k-1}(z - \theta_i)$. Then for any $\Delta \in K_v$ with

$$\mathrm{ord}_v(\Delta) \geq kV(E_v) + A_2 + Z \cdot (M + \log_v(R)) + M,$$

the roots of

$$f_k^*(z) := f_k(z) + \Delta = \prod_{i=0}^{k-1}(z - \theta_i^*)$$

belong to E_v and can be uniquely labeled in such a way that

$$\mathrm{ord}_v(\theta_i^* - \theta_i) \geq M$$

for each $i = 0, \ldots, m-1$.

Proof. Fix a root θ_J of $f_k(z)$, and expand

$$f_k(z) = b_1(z - \theta_J) + b_2(z - \theta_J)^2 + \ldots + b_{k-1}(z - \theta_J)^{k-1} + (z - \theta_J)^k.$$

Here $b_1 = \pm \prod_{i \neq J}(\theta_J - \theta_i)$, and for each $j \geq 2$,

$$b_j = \pm b_1 \sum_{\substack{i_1 < \ldots < i_{j-1} < m \\ \text{each } i_h \neq J}} \frac{1}{(\theta_J - \theta_{i_1}) \ldots (\theta_J - \theta_{i_{j-1}})}.$$

By our hypothesis, for each $j \geq 2$,

$$\mathrm{ord}_v(b_j) > \mathrm{ord}_v(b_1) - (j-1)M. \tag{8.34}$$

We now seek an upper bound for $\mathrm{ord}_v(b_1)$. Let I be an index, $0 \leq I < n$, for which $\mathrm{ord}_v(\theta_J - \alpha_I)$ is maximal. By Lemma 8.5 and the definition of a regular sequence, for each $i \neq I$,

$$\mathrm{ord}_v(\theta_J - \alpha_i) \leq \mathrm{ord}_v(\alpha_I - \alpha_i) < A_0 + \log_v(n).$$

Hence by (8.33), for each $\theta_i \in \Theta_0$, with $i \neq I, J$,

$$\mathrm{ord}_v(\theta_J - \theta_i) \leq \mathrm{ord}_v(\alpha_I - \alpha_i). \tag{8.35}$$

Fix $l \geq 1$, and let I_l be an index such that $\mathrm{ord}_v(\theta_J - \theta_{I_l})$ is maximal among the $\mathrm{ord}_v(\theta_J - \theta_i)$ with $\theta_i \in \Theta_l$. Since Θ_l is a regular sequence of length n_l, for each $\theta_i \in \Theta_l$, $i \neq I_l$,

$$\mathrm{ord}_v(\theta_{I_l} - \theta_i) < A_0 + \log_v(n_l).$$

Furthermore Lemma 8.5 gives

$$\mathrm{ord}_v(\theta_J - \theta_i) \leq \mathrm{ord}_v(\theta_{I_l} - \theta_i).$$

By Lemma 8.4 and our hypothesis on n,

$$\mathrm{ord}_v(\theta_J - \alpha_I) \geq \log_v(n) - A_1 \geq A_0 + \log_v(n_l)$$

so using (8.33) we find that for each $\theta_i \in \Theta_l$ with $i \neq I_l$,

$$\mathrm{ord}_v(\theta_J - \theta_i) = \mathrm{ord}_v(\alpha_I - \alpha_i). \tag{8.36}$$

To estimate $\mathrm{ord}_v(b_1)$, we now consider two possibilities. If $\theta_J \in \Theta_0$, then $I = J$, and we have

$$\begin{aligned}
\mathrm{ord}_v(b_1) &= \sum_{\substack{i=0 \\ i \neq J}}^{k-1} \mathrm{ord}_v(\theta_J - \theta_i) \\
&= \sum_{\substack{i=0 \\ i \neq J, I_1, \ldots, I_Z}}^{k-1} \mathrm{ord}_v(\theta_J - \theta_i) + \sum_{i \in \{I_1, \ldots, I_Z\}} \mathrm{ord}_v(\theta_J - \theta_i) \\
&= \sum_{\substack{i=0 \\ i \neq I}}^{k-1} \mathrm{ord}_v(\alpha_I - \alpha_i) - \sum_{i \in \{I_1, \ldots, I_Z\}} \mathrm{ord}_v(\alpha_I - \alpha_i) \\
&\quad + \sum_{i \in \{I_1, \ldots, I_Z\}} \mathrm{ord}_v(\theta_J - \theta_i) \\
&\leq kV(E_v) + A_2 + Z \cdot (M + \log_v(R)).
\end{aligned} \tag{8.37}$$

On the other hand, if $\theta_J \in \Theta_l$ for some $l \geq 1$, then $I_l = J$; I may or may not equal one of the I_l. In any case the sets $\{I, I_1, \ldots, I_Z\} \setminus \{I\}$ and $\{I, I_1, \ldots, I_Z\} \setminus \{J\}$ have the same number of elements, either $Z - 1$ or Z according as $I = I_l$ for some $l \geq 1$, or not. We have

$$\begin{aligned}
(8.38)\qquad \mathrm{ord}_v(b_1) &= \sum_{\substack{i=0\\ i\neq J}}^{k-1} \mathrm{ord}_v(\theta_J-\theta_i)\\
&= \sum_{\substack{i=0\\ i\neq I,I_1,\ldots,I_Z}}^{k-1} \mathrm{ord}_v(\theta_J-\theta_i) + \sum_{\substack{i\in\{I,I_1,\ldots,I_Z\}\\ i\neq J}} \mathrm{ord}_v(\theta_J-\theta_i)\\
&= \sum_{\substack{i=0\\ i\neq I}}^{k-1} \mathrm{ord}_v(\alpha_I-\alpha_i) - \sum_{\substack{i\in\{I,I_1,\ldots,I_Z\}\\ i\neq I}} \mathrm{ord}_v(\alpha_I-\alpha_i)\\
&\quad + \sum_{\substack{i\in\{I,I_1,\ldots,I_Z\}\\ i\neq J}} \mathrm{ord}_v(\theta_J-\theta_i)\\
&\leq kV(E_v)+A_2+Z\cdot(M+\log_v(R))
\end{aligned}$$

once again.

Since $f_k^*(z) = f_k(z)+\Delta$, and $\mathrm{ord}_v(\Delta) \geq kV(E_v)+A_2+Z\cdot(M+\log_v(R))+M$, by (8.34), (8.37) and (8.38) the Newton polygon of $f_k^*(z)$, expanded about θ_J, has a break at the point $(1,\mathrm{ord}_v(b_1))$ and there is a unique root θ_J^* of $f_k^*(z)$ for which

$$\mathrm{ord}_v(\theta_J^*-\theta_J) \geq M.$$

This holds for every J, so the lemma is established. ∎

We will apply Lemma 8.10 with $Z = T < A_3\log_v(n)$, the sets Θ_l for $l \geq 1$ being the $S_l^{\#}$ with length $n_l = l \leq T$, and with $M = A_4\log_v(n)$. Let W be the least positive integer such that

$$(8.39)\qquad W\cdot(-\log_v(h_v/r_v)) \geq A_2+T\cdot(M+\log_v(R))+M+2.$$

As in (8.21), a similar inequality holds for each $k \geq W$. There is a constant A_5 such that $W \leq A_5(\log_v(n))^2$ for all sufficiently large n. The purpose of Phase 4 is to carry out the patching process for $k = T+1,\ldots,W$.

For this, choose a regular subsequence Q_W of length W in E_v contained in S_0, which has the property that for each $\alpha \in Q_W$, and each $\theta \in S_1\cup\ldots\cup S_T\cup S_1^{\#}\cup\ldots\cup S_T^{\#}$,

$$\mathrm{ord}_v(\alpha-\theta) < A_0+\log_v(n).$$

For large enough n, the existence of such a subsequence follows from Lemma 8.6(B), applied to the regular sequence $S_0\cup S_1\cup\ldots\cup S_T$. Write

$$Q_W = \{\alpha_{i_0}^{(T)},\ldots,\alpha_{i_{W-1}}^{(T)}\},$$

listing the elements in their order as a regular sequence.

To patch $u_v^{(k-1)}(z)$ to $u_v^{(k)}(z)$, put

$$f_k(z) = \prod_{l=0}^{k-1}(z - \alpha_{i_l}^{(k-1)}), \qquad w_v^{(k)}(z) = u_v^{(k-1)}(z)/f_k(z)$$

(using the same indices as in Q_W) and set

$$u_v^{(k)}(z) = u_v^{(k-1)}(z) + \Delta_v^{(k)} w_v^{(k)}(z) = (f_k(z) + \Delta_v^{(k)}) w_v^{(k)}(z).$$

Since $\mathrm{ord}_v(\Delta_v^{(k)}) \geq k \log_v(h_v) \geq kV(E_v) + A_2 + A_0 + \lceil \log_v(n) \rceil$, Lemma 8.8 shows that the roots $\alpha_{i_l}^{(k)} := (\alpha_{i_l}^{(k-1)})^*$ of $f_k^*(z) = f_k(z) + \Delta_v^{(k)}$ satisfy

$$\mathrm{ord}_v(\alpha_{i_l}^{(k)} - \alpha_{i_l}^{(k-1)}) \geq A_0 + \lceil \log_v(n) \rceil,$$

so they retain their positions in a regular sequence of length n and do not move closer to the roots in $S_1^\#, \ldots, S_T^\#$; moreover $\{\alpha_{i_0}^{(l)}, \ldots, \alpha_{i_{W-1}}^{(l)}\}$ continues to form a regular sequence of length W in E_v. The remaining roots of $u_v^{(l)}(z)$ are the same as those of $u_v^{(l-1)}(z)$.

Thus, we can patch the coefficients $c_{v,k}^{(k)}$, $k = T+1, \ldots, W$.

Phase 5. *Completing the patching process* $(W + 1 \leq k \leq n)$. To patch the coefficients $c_{v,k}^{(k)}$ for $k = W+1, \ldots, n$, we apply Lemma 8.10 as indicated above, with $Z = T$ and $M = A_4 \log_v(n)$, taking Θ to be the set of roots of $u_v^{(k-1)}(z)$, and taking the Θ_l for $l = 1, \ldots, T$ to be the sets of roots obtained from $S_1^\#, \ldots, S_T^\#$ by the patching process (at the step $k = W + 1$ they are exactly the $S_l^\#$). Put

$$f_k(z) = \prod_{l=0}^{k-1}(z - \alpha_l^{(k-1)}), \qquad w_v^{(k)}(z) = u_v^{(k-1)}(z)/f_k(z),$$

and

$$u_v^{(k)}(z) = u_v^{(k-1)}(z) + \Delta_v^{(k)} w_v^{(k)}(z) = (f_k(z) + \Delta_v^{(k)}) w_v^{(k)}(z).$$

By the choice of W in (8.39), we have

$$\mathrm{ord}_v(\Delta_v^{(k)}) \geq k \log_v(h_v) \geq kV(E_v) + A_2 + T \cdot (M + \log_v(R)) + M.$$

By Lemma 8.10, the roots $\alpha_l^{(k)} := (\alpha_l^{(k-1)})^*$ of $f_k^*(z) = f_k(z) + \Delta_v^{(k)}$ satisfy

$$\mathrm{ord}_v(\alpha_l^{(k)} - \alpha_l^{(k-1)}) \geq A_4 \log_v(n) \qquad \text{for } l = 0, \ldots, k-1$$

while the remaining roots of $u_v^{(k)}(z)$ are the same as those of $u_v^{(k-1)}(z)$.

Thus the hypotheses of Lemma 8.10 continue hold, and the process carries through to the end. ∎

9. Local patching for irreducibility and ramification. In this section we will present two simple propositions which show that at a nonarchimedean place where E_v is the trivial set $\widehat{O}_v$, patching can be carried out in such a way that the final polynomial is irreducible over K_v, and the extension L_w/K_v obtained by adjoining one of its roots is either unramified, or totally ramified.

Let O_v be the ring of integers of K_v, let $\mathfrak{m}_v$ be the maximal ideal of O_v, let $\pi_v \in \mathfrak{m}_v$ be a uniformizing parameter, and let $k_v = O_v/\mathfrak{m}_v$ be the residue field.

PROPOSITION 9.1. *Suppose K_v is nonarchimedean, and $E_v = \widehat{O}_v$. Let $0 < h_v < r_v < \gamma(E_v) = 1$ be given, let L be any positive integer, and suppose $0 < B_v < 1$. For each n there is a monic polynomial $u_v^{(0)}(z) \in O_v[z]$ of degree n, whose roots belong to E_v and which can be patched with arbitrary $\Delta_v^{(k)} \in K_v$ satisfying*

$$|\Delta_v^{(k)}|_v \leq \begin{cases} B_v & \text{for } k \leq L, \\ h_v^k & \text{for } k > L, \end{cases}$$

in such a way that its roots remain in E_v, and $u_v^{(n)}(z) \pmod{\mathfrak{m}_v}$ is irreducible over $k_v[z]$. In particular, $u_v^{(n)}(z)$ is irreducible over $K_v[z]$, and for any root α of $u_v^{(n)}(z)$, if $L_w = K_v(\alpha)$ then L_w/K_v is unramified of degree n.

Proof. Let $\overline{u}(z) \in k_v[z]$ be any monic irreducible polynomial of degree n, and let $u_v^{(0)}(z) \in O_v[z]$ be a monic polynomial of degree n such that

$$u_v^{(0)}(z) \pmod{\mathfrak{m}_v} = \overline{u}(z).$$

Patch $u_v^{(0)}(z)$ successively to $u_v^{(1)}(z), u_v^{(2)}(z), \ldots, u_v^{(n)}(z)$ by setting $u_v^{(k)}(z) = u_v^{(k-1)}(z) + \Delta_v^{(k)} z^{n-k}$. As each $|\Delta_v^{(k)}|_v < 1$ for each k, at each step we have

$$u_v^{(k)}(z) \pmod{\mathfrak{m}_v} = \overline{u}(z). \quad \blacksquare$$

PROPOSITION 9.2. *Suppose K_v is nonarchimedean, and $E_v = \widehat{O}_v$. Let $0 < h_v < r_v < \gamma(E_v) = 1$ be given, let L be any positive integer, and suppose $0 < B_v < 1$. Let $u_v^{(0)}(z) \in O_v[z]$ be an Eisenstein polynomial of degree n. If n is sufficiently large, then $u_v^{(0)}(z)$ can be patched with arbitrary $\Delta_v^{(k)} \in K_v$ satisfying*

$$|\Delta_v^{(k)}|_v \leq \begin{cases} B_v & \text{for } k \leq L, \\ h_v^k & \text{for } k > L, \end{cases}$$

in such a way $u_v^{(n)}(z)$ remains an Eisenstein polynomial. In particular, $u_v^{(n)}(z)$ is irreducible over $K_v[z]$ and has all its roots in E_v, and for any root α of $u_v^{(n)}(z)$, if $L_w = K_v(\alpha)$ then L_w/K_v is totally ramified of degree n.

Proof. Let $u_v^{(0)}(z) \in O_v[z]$ be an Eisenstein polynomial of degree n. Patch $u_v^{(0)}(z)$ successively to $u_v^{(1)}(z), u_v^{(2)}(z), \ldots, u_v^{(n)}(z)$ by setting

$$u_v^{(k)}(z) = u_v^{(k-1)}(z) + \Delta_v^{(k)} z^{n-k}.$$

By hypothesis $|\Delta_v^{(k)}|_v < 1$ at each step; if $n > L$ is large enough that $h_v^n < |\pi_v|_v$, then $|\Delta_v^{(n)}|_v \le |\pi_v^2|_v$ and $u_v^{(n)}(z)$ will also be an Eisenstein polynomial. ∎

10. The global patching argument. We can now complete the proof of Theorem 2.3. For this, we will need the following version of the Strong Approximation Theorem, which can easily be derived from the Lemma on p. 66 of [4]. Let $\mathcal{S}$ be a finite set of places of K including all the archimedean places. Recall that the set of $\mathcal{S}$-*integers* of K is

$$K^{\mathcal{S}} := \{\kappa \in K : |\kappa|_v \le 1 \text{ for all } v \notin \mathcal{S}\}.$$

LEMMA 10.1. *Let K be a number field, and let $\mathcal{S}$ be a finite set of places of K containing all the archimedean places. There is a constant $C(K, \mathcal{S})$, depending only on K and $\mathcal{S}$, such that if numbers $C_v > 0$ are specified for each $v \in \mathcal{S}$, subject to the condition*

$$\prod_{v \in \mathcal{S}} C_v^{D_v} > C(K, \mathcal{S}),$$

then for any elements $c_v \in K_v$ $(v \in \mathcal{S})$ there is an $\mathcal{S}$-integer c with $|c - c_v|_v \le C_v$ for all $v \in \mathcal{S}$.

REMARK. If each C_v were required to belong to the value group of $K_v^\times$, then the constant $C(K, \mathcal{S})$ would depend only on K.

Proof of Theorem 2.3. Let $\widehat{S}$ be the finite set of places of K consisting of all archimedean v and all nonarchimedean v where $E_v \ne \widehat{O}_v$. By the hypotheses of Theorem 2.3 and the preliminary reductions (1)–(5), we can assume that $\gamma(\mathbb{E}) > 1$, that each E_v and U_v is stable under $\mathrm{Gal_c}(\mathbb{C}_v/K_v)$, and

(1) For each archimedean v, E_v is compact;

(2) For each nonarchimedean $v \in S$, $E_v = \bigcup_{i=1}^M a_i + \pi_v^{n_i} O_v$;

(3) For each nonarchimedean $v \in \widehat{S} \setminus S$, E_v is a PL-domain;

(4) For each $v \notin \widehat{S}$, $E_v = \widehat{O}_v$.

We are also given a finite set of places $S' = S_1' \cup S_2'$ of K, disjoint from $\widehat{S}$, such that each $v \in S_1'$ is to be inert, and each $v \in S_2'$ is to be totally ramified, in the extensions $K(\alpha)/K$. We can assume that S' is nonempty.

Using $\prod_{v\in\widehat{S}\cup S'}\gamma(E_v)^{D_v}=\gamma(\mathbb{E})>1$, fix numbers $0<h_v<r_v<\gamma(E_v)$ for each $v\in\widehat{S}\cup S'$, with

$$H:=\prod_{v\in\widehat{S}\cup S'}h_v^{D_v}>1.$$

For each $v\in\widehat{S}$, let M_v be the constant from the corresponding local patching proposition 5.1, 6.1, 7.1, or 8.1.

Fix an integer $L>0$ subject to the appropriate local condition (5.1), (6.1), (7.1), or (8.1) for each $v\in\widehat{S}$: namely

$$\begin{cases}\left(\dfrac{h_v}{r_v}\right)^L\dfrac{M_v}{1-h_v/r_v}<\dfrac{1}{8} & \text{if } K_v\cong\mathbb{C},\ \text{or if } K_v\cong\mathbb{R} \text{ but } E_v\subsetneq\mathbb{R},\\ \left(\dfrac{h_v}{r_v}\right)^L\dfrac{M_v}{1-h_v/r_v}<\dfrac{1}{16} & \text{if } K_v\cong\mathbb{R} \text{ and } E_v\subset\mathbb{R},\\ \left(\dfrac{h_v}{r_v}\right)^L M_v\le\dfrac{1}{2} & \text{if } K_v \text{ is nonarchimedean and } E_v \text{ is a PL-domain},\\ L\cdot\left(\dfrac{h_v}{r_v}\right)^L M_v\le 1 & \text{if } K_v \text{ is nonarchimedean and } E_v\subset K_v;\end{cases}$$

and also subject to the global condition

$$H^L>C(K,\widehat{S}\cup S'), \tag{10.1}$$

where $C(K,\widehat{S}\cup S')$ is the constant from Lemma 10.1 above.

Next choose constants B_v, for $v\in\widehat{S}\cup S'$, as follows. For each nonarchimedean $v\in\widehat{S}$, let B_v be the constant specified in Proposition 7.1 or 8.1; note that B_v depends on L. For each $v\in S'$, take $B_v=1/2$. Then, recalling that in Propositions 5.1 and 6.1 the constants B_v can be specified arbitrarily, choose numbers $B_v>0$ for the archimedean $v\in\widehat{S}$ in such a way that

$$\prod_{v\in\widehat{S}\cup S'}B_v^{D_v}>C(K,\widehat{S}\cup S'). \tag{10.2}$$

Finally, let $n\ge L$ be any integer sufficiently large that for each $v\in\widehat{S}\cup S'$ the appropriate local patching proposition 5.1, 6.1, 7.1, 8.1, 9.1, or 9.2 applies, relative to the given E_v, U_v, L, and B_v.

For each $v\in\widehat{S}\cup S'$, let $u_v^{(0)}(z)\in K_v[z]$ be the monic polynomial of degree n given by the corresponding local patching proposition. We will patch these to a common global polynomial $u(z)\in K[z]$. Suppose inductively there are global $\widehat{S}\cup S'$-integers $c_l\in K$, for $l=1,\dots,k-1$, such that the coefficients of the polynomials

$$u_v^{(k-1)}(z)=z^n+\sum_{l=1}^{n}c_{v,l}^{(k-1)}z^{n-l}$$

satisfy

$$c_{v,l}^{(k-1)} = c_l$$

for all $v \in \widehat{S} \cup S'$ and all $l = 1, \ldots, k-1$. If $k \le L$, (10.2) together with Proposition 10.1 shows that there is a global $\widehat{S} \cup S'$-integer c_k satisfying

$$|c_k - c_{v,k}^{(k-1)}|_v \le B_v$$

for each v. If $k > L$, then by (10.1),

$$\prod_{v \in S} (h_v^k)^{D_v} = H^k > C(K, \widehat{S} \cup S')$$

and Proposition 10.1 shows that there is a global $\widehat{S} \cup S'$-integer c_k such that

$$|c_k - c_{v,k}^{(k-1)}|_v \le h_v^k$$

for each v. In either case, put $\Delta_v^{(k)} = c_k - c_{v,k}^{(k-1)}$ for each v, and patch $u_v^{(k-1)}(z)$ to $u_v^{(k)}(z)$ using the appropriate local patching construction.

At the end of the patching process, we have a global polynomial

$$u(z) = z^n + \sum_{k=1}^{n} c_k z^{n-k}$$

with $\widehat{S} \cup S'$-integer coefficients such that $u(z) = u_v^{(n)}(z)$ for each $v \in \widehat{S} \cup S'$. By the local patching constructions, the roots of $u(z)$ belong to U_v for each $v \in \widehat{S}$, and to K_v for each $v \in S$. For each $v \in S_1'$, $u(z) \pmod{\mathfrak{m}_v}$ is irreducible, and for each $v \in S_2'$, $u(z)$ is an Eisenstein polynomial. In particular, $u(z)$ is irreducible; and if $\alpha \in \widetilde{K}$ is any root of $u(z)$, then each $v \in S_1'$ is inert in $K(\alpha)/K$, and each $v \in S_2'$ is totally ramified. Finally, because the c_k are S-integers, the roots of $u(z)$ belong to $U_v = \widehat{O}_v$ for each $v \notin \widehat{S}$. This completes the proof. ■

11. Appendix I. Inner capacities of archimedean sets. In this appendix we will prove the following lemma, which is presumably well known but seems not to appear in the literature.

LEMMA 11.1. *Let $E \subset \mathbb{C}$ be a compact set which is the closure of its interior E^0, and whose boundary is a finite union of smooth arcs. Then*

$$\gamma(E^0) = \gamma(E).$$

Proof. Put $F_0 = \emptyset$. For each $n \ge 1$, let $F_n \subset E^0$ be a compact set containing F_{n-1} and all $z \in E$ with distance at least $1/n$ from the boundary ∂E, so $\bigcup_{n=1}^{\infty} F_n = E^0$. The potential function and Robin constant of F_n are

$$u_n(z) = u(z, F_n) = \int -\log(|z-w|)\, d\mu_n(w),$$

$$V(F_n) = \iint -\log(|z-w|)\, d\mu_n(z)\, d\mu_n(w).$$

Let $u(z)$ be the potential function, and $V(E)$ the Robin constant, of E. Put

$$\widehat{V}(E) = \lim_{n\to\infty} V(F_n).$$

The limit is decreasing, since the F_n are increasing. We will prove that $\widehat{V}(E) = V(E)$, so

$$\lim_{n\to\infty} \gamma(F_n) = \gamma(E).$$

Let μ_n be the equilibrium distribution of F_n. Our strategy is to show that the sequence $\{\mu_n\}$ converges weakly to the equilibrium distribution μ of E. In any case, after passing to a subsequence of the F_n, we can suppose that $\{\mu_n\}$ converges weakly to some measure $\widehat{\mu}$. Clearly $\widehat{\mu}$ is a probability measure supported on ∂E. We will now study the properties of the potential function

$$\widehat{u}(z) = \int -\log(|z-w|)\, d\widehat{\mu}(w)$$

associated to $\widehat{\mu}$.

Step 1. We claim that $\widehat{u}(z) = \widehat{V}(E)$ for all $z \in E^0$. To see this, fix $z_0 \in E^0$, and let $\delta = \inf_{x\in\partial E}(|z_0 - x|)$ be the distance from z to ∂E. Let N be large enough that $1/N < \delta$. Then for all $n \geq N$, z_0 is in the interior of F_n, and hence $u_n(z_0) = V(F_n)$. The function $\log(|z_0 - w|)$ is continuous and bounded on the set

$$\partial E(1/N) := \{z \in \mathbb{C} : |z - x| < 1/N \text{ for some } x \in \partial E\},$$

so by weak convergence,

$$\begin{aligned} \widehat{u}(z_0) &= \int -\log(|z_0-w|)\, d\widehat{\mu}(w) = \lim_{n\to\infty} \int -\log(|z_0-w|)\, d\mu_n(w) \\ &= \lim_{n\to\infty} u_n(z_0) = \lim_{n\to\infty} V(F_n) = \widehat{V}(E). \end{aligned}$$

Step 2. We claim that $\widehat{u}(z) \leq \widehat{V}(E)$ for all $z \in E$. Indeed, for any M and n,

$$\begin{aligned} u_n^{(M)}(z) &:= \int \min(M, -\log(|z-w|))\, d\mu_n(w) \\ &\leq \int -\log(|z-w|)\, d\mu_n(w) = u_n(z) \leq V(F_n), \end{aligned}$$

and so by weak convergence

$$\begin{aligned} \widehat{u}^{(M)}(z) &= \int \min(M, -\log(|z-w|))\, d\widehat{\mu}(w) \\ &= \lim_{n\to\infty} u_n^{(M)}(z) \leq \lim_{n\to\infty} V(F_n) = \widehat{V}(E). \end{aligned}$$

Hence, letting $M \to \infty$, we obtain

$$\widehat{u}(z) = \lim_{M\to\infty} \widehat{u}^{(M)}(z) \leq \widehat{V}(E).$$

STEP 3. Third, we claim that $\widehat{u}(z) = \widehat{V}(E)$ for all $z \in E$ except possibly a finite set of points. Here we use the hypotheses that ∂E is a finite union of smooth arcs, and that E is the closure of E^0. Take any point $z_0 \in \partial E$ which is not an intersection point of two arcs. Then there is a well-defined normal line $\mathcal{N}$ to ∂E at z_0. For small enough r, the ball $B(z_0, r)$ will not meet any of the arcs comprising ∂E except for the one containing z_0. Also for small enough r, if $w \in \partial E \cap B(z_0, r)$ with $w \neq z_0$, and $x \in \mathcal{N} \cap E^0 \cap B(z_0, r)$ then the angle between the segments $\overline{z_0 x}$ and $\overline{z_0 w}$ will be at least $\pi/4$. Fix an r satisfying these two conditions. We assert that there is a constant C such that for all $x \in \mathcal{N} \cap E^0 \cap B(z_0, r/2)$, and all $w \in \partial E$,

$$-\log(|x - w|) \leq -\log(|z_0 - w|) + C. \tag{11.1}$$

For $w \in \partial E \setminus B(z_0, r/2)$, and $x \in \mathcal{N} \cap E^0 \cap B(z_0, r)$, both $-\log(|x - w|)$ and $-\log(|z_0 - w|)$ are uniformly bounded, so the assertion is trivial. For any fixed $w \in \partial E \cap B(z_0, r)$ with $w \neq z_0$, if $x_0 \in \mathcal{N} \cap E \cap B(z_0, r)$ is the point nearest to z, then by our hypothesis we have $|x_0 - w| \geq |z_0 - w| \sin(\pi/4)$, and hence for any $x \in \mathcal{N} \cap E^0 \cap B(z_0, r)$,

$$-\log(|x - w|) \leq -\log(|x_0 - w|) \leq -\log(|z_0 - w|) + \log(\sqrt{2}).$$

Finally, if $w = z_0$ the assertion is again trivial.

By Step 2,

$$\int (-\log(|z_0 - w|) + C)\, d\widehat{\mu}(w) \leq \widehat{V}(E) + C < \infty.$$

Let $x_1, x_2, \ldots$ be a sequence of points in $\mathcal{N} \cap E^0 \cap B(z_0, r)$ with $\lim_{n\to\infty} x_n = z_0$. Then pointwise, for each $z \in \partial E$, $\lim_{n\to\infty} |z - x_n| = |z - z_0|$. By Lebesgue's dominated convergence theorem, (11.1) and Step 1,

$$\begin{aligned}\widehat{u}(z_0) &:= \int -\log(|z_0 - w|)\, d\widehat{\mu}(w) = \lim_{n\to\infty} \int -\log(|x_n - w|)\, d\widehat{\mu}(w) \\ &= \lim_{n\to\infty} \widehat{u}(x_n) = \widehat{V}(E).\end{aligned}$$

Thus, $\widehat{u}(z) = \widehat{V}(E)$ for all $z \in E$, except possibly on set of capacity 0 (the finite set of intersection points of the arcs in ∂E).

STEP 4. We can now show that $V(E) = \widehat{V}(E)$ and $\widehat{\mu} = \mu$, the equilibrium distribution of E. Then since $u_E(z) = V(E)$ for all $z \in E$ except possibly on a set of capacity 0, which has μ-measure 0, and $\widehat{u}(z) = \widehat{V}(E)$ for all $z \in E$ except possibly on a set of capacity 0, which has $\widehat{\mu}$-measure 0 (see [13, Theorem III.7, p. 56]), Fubini's theorem gives

$$\begin{aligned}V(E) &= \int u_E(z)\, d\mu(z) = \iint -\log(|z - w|)\, d\mu(z)\, d\widehat{\mu}(w) \\ &= \int \widehat{u}(w)\, d\widehat{\mu}(z) = \widehat{V}(E).\end{aligned}$$

Furthermore,

$$\iint -\log(|z-w|)\,d\widehat{\mu}(z)\,d\widehat{\mu}(w) = \int \widehat{u}(w)\,d\widehat{\mu}(z) = V(E).$$

This is the defining property of μ, and μ is unique. Hence $\widehat{\mu} = \mu$. ∎

12. Appendix II. Proof of a lemma of Cantor. In this section we will give a proof of the key fact needed for the local patching construction for nonarchimedean PL-domains, the fact that finite unions of balls are PL-domains. The proof follows an argument sketched by Cantor ([3, Lemma 3.2.3]): Cantor's proof, though basically correct, was incomplete.

We remark that the result in now known in a much more general setting, for algebraic curves (see [11, Theorem 4.2.12, p. 244]). It has also been extended for the affine line over a valuation field with a valuation group of arbitrary rank (see [7]). Related results have been proved by Fieseler and by Matignon in the context of rigid analysis. However, the argument given here leads to an explicit construction of the polynomial, which is not true in the general case.

Let v be a nonarchimedean place of K. Suppose $a \in \mathbb{C}_v$, and $r > 0$. Recall that we write

$$B(a,r) = \{z \in \mathbb{C}_v : |z-a|_v \le r\},$$
$$\partial B(a,r) = \{z \in \mathbb{C}_v : |z-a|_v = r\},$$
$$B(a,r)^- = \{z \in \mathbb{C}_v : |z-a|_v < r\}.$$

By a finite union of balls, we mean a set of the form

$$E_v = \bigcup_{i=1}^{M} B(a_i, r_i)$$

where each r_i belongs to the value group of $\mathbb{C}_v^\times$. By the ultrametric inequality, we can assume the $B(a_i, r_i)$ are pairwise disjoint. Fix such a set E_v.

Let ζ be a point in $\mathbb{P}^1(\mathbb{C}_v)\backslash E_v$. Our goal is to show that there is a rational function $f(z)$, whose only poles are at ζ, and whose zeros are supported on the a_i's, such that

$$E_v = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |f(z)|_v \le 1\}.$$

To motivate the construction, first suppose that $\zeta = \infty$, and that $f(z)$ could be taken to be polynomial of the form

$$f(z) = \pi_v^{nw_0} \prod_{i=1}^{M} (z-a_i)^{nw_i} \tag{12.1}$$

where n is the degree of $f(z)$, and the w_i are rational numbers with $\sum_{i=1}^{M} w_i = 1$ such that $w_i \ge 0$ for $i = 1, \ldots, M$, and π_v is a uniformizing parameter for K_v. Write

$$\Theta_{ij} = \begin{cases} \log_v(|a_i - a_j|_v) & \text{if } i \neq j, \\ \log_v(r_i) & \text{if } i = j, \end{cases}$$

noting that since the balls $B(a_i, r_i)$ are disjoint we have $\Theta_{ij} > \Theta_{ii}$ if $i \neq j$.

The numbers w_i are subject to several constraints. To see them, take $x \in \partial B(a_i, r_i)$. Then $\log_v(|x - a_j|_v) = \Theta_{ij}$ for $j = 1, \dots, M$, and by hypothesis $\log_v(|f(x)|_v) = 0$. This leads to the system of equations

$$0 = w_0 + \sum_{j=1}^{M} \Theta_{ij} w_j \quad \text{for } i = 1, \dots, M. \tag{12.2}$$

In addition we have the relation

$$1 = 0 \cdot w_0 + \sum_{j=1}^{M} 1 \cdot w_j. \tag{12.3}$$

If we write

$$\begin{cases} \Theta_{00} = 0, \\ \Theta_{0j} = 1 \ \text{ for } j = 1, \dots, M, \\ \Theta_{i0} = 1 \ \text{ for } i = 1, \dots, M, \end{cases}$$

and put $\Theta = (\Theta_{ij})_{0 \leq i,j \leq M}$, then the system of equations (12.2), (12.3) becomes

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \Theta \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_m \end{bmatrix}.$$

Next, suppose $\zeta \neq \infty$. In this case one could hope $f(z)$ would have the form

$$f(z) = \pi_v^{nw_0} \cdot \frac{\prod_{i=1}^{M} (z - a_i)^{nw_i}}{(z - \zeta)^n}. \tag{12.4}$$

Taking $x \in \partial B(a_i, r_i)$ and using $|f(x)|_v = 1$, we find

$$0 = nw_0 + \Big(\sum_{j=1}^{M} \log_v(|x - a_i|_v) n w_j \Big) - n \log_v(|x - \zeta|_v)$$

which, since $|x - \zeta|_v = |\zeta - a_i|_v$, can be written as

$$\log_v(|\zeta - a_i|_v) = w_0 + \sum_{j=1}^{M} \log_v(|x - a_i|_v) w_j$$

for each i. Thus we are led to the system of equations

$$\begin{bmatrix} 1 \\ \log_v(|\zeta - a_1|_v) \\ \vdots \\ \log_v(|\zeta - a_M|_v) \end{bmatrix} = \Theta \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_M \end{bmatrix}. \tag{12.5}$$

Cantor's idea was to reverse this construction: if the matrix Θ could be shown to be nonsingular, then the w_i's would be uniquely defined by the equations above; and if they could be shown to be positive, then it would be natural to expect that they could be used to construct a function $f(z)$ with the desired property.

Write $\mathbf{1}_M$ for the M-element column vector whose entries are all 1's, and write $\mathbf{J}_M$ for the $M \times M$ matrix whose entries are all 1's. For any matrix A, write ${}^{\mathrm{t}}A$ for its transpose. Observe that

$$\mathbf{J}_M = \mathbf{1}_M {}^{\mathrm{t}}\mathbf{1}_M.$$

Further, write $\Theta' = (\Theta_{ij})_{1 \le i,j \le M}$, so that in terms of block matrices,

$$\Theta = \begin{bmatrix} 0 & {}^{\mathrm{t}}\mathbf{1}_M \\ \mathbf{1}_M & \Theta' \end{bmatrix}.$$

Cantor's Lemma 3.2.3 can be formulated as follows:

Lemma 12.1. *The matrix Θ is nonsingular, and if $w_0, w_1, \ldots, w_m$ are defined by*

$$\begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_M \end{bmatrix} = \Theta^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \textit{resp.} \quad \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_M \end{bmatrix} = \Theta^{-1} \begin{bmatrix} 1 \\ \log_v(|\zeta - a_1|_v) \\ \vdots \\ \log_v(|\zeta - a_M|_v) \end{bmatrix},$$

according as $\zeta = \infty$ or $\zeta \ne \infty$, then $w_1, \ldots, w_m > 0$ and $\sum_{i=1}^m w_i = 1$; and if $f(z)$ is defined by (12.1), *respectively* (12.4), *then*

$$E_v = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |f(z)|_v \le 1\}.$$

Furthermore, we have the following formula for Θ^{-1}: let δ be any number satisfying $\delta > \max_{1 \le i,j \le m}(\Theta_{ij})$. Then

(A) *the symmetric matrix $\mathbf{A}_\delta := \Theta' - \delta \mathbf{J}_M$ is negative definite*;

(B) *if we define α by $\alpha^{-1} = -{}^{\mathrm{t}}\mathbf{1}_M \mathbf{A}_\delta^{-1} \mathbf{1}_M$ and put $\mathbf{b} = -\mathbf{A}_\delta^{-1}\mathbf{1}_M$, then $\alpha > 0$ and*

$$\Theta^{-1} = \begin{bmatrix} \alpha - \delta & \alpha\,{}^{\mathrm{t}}\mathbf{b} \\ \alpha \mathbf{b} & \mathbf{A}_\delta^{-1} + \alpha \mathbf{b}\,{}^{\mathrm{t}}\mathbf{b} \end{bmatrix}.$$

Proof. We first prove (A) and (B), by induction on M. Fix δ, and write $\mathbf{A} = \mathbf{A}_\delta$.

If $M = 1$, then $\mathbf{A} = \Theta_{11} - \delta < 0$, so (A) holds, and we find

$$b = \frac{-1}{\Theta_{11} - \delta}, \qquad \alpha = \frac{1}{b} = \delta - \Theta_{11} > 0, \qquad \Theta = \begin{bmatrix} 0 & 1 \\ 1 & \Theta_{11} \end{bmatrix}.$$

The matrix on the right side of (B) is

$$\begin{bmatrix} \alpha - \delta & \alpha b \\ \alpha b & 1/A + \alpha b^2 \end{bmatrix} = \begin{bmatrix} -\Theta_{11} & 1 \\ 1 & \dfrac{1}{\Theta_{11} - \delta} + (\delta - \Theta_{11})\left(\dfrac{-1}{\Theta_{11} - \delta}\right)^2 \end{bmatrix}$$
$$= \begin{bmatrix} -\Theta_{11} & 1 \\ 1 & 0 \end{bmatrix}$$

which equals Θ^{-1}, so (B) holds as well.

Next take $M > 1$ and suppose (A) and (B) have been proven for all $M' < M$. Put

$$\delta' = \max_{1 \le i,j \le M} \Theta_{ij}$$

and define an equivalence relation on $\{a_1, \ldots, a_M\}$ by $a_i \sim a_j$ if $\Theta_{ij} < \delta'$. (To see this is an equivalence relation, note that since $M \ge 2$, for each i there is a $j \neq i$, and $\Theta_{ii} < \Theta_{ij} \le \delta'$. This yields reflexivity. Symmetry is clear, and transitivity follows from the ultrametric inequality.) After relabeling the a_i's if necessary, we can assume that the equivalence classes consist of elements whose indices are consecutive blocks of integers. By the definition of δ', there are at least two equivalence classes.

The matrix Θ' thus has the form

$$\begin{bmatrix} \Theta'_1 & \delta' & \ldots & \delta' \\ \delta' & \Theta'_2 & & \delta' \\ \vdots & \vdots & \ddots & \vdots \\ \delta' & \delta' & & \Theta'_r \end{bmatrix}$$

with blocks corresponding to the various equivalence classes down the diagonal, and entries δ' everywhere else. By the definition of the relation $\sim$, each entry in each Θ'_k is strictly less than δ'. Consequently the matrix $\mathbf{A}' := \Theta' - \delta'\mathbf{J}_M$ has the block diagonal form $\mathrm{diag}(\mathbf{A}_1, \ldots, \mathbf{A}_r)$, where $\mathbf{A}_k = \Theta'_k - \delta'\mathbf{J}_{m_k}$ if m_k is the number of elements of the kth equivalence class. Since δ' is strictly greater than every entry in Θ'_k, each $\mathbf{A}_k$ satisfies the hypotheses of the lemma, and by induction is negative definite. Therefore

$$\mathbf{A} = \mathbf{A}' - (\delta - \delta')\mathbf{J}_M$$

is also negative definite, and in particular is nonsingular. This yields (A).

To prove (B), write $\mathbf{A} = \mathbf{A}_\delta$, $\mathbf{1} = \mathbf{1}_M$, $\mathbf{J} = \mathbf{J}_M$, $\mathbf{b} = -\mathbf{A}^{-1}\mathbf{1}_M$, and let

$$\nabla = \begin{bmatrix} \alpha - \delta & \alpha\,{}^{\mathrm{t}}\mathbf{b} \\ \alpha\mathbf{b} & \mathbf{A}^{-1} + \alpha\mathbf{b}\,{}^{\mathrm{t}}\mathbf{b} \end{bmatrix};$$

note that $\alpha^{-1} = -{}^{\mathrm{t}}\mathbf{1}\mathbf{A}^{-1}\mathbf{1} = {}^{\mathrm{t}}\mathbf{1}\mathbf{b}$. We compute

$$\begin{aligned} \Theta \cdot \nabla &= \begin{bmatrix} 0 & {}^{\mathrm{t}}\mathbf{1} \\ \mathbf{1} & \Theta' \end{bmatrix} \cdot \begin{bmatrix} \alpha - \delta & \alpha\,{}^{\mathrm{t}}\mathbf{b} \\ \alpha\mathbf{b} & \mathbf{A}^{-1} + \alpha\mathbf{b}\,{}^{\mathrm{t}}\mathbf{b} \end{bmatrix} \\ &= \begin{bmatrix} \alpha\,{}^{\mathrm{t}}\mathbf{1}\mathbf{b} & {}^{\mathrm{t}}\mathbf{1}\mathbf{A}^{-1} + \alpha\,{}^{\mathrm{t}}\mathbf{1}\mathbf{b}\,{}^{\mathrm{t}}\mathbf{b} \\ (\alpha - \delta)\mathbf{1} + \alpha\Theta'\mathbf{b} & \alpha\mathbf{1}\,{}^{\mathrm{t}}\mathbf{b} + \Theta'(\mathbf{A}^{-1} + \alpha\mathbf{b}\,{}^{\mathrm{t}}\mathbf{b}) \end{bmatrix}. \end{aligned}$$

Since $\mathbf{A}$ is negative definite, so is $\mathbf{A}^{-1}$, and consequently $\alpha > 0$. Now let us compute the quantities in the four corners of $\Theta \cdot \nabla$. For the first three:

$$\begin{aligned} \alpha\,{}^{\mathrm{t}}\mathbf{1}\mathbf{b} &= \alpha \cdot \alpha^{-1} = 1; \\ {}^{\mathrm{t}}\mathbf{1}\mathbf{A}^{-1} + \alpha\,{}^{\mathrm{t}}\mathbf{1}\mathbf{b}\,{}^{\mathrm{t}}\mathbf{b} &= {}^{\mathrm{t}}\mathbf{1}\mathbf{A}^{-1} + \alpha\,{}^{\mathrm{t}}\mathbf{1}(-\mathbf{A}^{-1}\mathbf{1})(-{}^{\mathrm{t}}\mathbf{1}\mathbf{A}^{-1}) \\ &= {}^{\mathrm{t}}\mathbf{1}\mathbf{A}^{-1} + \alpha({}^{\mathrm{t}}\mathbf{1}\mathbf{A}^{-1}\mathbf{1})\,({}^{\mathrm{t}}\mathbf{1}\mathbf{A}^{-1}) \\ &= {}^{\mathrm{t}}\mathbf{1}\mathbf{A}^{-1} + \alpha \cdot (-\alpha^{-1})\,{}^{\mathrm{t}}\mathbf{1}\mathbf{A}^{-1} = \mathbf{0}; \\ (\alpha - \delta)\mathbf{1} + \alpha\Theta'\mathbf{b} &= \alpha\mathbf{1} - \delta\mathbf{1} + \alpha\Theta'(-\mathbf{A}^{-1}\mathbf{1}) \\ &= \alpha\mathbf{1} - \delta\mathbf{1} + \alpha(\mathbf{A} + \delta\mathbf{J})(-\mathbf{A}^{-1}\mathbf{1}) \\ &= \alpha\mathbf{1} - \delta\mathbf{1} - \alpha\mathbf{A}\mathbf{A}^{-1}\mathbf{1} - \delta\alpha\mathbf{1}\,{}^{\mathrm{t}}\mathbf{1}\mathbf{A}^{-1}\mathbf{1} \\ &= -\delta\mathbf{1} - (\delta\mathbf{1})(\alpha\,{}^{\mathrm{t}}\mathbf{1}\mathbf{A}^{-1}\mathbf{1}) = -\delta\mathbf{1} + \delta\mathbf{1} = \mathbf{0}. \end{aligned}$$

For the fourth, note first that

$$\begin{aligned} &\alpha\mathbf{1}\,{}^{\mathrm{t}}\mathbf{b} + \Theta'(\mathbf{A}^{-1} + \alpha\mathbf{b}\,{}^{\mathrm{t}}\mathbf{b}) \\ &\quad = \alpha\mathbf{1}({}^{\mathrm{t}}(-\mathbf{A}^{-1}\mathbf{1})) + (\mathbf{A} + \delta\mathbf{J})(\mathbf{A}^{-1} + \alpha(-\mathbf{A}^{-1}\mathbf{1})\,{}^{\mathrm{t}}(-\mathbf{A}^{-1}\mathbf{1})) \\ &\quad = -\alpha\mathbf{1}\,{}^{\mathrm{t}}\mathbf{1}\mathbf{A}^{-1} + \mathbf{A}\mathbf{A}^{-1} + \delta\mathbf{J}\mathbf{A}^{-1} + \alpha\mathbf{A}\mathbf{A}^{-1}\mathbf{1}\,{}^{\mathrm{t}}\mathbf{1}\mathbf{A}^{-1} + \alpha\delta\mathbf{J}\mathbf{A}^{-1}\mathbf{1}\,{}^{\mathrm{t}}\mathbf{1}\mathbf{A}^{-1} \\ &\quad = \mathbf{I}_M + \delta\mathbf{J}\mathbf{A}^{-1} + \alpha\delta\mathbf{J}\mathbf{A}^{-1}\mathbf{J}\mathbf{A}^{-1}. \end{aligned}$$

But, expanding $\mathbf{J}$ by columns and rows,

$$\begin{aligned} \mathbf{J}\mathbf{A}^{-1}\mathbf{J}\mathbf{A}^{-1} &= \mathbf{J}(\mathbf{A}^{-1}\mathbf{J})\mathbf{A}^{-1} \\ &= \mathbf{J}[\mathbf{A}^{-1}\mathbf{1}, \ldots, \mathbf{A}^{-1}\mathbf{1}]\mathbf{A}^{-1} = \mathbf{J}[-\mathbf{b}, \ldots, -\mathbf{b}]\mathbf{A}^{-1} \\ &= -\begin{bmatrix} {}^{\mathrm{t}}\mathbf{1} \\ \vdots \\ {}^{\mathrm{t}}\mathbf{1} \end{bmatrix} [\mathbf{b}, \ldots, \mathbf{b}]\mathbf{A}^{-1} = -\begin{bmatrix} {}^{\mathrm{t}}\mathbf{1}\mathbf{b} \ldots {}^{\mathrm{t}}\mathbf{1}\mathbf{b} \\ \vdots \qquad \vdots \\ {}^{\mathrm{t}}\mathbf{1}\mathbf{b} \ldots {}^{\mathrm{t}}\mathbf{1}\mathbf{b} \end{bmatrix} \mathbf{A}^{-1} \\ &= -\alpha^{-1}\mathbf{J}\mathbf{A}^{-1}. \end{aligned}$$

Consequently, $\delta\mathbf{J}\mathbf{A}^{-1} + \alpha\delta\mathbf{J}\mathbf{A}^{-1}\mathbf{J}\mathbf{A}^{-1} = \mathbf{0}$, so the lower right corner of $\Theta \cdot \nabla$ is the identity matrix $\mathbf{I}_M$. Thus Θ is invertible, and $\Theta^{-1} = \nabla$. This yields (B).

So far we have followed Cantor's argument; now we leave it. We aim to prove the existence of a function $f(z)$ with poles only at ζ such that $E_v = \bigcup_{i=1}^{M} B(a_i, r_i) = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |f(z)|_v \le 1\}$.

Suppose first that $\zeta = \infty$. We claim that if $w_0, w_1, \ldots, w_M$ are defined by

$$\begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_M \end{bmatrix} = \Theta^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{12.6}$$

then the w_i are rational, $w_1, \ldots, w_M$ are positive, $\sum_{i=1}^{M} w_i = 1$, and the polynomial $f(z)$ defined by (12.1) satisfies the required conditions (where we take the degree n of $f(z)$ to be a number such that each nw_i is integral).

The w_i are rational since every element in Θ is. To prove the remaining assertions we proceed by induction on M.

If $M = 1$, then

$$\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \Theta^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\Theta_{11} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\Theta_{11} \\ 1 \end{bmatrix}$$

where $w_0 = \Theta_{11} = -\log_v(r_1)$. Thus $w_1 = 1 > 0$ and as a trivial sum $w_1 = 1$. Take n so that nw_0 is integral. If we put

$$f(z) = \pi_v^{nw_0}(z - a_1)^n$$

then it is obvious that $B(a_1, r_1) = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |f(z)|_v \le 1\}$.

Now take $M > 1$ and suppose the assertions proved for $M' < M$. Let the equivalence relation $\sim$ on $\{a_1, \ldots, a_M\}$ be as defined above, and keep the other notations established there. Suppose $\{a_1, \ldots, a_{m_1}\}$ is the first equivalence class. Put

$$E_v' = \bigcup_{i=1}^{m_1} B(a_i, r_i), \qquad E_v'' = \bigcup_{i=m_1+1}^{M} B(a_i, r_i).$$

By induction, the numbers $w_0', w_1', \ldots, w_{m_1}'$ and $w_0'', w_{m_1+1}'', \ldots, w_M''$ defined for the sets E_v', E_v'' by the equations

$$\begin{bmatrix} w_0' \\ w_1' \\ \vdots \\ w_{m_1}' \end{bmatrix} = (\Theta')^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} w_0'' \\ w_{m_1+1}'' \\ \vdots \\ w_M'' \end{bmatrix} = [\Theta'']^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

satisfy the required conditions, and the polynomials $f_1(z)$, $f_2(z)$ of respective degrees n_1, n_2 given by (12.1) serve to define E_v', E_v'' as PL-domains. After raising $f_1(z)$ and $f_2(z)$ to appropriate powers, we can assume that $n_1 = n_2$.

We claim that there exist rational numbers w_0, α_1, and α_2 with $\alpha_1, \alpha_2 > 0$ and $\alpha_1 + \alpha_2 = 1$, such that

$$\widehat{\mathbf{w}} := \begin{bmatrix} w_0 \\ \alpha_1 w_1' \\ \vdots \\ \alpha_1 w_{m_1}' \\ \alpha_2 w_{m_1+1}'' \\ \vdots \\ \alpha_2 w_M'' \end{bmatrix}$$

satisfies the equation

$$\begin{bmatrix} 1 \\ \vec{0} \end{bmatrix} = \Theta \widehat{\mathbf{w}}. \tag{12.7}$$

Since such a solution is unique, the entries in $\widehat{\mathbf{w}}$ must be the numbers w_i in (12.6), and our claims that $w_1, \ldots, w_m$ are positive and $\sum_{i=1}^m w_i = 1$ will be established.

Letting $\widehat{\mathbf{w}}$ be as above, with w_0, α_1 and α_2 as variables, in view of the block structure of the matrix Θ, and the equations satisfied by the w_j' and the w_j'', (12.7) reads

$$\begin{bmatrix} 1 \\ \vec{0} \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_2 \\ w_0 + \alpha_1(-w_0') + \alpha_2 \delta' \\ \vdots \\ w_0 + \alpha_1(-w_0') + \alpha_2 \delta' \\ w_0 + \alpha_1 \delta' + \alpha_2(-w_0'') \\ \vdots \\ w_0 + \alpha_1 \delta' + \alpha_2(-w_0'') \end{bmatrix}.$$

Consequently, in order to obtain (12.7), we need w_0, α_1 and α_2 to satisfy

$$\begin{aligned} \alpha_1 + \alpha_2 &= 1, \\ w_0 - w_0'\alpha_1 + \delta'\alpha_2 &= 0, \\ w_0 + \delta'\alpha_1 - w_0''\alpha_2 &= 0. \end{aligned}$$

Solving, we find

$$\alpha_1 = \frac{\delta' + w_0''}{(\delta' + w_0') + (\delta' + w_0'')}, \qquad \alpha_2 = \frac{\delta' + w_0'}{(\delta' + w_0') + (\delta' + w_0'')},$$

$$w_0 = \frac{w_0' w_0'' - \delta'\delta'}{(\delta' + w_0') + (\delta' + w_0'')}.$$

Thus, our assertions hold if $\delta' + w_0'$ and $\delta' + w_0''$ are both positive. However, that is the case, since $-w_0' = \sum_{j=1}^{m_1} \Theta_{ij} w_j'$ is an average (with positive coefficients) of numbers Θ_{ij} which are strictly less than δ', and similarly w_0'' is an average of numbers $\Theta_{ij} \le \delta'$, with $\Theta_{ii} < \delta'$.

Now let N be a common multiple of the denominators of α_1 and α_2, and

$$F(z) = f_1(z)^{N\alpha_1} f_2(z)^{N\alpha_2}.$$

By the ultrametric inequality and the definition of the equivalence classes for $\sim$, $|f_1(z)|_v$ is constant on E_v'', and similarly $|f_2(z)|_v$ is constant on E_v'. Hence $|F(z)|_v$ takes the same value, say R, at each point of the "boundary" $\partial E_v = \bigcup_{i=1}^M \partial B(a_i, r_i)$. By the construction of $f_1(z)$ and $f_2(z)$ it follows that

$$E_v = E_v' \cup E_v'' \subset \{z \in \mathbb{P}^1(\mathbb{C}_v) : |F(z)|_v \le R\}.$$

Let r' be such that $\delta' = \log_v(r')$. For the same reasons as above, $|f_1(z)|_v$ is constant on each "open" ball $B(a_j, r')^-$ with $a_j \not\sim a_1$, and $|f_2(z)|_v$ is constant on $B(a_1, r')^-$. Suppose now that $x \notin E_v$. If x belongs to $B(a_1, r')^-$, fix $x_1 \in \partial B(a_1, r_1)$. By the definition of $f_1(z)$ and the discussion above,

$$|f_1(x)|_v > |f_1(x_1)|_v, \quad \text{while} \quad |f_2(x)|_v = |f_2(x_1)|_v.$$

Consequently $|F(x)|_v > |F(x_1)|_v = R$. Similarly if x belongs to any other open ball $B(a_j, r')^-$ then $|F(x)|_v > R$. If x belongs to none of these open balls, then $|x - a_i|_v \ge r'$ for all i, and so fixing $x_1 \in \partial B(a_1, r_1)$ we see that $|f_1(x)|_v > |f_1(x_1)|_v$ and $|f_2(x)|_v \ge |f_2(x_1)|_v$. Hence again $|F(x)|_v > R$. We have thus shown that

$$E_v = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |F(z)|_v \le R\}.$$

By scaling $F(z)$ appropriately we can obtain a polynomial $f(z)$ for which

$$E_v = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |f(z)|_v \le 1\}.$$

Since the roots of $f(z)$ have the same relative multiplicities as those of $F(z)$ (namely $w_1, \ldots, w_M$), while

$$w_0 = -\sum_{j=1}^{M} \Theta_{ij} w_j \quad \text{for any } i,$$

we have shown that the polynomial defined by (12.1) exhibits E_v as a PL-domain. This completes the proof when $\zeta = \infty$.

Finally, suppose $\zeta \ne \infty$. The linear fractional transformation $h(z) = 1/(z - \zeta)$ takes ζ to ∞, and takes a ball $B(a, r)$ not containing ζ to the ball $B(h(a), r/|\zeta - a|_v^2)$. Consequently, $h(E_v)$ is again a finite union of balls. If $g(z)$ is the polynomial constructed by (12.1) for the set $h(E_v)$, with zeros supported on the points $h(a_i)$, then $f(z) = g(h(z))$ will be a rational function with poles only at ζ and zeros supported on the a_i, such that

$$E_v = \{z \in \mathbb{P}^1(\mathbb{C}_v) : |f(z)|_v \le 1\}.$$

After scaling $f(z)$ by a number $u \in \mathbb{C}_v$ with $|u|_v = 1$, we can assume $f(z)$ has the form (12.4) with $w_1, \ldots, w_M$ positive. If we now repeat the argument leading to (12.5), it follows that $w_0, w_1, \ldots, w_M$ satisfy

$$\begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_M \end{bmatrix} = \Theta^{-1} \begin{bmatrix} 1 \\ \log_v(|\zeta - a_1|_v) \\ \vdots \\ \log_v(|\zeta - a_M|_v) \end{bmatrix}. \ \blacksquare$$

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