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On some combinatorial relations for Tornheim's double series

by

Hirofumi Tsumura (Tokyo)

1. Introduction. *Tornheim's double series* $T(r,s,t)$ is defined by

$$T(r,s,t)=\sum_{m,n=1}^{\infty}\frac{1}{m^r n^s (m+n)^t}, \tag{1.1}$$

where r,s,t are nonnegative integers with $r+t>1$, $s+t>1$ and $r+s+t>2$, and was introduced in [5]. Tornheim showed that $T(r,s,N-r-s)$ is a polynomial in $\{\zeta(j)\mid 2\le j\le N\}$ with rational coefficients when N is odd and $N\ge 3$ (see [5, Theorem 7]). Recently Huard, Williams and Zhang Nan-Yue gave an explicit formula for $T(r,s,N-r-s)$ as a rational linear combination of the products $\zeta(2j)\zeta(N-2j)$ $(0\le j\le (N-3)/2)$ when N is odd, $N\ge 3$, and r,s are nonnegative integers satisfying $1\le r+s\le N-1$, $r\le N-2$, and $s\le N-2$ (see [2, Theorem 2]). On the other hand, it is an open problem to determine an explicit formula for $T(r,s,N-r-s)$ when N is even. Indeed, only a few cases have been determined (see e.g. [2, p. 116]). In [4], Subbarao and Sitaramachandrarao gave an explicit formula for

$$T(2k,2p,2q)+T(2k,2q,2p)+T(2p,2q,2k) \tag{1.2}$$

for $k,p,q\ge 1$. This can be viewed as a generalization of Mordell's result for $T(2k,2k,2k)$ given in [3].

The purpose of this paper is to give some relations for $T(r,s,t)$ and their alternating analogues. Indeed, as a generalization of (1.2), we determine

$$T(r,s,t)+(-1)^r T(r,t,s)+(-1)^{r+t}T(t,s,r), \tag{1.3}$$

where $r,s\ge 0$ with $r+s\ge 2$ and $t\ge 2$ (see Theorem 1). In particular when $N=r+s+t$ is odd, we recover Tornheim's theorem mentioned above (see Remark after Theorem 1). When N is even, for example, we give an explicit

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formula for

$$T(2k+1,2p,2q+1)-T(2k+1,2q+1,2p)+T(2p,2q+1,2k+1)$$

(see Theorem 2). As an application, we give some relations for alternating analogues of $T(r,s,t)$ (see Proposition 3). In particular, we give an explicit formula for

$$\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}}{m^{2k+1}n^{2k+1}(m+n)^{2k+1}} \tag{1.4}$$

for $k\in\mathbb{N}\cup\{0\}$ (see Corollary 3). The problem of evaluating (1.4) was posed in [4, §5].

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2. Preliminaries. Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{Z}$ the ring of rational integers, and $\mathbb{R}$ the field of real numbers. Throughout this paper we fix $\delta\in\mathbb{R}$ with $\delta>0$. For $u\in\mathbb{R}$ with $1\le u\le 1+\delta$ and $s\in\mathbb{Z}$, define

$$\phi(s;u):=\sum_{m=1}^{\infty}\frac{(-u)^{-m}}{m^s}. \tag{2.1}$$

If $u>1$ then $\phi(s;u)$ is convergent for any $s\in\mathbb{Z}$. In the case when $u=1$, let $\phi(s):=\phi(s;1)=(2^{1-s}-1)\zeta(s)$. Corresponding to $\phi(s;u)$, we define a set $\{\varepsilon_m(u)\}$ of numbers by

$$G(x;u):=\frac{(1+u)e^x}{e^x+u}=\sum_{m=0}^{\infty}\varepsilon_m(u)\,\frac{x^m}{m!}. \tag{2.2}$$

When $u=1$, it is well known that

$$G(x;1)=\frac{2e^x}{e^x+1}=\sum_{m=0}^{\infty}E_m(1)\,\frac{x^m}{m!},$$

where $E_m(x)$ is the mth Euler polynomial (see e.g. [1]). Hence

$$\varepsilon_{2j}(1)=E_{2j}(1)=0 \qquad (j\in\mathbb{N}), \tag{2.3}$$

and if $u\in[1,1+\delta]$, then

$$\liminf_{m\to\infty}\left(\frac{|\varepsilon_m(u)|}{m!}\right)^{-1/m}\ge\pi. \tag{2.4}$$

From (2.1) and (2.2), we have

LEMMA 1. *$\phi(-k;u) = -\varepsilon_k(u)/(1+u)$ for $k \in \mathbb{N}\cup\{0\}$ and $u \in (1, 1+\delta]$.*

By Lemma 1, we have

$$(2.5)\qquad I_{2k}(\theta;u) := \sum_{m=1}^{\infty} \frac{(-u)^{-m}\cos(m\theta)}{m^{2k}} - \sum_{j=0}^{k} \phi(2k-2j;u)\,\frac{(i\theta)^{2j}}{(2j)!} = -\frac{1}{1+u}\sum_{m=1}^{\infty} \varepsilon_{2m}(u)\,\frac{(i\theta)^{2m+2k}}{(2m+2k)!}$$

and

$$(2.6)\qquad J_{2k+1}(\theta;u) := i\sum_{m=1}^{\infty} \frac{(-u)^{-m}\sin(m\theta)}{m^{2k+1}} - \sum_{j=0}^{k} \phi(2k-2j;u)\,\frac{(i\theta)^{2j+1}}{(2j+1)!} = -\frac{1}{1+u}\sum_{m=1}^{\infty} \varepsilon_{2m}(u)\,\frac{(i\theta)^{2m+2k+1}}{(2m+2k+1)!}$$

for $k, l \in \mathbb{N}\cup\{0\}$. Combining (2.3) and (2.4), we obtain

LEMMA 2. *Suppose $k \in \mathbb{N}\cup\{0\}$ and $\theta \in (-\pi,\pi)$. Then $I_{2k}(\theta;u)$ and $J_{2k+1}(\theta;u)$ are uniformly convergent with respect to $u \in (1, 1+\delta]$, and satisfy $I_{2k}(\theta;u) \to 0$ and $J_{2k+1}(\theta;u) \to 0$ as $u \to 1$.*

For $u \in (1, 1+\delta]$, $s \in \mathbb{Z}$ and $k, l \in \mathbb{N}\cup\{0\}$, define

$$(2.7)\qquad R(k,s,l;u) := \sum_{m,n=1}^{\infty} \frac{(-u)^{-(2m+n)}}{m^k n^s (m+n)^l},$$

$$(2.8)\qquad S(k,l,s;u) := \sum_{m,n=1}^{\infty} \frac{(-u)^{-(m+n)}}{m^k n^l (m+n)^s}.$$

LEMMA 3. *Suppose $k, l \in \mathbb{N}\cup\{0\}$, $\theta \in (-\pi,\pi)$ and $u \in (1, 1+\delta]$. Then*

$$(2.9)\qquad \sum_{m=1}^{\infty} \frac{(-u)^{-m}e^{im\theta}}{m^k}\cdot\sum_{n=1}^{\infty} \frac{(-u)^{-n}e^{in\theta}}{n^l} = \sum_{m=0}^{\infty} S(k,l,-m;u)\,\frac{(i\theta)^m}{m!},$$

$$(2.10)\qquad \sum_{m=1}^{\infty} \frac{(-u)^{-m}e^{im\theta}}{m^k}\cdot\sum_{n=1}^{\infty} \frac{(-u)^{-n}e^{-in\theta}}{n^l} = \sum_{m=0}^{\infty} \{R(l,-m,k;u) + (-1)^m R(k,-m,l;u)\}\,\frac{(i\theta)^m}{m!} + \sum_{m=1}^{\infty} \frac{u^{-2m}}{m^{k+l}}.$$

Proof. By (2.8), we immediately obtain (2.9). The left-hand side of (2.10) is equal to

$$\sum_{\substack{m,n=1\\ m>n}}^{\infty}\frac{(-u)^{-(m+n)}e^{i(m-n)\theta}}{m^k n^l}+\sum_{m=1}^{\infty}\frac{u^{-2m}}{m^{k+l}}+\sum_{\substack{m,n=1\\ m<n}}^{\infty}\frac{(-u)^{-(m+n)}e^{-i(n-m)\theta}}{m^k n^l}$$
$$=\sum_{n,j=1}^{\infty}\frac{(-u)^{-(2n+j)}e^{ij\theta}}{n^l(n+j)^k}+\sum_{m=1}^{\infty}\frac{u^{-2m}}{m^{k+l}}+\sum_{m,j=1}^{\infty}\frac{(-u)^{-(2m+j)}e^{-ij\theta}}{m^k(m+j)^l}.$$

Hence, by (2.7), we obtain (2.10). ■

Lemma 4. *For* $k,l\in\mathbb{N}\cup\{0\}$, $\theta\in(-\pi,\pi)$ *and* $u\in(1,1+\delta]$,

$$I_{2k}(\theta;u)\left(\sum_{n=1}^{\infty}\frac{(-u)^{-n}e^{in\theta}}{n^l}\right) \tag{2.11}$$
$$=\sum_{m=0}^{\infty}\frac{1}{2}\{S(2k,l,-m;u)+R(2k,-m,l;u)+(-1)^m R(l,-m,2k;u)\}\frac{(i\theta)^m}{m!}$$
$$-\sum_{m=0}^{\infty}\sum_{j=0}^{k}\binom{m}{2j}\phi(2k-2j;u)\phi(l+2j-m;u)\frac{(i\theta)^m}{m!}+\frac{1}{2}\sum_{m=1}^{\infty}\frac{u^{-2m}}{m^{2k+l}},$$

$$J_{2k+1}(\theta;u)\left(\sum_{n=1}^{\infty}\frac{(-u)^{-n}e^{in\theta}}{n^l}\right) \tag{2.12}$$
$$=\sum_{m=0}^{\infty}\frac{1}{2}\{S(2k+1,l,-m;u)-R(2k+1,-m,l;u)$$
$$-(-1)^m R(l,-m,2k+1;u)\}\frac{(i\theta)^m}{m!}$$
$$-\sum_{m=0}^{\infty}\sum_{j=0}^{k}\binom{m}{2j+1}\phi(2k-2j;u)\phi(l+2j+1-m;u)\frac{(i\theta)^m}{m!}$$
$$-\frac{1}{2}\sum_{m=1}^{\infty}\frac{u^{-2m}}{m^{2k+1+l}}.$$

Proof. By the binomial theorem, we have

$$\left(\sum_{j=0}^{k}\phi(2k-2j;u)\frac{(i\theta)^{2j}}{(2j)!}\right)\left(\sum_{n=1}^{\infty}\frac{(-u)^{-n}e^{in\theta}}{n^l}\right)$$
$$=\left(\sum_{j=0}^{k}\phi(2k-2j;u)\frac{(i\theta)^{2j}}{(2j)!}\right)\left(\sum_{m=0}^{\infty}\phi(l-m;u)\frac{(i\theta)^m}{m!}\right)$$
$$=\sum_{N=0}^{\infty}\sum_{j=0}^{k}\binom{N}{2j}\phi(2k-2j;u)\phi(l+2j-N;u)\frac{(i\theta)^N}{N!}$$

and

$$\left(\sum_{j=0}^{k}\phi(2k-2j;u)\,\frac{(i\theta)^{2j+1}}{(2j+1)!}\right)\left(\sum_{n=1}^{\infty}\frac{(-u)^{-n}e^{in\theta}}{n^l}\right)$$
$$=\sum_{N=0}^{\infty}\sum_{j=0}^{k}\binom{N}{2j+1}\phi(2k-2j;u)\phi(l+2j+1-N;u)\,\frac{(i\theta)^N}{N!}.$$

By Lemma 3, we immediately obtain (2.11) and (2.12). ■

For $k,l\in\mathbb{N}\cup\{0\},\ u\in(1,1+\delta]$ and $r\in\mathbb{Z}$, we define

$$\begin{aligned}A(r;2k,l;u)&\\ :=\frac{1}{2}&\{S(2k,l,r;u)+R(2k,r,l;u)+(-1)^rR(l,r,2k;u)\}\\ &-\sum_{j=0}^{k}\binom{r+2j-1}{2j}\phi(2k-2j;u)\phi(l+2j+r;u)\end{aligned}\tag{2.13}$$

and

$$\begin{aligned}A(r;2k+1,l;u)&\\ :=\frac{1}{2}&\{S(2k+1,l,r;u)-R(2k+1,r,l;u)-(-1)^rR(l,r,2k+1;u)\}\\ &+\sum_{j=0}^{k}\binom{r+2j}{2j+1}\phi(2k-2j;u)\phi(l+2j+1+r;u).\end{aligned}\tag{2.14}$$

Lemma 5. *Suppose $p,q\in\mathbb{N}\cup\{0\}$ with $p+q\geq 2$. Then*

$$\liminf_{m\to\infty}\left(\frac{A(-m;p,q;u)}{m!}\right)^{-1/m}\geq\pi\qquad(u\in(1,1+\delta]),\tag{2.15}$$

$$\lim_{u\to 1}A(-m;p,q;u)=0\qquad(m\in\mathbb{N}),\tag{2.16}$$

$$\lim_{u\to 1}A(0;p,q;u)=\frac{(-1)^{p+1}}{2}\,\zeta(p+q).\tag{2.17}$$

Proof. It follows from (2.4) and Lemma 2 that (2.11) and (2.12) are uniformly convergent with respect to $u\in(1,1+\delta]$ when $k,l\in\mathbb{N}\cup\{0\}$ with $2k+l\geq 2$ and $\theta\in(-\pi,\pi)$ and that these tend to 0 as $u\to 1$. Hence each coefficient of $(i\theta)^m/m!$ on the right-hand side of (2.11) and (2.12) tends to 0 as $u\to 1$. Then it follows from the well-known relation

$$\binom{-x}{y}=(-1)^y\binom{x+y-1}{y}$$

that (2.15)–(2.17) hold. ■

3. Relations for $T(r,s,t)$. We begin by proving the following combinatorial relations.

LEMMA 6. *For $c,s\in\mathbb{N}\cup\{0\}$,*

$$\sum_{\nu=0}^{c}\frac{(-\theta)^\nu}{\nu!}\binom{s+c-\nu}{c-\nu}\frac{\cos^{(\nu)}(x\theta)}{x^{s+1+c-\nu}} = \sum_{N=0}^{\infty}\binom{s+c-2N}{c}\frac{1}{x^{s+1+c-2N}}\cdot\frac{(i\theta)^{2N}}{(2N)!}, \tag{3.1}$$

where $\cos^{(\nu)}(\theta)$ is the νth derivative of $\cos\theta$.

Proof. Let $f(x;s,\theta)=\cos(x\theta)x^{-s-1}$. In order to prove (3.1), we have only to calculate the cth derivative of $f(x;s,\theta)$ with respect to x as follows:

$$\frac{d^c}{dx^c}f(x;s,\theta)=\sum_{\nu=0}^{c}\binom{c}{\nu}\{\cos(x\theta)\}^{(\nu)}\cdot\{x^{-s-1}\}^{(c-\nu)}$$

and

$$\frac{d^c}{dx^c}f(x;s,\theta)=\frac{d^c}{dx^c}\bigg\{\sum_{N=0}^{\infty}\frac{(i\theta)^{2N}}{(2N)!}\,x^{2N-s-1}\bigg\},$$

by using the laws of differentiation and the Maclaurin expansion of $\cos\theta$. ■

By applying this lemma, we obtain

$$\sum_{\nu=0}^{c}\frac{(-\theta)^\nu}{\nu!}\binom{s+c-\nu}{c-\nu}\sum_{m=1}^{\infty}\frac{(-u)^{-m}\cos^{(\nu)}(m\theta)}{m^{s+1+c+l-\nu}} = \sum_{N=0}^{\infty}\binom{s+c-2N}{c}\sum_{m=1}^{\infty}\frac{(-u)^{-m}}{m^{s+1+c+l-2N}}\cdot\frac{(i\theta)^{2N}}{(2N)!} \tag{3.2}$$

for $c,l,s\in\mathbb{N}\cup\{0\}$. Hence we can prove

LEMMA 7. *Suppose $k,l\in\mathbb{N}\cup\{0\}$, $d\in\mathbb{N}$, $\theta\in(-\pi,\pi)$ and $u\in(1,1+\delta]$. Then*

$$\frac{1}{2}\sum_{m,n=1}^{\infty}\bigg\{\frac{(-u)^{-(m+n)}\cos((m+n)\theta)}{m^{2k}n^{l}(m+n)^{d}}+\frac{(-u)^{-(2m+n)}\cos(n\theta)}{m^{2k}n^{d}(m+n)^{l}} + (-1)^d\frac{(-u)^{-(2m+n)}\cos(n\theta)}{m^{l}n^{d}(m+n)^{2k}}\bigg\} \tag{3.3}$$

$$- \sum_{j=0}^{k} \phi(2k-2j;u) \sum_{\nu=0}^{2j} \frac{(-\theta)^{\nu}}{\nu!} \binom{d-1+2j-\nu}{2j-\nu}$$

$$\times \sum_{m=1}^{\infty} \frac{(-u)^{-m} \cos^{(\nu)}(m\theta)}{m^{d+2j+l-\nu}}$$

$$= \sum_{N=0}^{\infty} A(d-2N;2k,l;u) \frac{(i\theta)^{2N}}{(2N)!},$$

$$\begin{aligned}(3.4)\quad & \frac{1}{2} \sum_{m,n=1}^{\infty} \left\{ \frac{(-u)^{-(m+n)} \cos((m+n)\theta)}{m^{2k+1} n^{l} (m+n)^{d}} - \frac{(-u)^{-(2m+n)} \cos(n\theta)}{m^{2k+1} n^{d} (m+n)^{l}} \right. \\ & \qquad \left. -(-1)^{d} \frac{(-u)^{-(2m+n)} \cos(n\theta)}{m^{l} n^{d} (m+n)^{2k+1}} \right\} \\ & + \sum_{j=0}^{k} \phi(2k-2j;u) \sum_{\nu=0}^{2j+1} \frac{(-\theta)^{\nu}}{\nu!} \binom{d+2j-\nu}{2j+1-\nu} \\ & \times \sum_{m=1}^{\infty} \frac{(-u)^{-m} \cos^{(\nu)}(m\theta)}{m^{d+2j+1+l-\nu}} \\ & = \sum_{N=0}^{\infty} A(d-2N;2k+1,l;u) \frac{(i\theta)^{2N}}{(2N)!}.\end{aligned}$$

In particular when $d \geq 2$, (3.3) *and* (3.4) *hold for* $\theta \in [-\pi, \pi]$.

Proof. Suppose $k, l \in \mathbb{N} \cup \{0\}$, $d \in \mathbb{N}$ and $\theta \in (-\pi, \pi)$. By (2.7) and (2.8), we have

$$(3.5)\quad \sum_{m,n=1}^{\infty} \frac{(-u)^{-(2m+n)} \cos(n\theta)}{m^{p} n^{q} (m+n)^{r}} = \sum_{m=0}^{\infty} R(p,q-2m,r;u) \frac{(i\theta)^{2m}}{(2m)!}$$

and

$$(3.6)\quad \sum_{m,n=1}^{\infty} \frac{(-u)^{-(m+n)} \cos((m+n)\theta)}{m^{p} n^{q} (m+n)^{r}} = \sum_{m=0}^{\infty} S(p,q,r-2m;u) \frac{(i\theta)^{2m}}{(2m)!}.$$

On the other hand, by applying (3.2) in the case $(c,s) = (2j, d-1)$, we have

$$\begin{aligned}(3.7)\quad & \sum_{\nu=0}^{2j} \frac{(-\theta)^{\nu}}{\nu!} \binom{d-1+2j-\nu}{2j-\nu} \sum_{m=1}^{\infty} \frac{(-u)^{-m} \cos^{(\nu)}(m\theta)}{m^{d+2j+l-\nu}} \\ & = \sum_{N=0}^{\infty} \binom{2N-d}{2j} \phi(l+2j+d-2N;u) \frac{(i\theta)^{2N}}{(2N)!}.\end{aligned}$$

Combining (2.13) and (3.5)–(3.7), we obtain the proof of (3.3). In particular when $d \geq 2$, it follows from (2.15) that (3.3) holds for $\theta \in [-\pi, \pi]$. In the same way as above, we can verify (3.4). ■

From (2.13), (2.14) and (2.17), we can define

$$A(r;p,q) := \lim_{u \to 1} A(r;p,q;u) \tag{3.8}$$

for $p, q, r \in \mathbb{N} \cup \{0\}$ with $p + q \geq 2$. With this notation, we obtain

PROPOSITION 1. *Suppose $k, l \in \mathbb{N} \cup \{0\}$, $d \in \mathbb{N}$ with $d \geq 2$ and $\theta \in [-\pi, \pi]$. When $2k + l \geq 2$,*

$$\begin{aligned}\frac{1}{2} \sum_{m,n=1}^{\infty} \bigg\{ & \frac{(-1)^{m+n} \cos((m+n)\theta)}{m^{2k} n^l (m+n)^d} + \frac{(-1)^n \cos(n\theta)}{m^{2k} n^d (m+n)^l} \\ & + \frac{(-1)^{d+n} \cos(n\theta)}{m^l n^d (m+n)^{2k}} \bigg\} \\ & - \sum_{j=0}^{k} \phi(2k-2j) \sum_{\nu=0}^{2j} \frac{(-\theta)^\nu}{\nu!} \binom{d-1+2j-\nu}{2j-\nu} \\ & \times \sum_{m=1}^{\infty} \frac{(-1)^m \cos^{(\nu)}(m\theta)}{m^{d+2j+l-\nu}} \\ = & \sum_{N=0}^{[d/2]} A(d-2N; 2k, l) \frac{(i\theta)^{2N}}{(2N)!}.\end{aligned} \tag{3.9}$$

When $2k + l \geq 1$,

$$\begin{aligned}\frac{1}{2} \sum_{m,n=1}^{\infty} \bigg\{ & \frac{(-1)^{m+n} \cos((m+n)\theta)}{m^{2k+1} n^l (m+n)^d} - \frac{(-1)^n \cos(n\theta)}{m^{2k+1} n^d (m+n)^l} \\ & - \frac{(-1)^{d+n} \cos(n\theta)}{m^l n^d (m+n)^{2k+1}} \bigg\} \\ & + \sum_{j=0}^{k} \phi(2k-2j) \sum_{\nu=0}^{2j+1} \frac{(-\theta)^\nu}{\nu!} \binom{d+2j-\nu}{2j+1-\nu} \\ & \times \sum_{m=1}^{\infty} \frac{(-1)^m \cos^{(\nu)}(m\theta)}{m^{d+2j+1+l-\nu}} \\ = & \sum_{N=0}^{[d/2]} A(d-2N; 2k+1, l) \frac{(i\theta)^{2N}}{(2N)!}.\end{aligned} \tag{3.10}$$

Proof. By (2.15), we see that both sides of (3.3) and (3.4) are uniformly convergent with respect to $u \in (1, 1+\delta]$ for $\theta \in [-\pi, \pi]$ when $d \geq 2$. Hence, by letting $u \to 1$ on both sides of (3.3) and (3.4), and by using (2.16), (2.17) and (3.8), we obtain the proofs of (3.9) and (3.10). ■

By applying Proposition 1 in the case $\theta = \pi$, we have the following relations for $T(r,s,t)$.

COROLLARY 1. *Suppose* $k, l \in \mathbb{N} \cup \{0\}$ *and* $d \in \mathbb{N}$ *with* $d \geq 2$. *When* $2k + l \geq 2$,

$$\begin{aligned}(3.11)\quad & \frac{1}{2}\{T(2k,l,d) + T(2k,d,l) + (-1)^d T(l,d,2k)\} \\ & - \sum_{j=0}^{k} \phi(2k-2j) \sum_{\mu=0}^{j} \frac{(i\pi)^{2\mu}}{(2\mu)!} \binom{d-1+2j-2\mu}{2j-2\mu} \zeta(d+2j+l-2\mu) \\ &= \sum_{N=0}^{[d/2]} A(d-2N; 2k, l) \frac{(i\pi)^{2N}}{(2N)!}.\end{aligned}$$

When $2k + l \geq 1$,

$$\begin{aligned}(3.12)\quad & \frac{1}{2}\{T(2k+1,l,d) - T(2k+1,d,l) - (-1)^d T(l,d,2k+1)\} \\ & + \sum_{j=0}^{k} \phi(2k-2j) \sum_{\mu=0}^{j} \frac{(i\pi)^{2\mu}}{(2\mu)!} \binom{d+2j-2\mu}{2j+1-2\mu} \zeta(d+2j+1+l-2\mu) \\ &= \sum_{N=0}^{[d/2]} A(d-2N; 2k+1, l) \frac{(i\pi)^{2N}}{(2N)!}.\end{aligned}$$

PROPOSITION 2. *Suppose* $k, l \in \mathbb{N} \cup \{0\}$, $d \in \mathbb{N}$ *with* $d \geq 3$ *and* $\theta \in [-\pi, \pi]$. *When* $2k + l \geq 2$,

$$\begin{aligned}(3.13)\quad & -\frac{1}{2} \sum_{m,n=1}^{\infty} \left\{ \frac{(-1)^{m+n} \sin((m+n)\theta)}{m^{2k} n^{l} (m+n)^{d-1}} + \frac{(-1)^{n} \sin(n\theta)}{m^{2k} n^{d-1} (m+n)^{l}} \right. \\ & \left. \qquad + \frac{(-1)^{d+n} \sin(n\theta)}{m^{l} n^{d-1} (m+n)^{2k}} \right\} \\ & - \sum_{j=0}^{k} \phi(2k-2j) \sum_{\nu=0}^{2j} \frac{(-\theta)^{\nu}}{\nu!} \binom{d-2+2j-\nu}{2j-\nu} \\ & \times \sum_{m=1}^{\infty} \frac{(-1)^m \cos^{(\nu+1)}(m\theta)}{m^{d+2j+l-\nu-1}} \\ &= \sum_{N=1}^{[d/2]} A(d-2N; 2k, l) \frac{(-1)^N \theta^{2N-1}}{(2N-1)!}.\end{aligned}$$

When $2k+l\geq 1$,

$$
(3.14)\qquad -\frac{1}{2}\sum_{m,n=1}^{\infty}\left\{\frac{(-1)^{m+n}\sin((m+n)\theta)}{m^{2k+1}n^{l}(m+n)^{d-1}}-\frac{(-1)^{n}\sin(n\theta)}{m^{2k+1}n^{d-1}(m+n)^{l}}-\frac{(-1)^{d+n}\sin(n\theta)}{m^{l}n^{d-1}(m+n)^{2k+1}}\right\}
$$
$$
+\sum_{j=0}^{k}\phi(2k-2j)\sum_{\nu=0}^{2j+1}\frac{(-\theta)^{\nu}}{\nu!}\binom{d-1+2j-\nu}{2j+1-\nu}\times\sum_{m=1}^{\infty}\frac{(-1)^{m}\cos^{(\nu+1)}(m\theta)}{m^{d+2j+l-\nu}}
$$
$$
=\sum_{N=1}^{[d/2]}A(d-2N;2k+1,l)\,\frac{(-1)^{N}\theta^{2N-1}}{(2N-1)!}.
$$

Proof. We differentiate both sides of (3.3) and (3.4) with respect to θ. For example, we can easily verify that

$$
\frac{d}{d\theta}\left(\sum_{\nu=0}^{2j}\frac{(-\theta)^{\nu}}{\nu!}\binom{d-1+2j-\nu}{2j-\nu}\sum_{m=1}^{\infty}\frac{(-u)^{-m}\cos^{(\nu)}(m\theta)}{m^{d+2j+l-\nu}}\right)
=\sum_{\nu=0}^{2j}\frac{(-\theta)^{\nu}}{\nu!}\binom{d-2+2j-\nu}{2j-\nu}\sum_{m=1}^{\infty}\frac{(-u)^{-m}\cos^{(\nu+1)}(m\theta)}{m^{d+2j+l-\nu-1}},
$$

by using the well-known relation

$$
-\binom{x-1}{y-1}+\binom{x}{y}=\binom{x-1}{y}.
$$

Since $d\geq 3$, both sides of the resulting equations are uniformly convergent with respect to $u\in(1,1+\delta]$ for $\theta\in[-\pi,\pi]$. By using the same method as in the proof of Proposition 1, we can prove (3.13) and (3.14). ■

By applying Proposition 2 in the case $\theta=\pi$, we have the following relations.

Corollary 2. *Suppose* $k,l\in\mathbb{N}\cup\{0\}$ *and* $d\in\mathbb{N}$ *with* $d\geq 3$. *When* $2k+l\geq 2$,

$$
\sum_{j=0}^{k}\phi(2k-2j)\sum_{\mu=0}^{j-1}\frac{(i\pi)^{2\mu}}{(2\mu+1)!}\binom{d-3+2j-2\mu}{2j-2\mu-1}\zeta(d-2+2j+l-2\mu)
=\sum_{\nu=0}^{[d/2]-1}A(d-2-2\nu;2k,l)\,\frac{(i\pi)^{2\nu}}{(2\nu+1)!}.
$$

When $2k+l\geq 1$,

$$\sum_{j=0}^{k}\phi(2k-2j)\sum_{\mu=0}^{j}\frac{(i\pi)^{2\mu}}{(2\mu+1)!}\binom{d-2+2j-2\mu}{2j-2\mu}\zeta(d-1+2j+l-2\mu)$$
$$=-\sum_{\nu=0}^{[d/2]-1}A(d-2-2\nu;2k+1,l)\,\frac{(i\pi)^{2\nu}}{(2\nu+1)!}.$$

It follows from Corollary 2 and (2.17) that $A(p;2k,l)$ and $A(p;2k+1,l)$ are inductively determined as polynomials in $\zeta(j)$ with rational coefficients for $p\in\mathbb{N}\cup\{0\}$. Hence, by Corollary 1, we obtain

THEOREM 1. *Suppose* $r,s\in\mathbb{N}\cup\{0\}$ *with* $r+s\geq 2$ *and* $t\in\mathbb{N}$ *with* $t\geq 2$. *Then*

$$T(r,s,t)+(-1)^{r}T(r,t,s)+(-1)^{r+t}T(t,s,r)$$

is a polynomial in $\{\zeta(j)\mid 2\leq j\leq r+s+t\}$ *with rational coefficients.*

REMARK. By applying Theorem 1 in the case when $r+s+t$ is odd, we immediately obtain Tornheim's theorem mentioned in Section 1. In particular, an explicit formula for (1.2) can be deduced from Theorem 1. But this is more complicated than in [4, Theorem 4.1].

4. Explicit formulas. In this section, we give some explicit formulas for $T(r,s,t)$ and their alternating series. We need the following elementary lemma which can be proved by formal calculation.

LEMMA 8. *Suppose* $\{P_m\}$ *and* $\{Q_m\}$ *are sequences which satisfy*

$$\sum_{\nu=0}^{m}P_{m-\nu}\,\frac{(i\pi)^{2\nu}}{(2\nu+1)!}=Q_m\qquad\text{for any } m\in\mathbb{N}\cup\{0\}.$$

Then

$$P_m=-2\sum_{\nu=0}^{m}\phi(2m-2\nu)Q_\nu\qquad\text{for any } m\in\mathbb{N}\cup\{0\}.$$

Proof. By the assumption, we have

$$\Big(\sum_{m=0}^{\infty}P_m x^{2m}\Big)\frac{e^{i\pi x}-e^{-i\pi x}}{2}=\sum_{m=0}^{\infty}\Big\{\sum_{\nu=0}^{m}P_{m-\nu}\,\frac{(i\pi)^{2\nu+1}}{(2\nu+1)!}\Big\}x^{2m+1}$$
$$=(i\pi x)\sum_{m=0}^{\infty}\,Q_m x^{2m}.$$

Recalling the definition of the Bernoulli numbers (see e.g. [1, Chap. 1]), we can verify that

$$\frac{(2i\pi x)e^{i\pi x}}{e^{2i\pi x}-1}=\sum_{n=0}^{\infty}(2-2^{2n})B_{2n}\,\frac{(i\pi x)^{2n}}{(2n)!}=\sum_{n=0}^{\infty}(2-2^{2-2n})\zeta(2n)x^{2n}.$$

Since $\phi(s)=(2^{1-s}-1)\zeta(s)$, we have

$$\sum_{m=0}^{\infty}P_m x^{2m}=\Big(\sum_{k=0}^{\infty}Q_k x^{2k}\Big)\Big(-2\sum_{n=0}^{\infty}\phi(2n)x^{2n}\Big),$$

and the asserted formula follows. ■

Proposition 3. *Suppose* $k,l,m\in\mathbb{N}\cup\{0\}$ *with* $2k+l\geq 1$. *Then*

$$\begin{aligned}(4.1)\quad & A(2m+1;2k+1,l)\\ &=2\sum_{\nu=0}^{m}\phi(2m-2\nu)\sum_{j=0}^{k}\phi(2k-2j)\\ &\quad\times\sum_{\mu=0}^{j}\frac{(i\theta)^{2\mu}}{(2\mu+1)!}\binom{2\nu+1+2j-2\mu}{2j-2\mu}\zeta(2\nu+2+2j+l-2\mu).\end{aligned}$$

Proof. Corollary 2 with $\theta=\pi$ and $d=2q+3$, where $q\geq 0$, gives

$$\begin{aligned}&\sum_{j=0}^{k}\phi(2k-2j)\sum_{\mu=0}^{j}\frac{(i\pi)^{2\mu}}{(2\mu+1)!}\binom{2q+1+2j-2\mu}{2j-2\mu}\zeta(2q+2+2j+l-\mu)\\ &\qquad=-\sum_{\nu=0}^{q}A(2q+1-2\nu;2k+1,l)\,\frac{(i\pi)^{2\nu}}{(2\nu+1)!}.\end{aligned}$$

Then, by applying Lemma 8 with $P_m=A(2m+1;2k+1,l)$, and

$$\begin{aligned}Q_m=&-\sum_{j=0}^{k}\phi(2k-2j)\sum_{\mu=0}^{j}\frac{(i\pi)^{2\mu}}{(2\mu+1)!}\\ &\times\binom{2m+1+2j-2\mu}{2j-2\mu}\zeta(2m+2+2j+l-2\mu),\end{aligned}$$

for $m\geq 0$, we obtained the asserted formula for $A(2m+1;2k+1,l)$. ■

By applying Proposition 3 in the case $m=k$ and $l=2k+1$ for $k\in\mathbb{N}\cup\{0\}$, and using (2.14) and (3.8), we have the following formula. This gives a partial answer to the Subbarao–Sitaramachandrarao problem posed in [4, §5].

COROLLARY 3. *For $k \in \mathbb{N} \cup \{0\}$,*

$$\sum_{m,n=1}^{\infty} \frac{(-1)^{m+n}}{m^{2k+1} n^{2k+1} (m+n)^{2k+1}}$$

$$= -2 \sum_{j=0}^{k} \binom{2k+2j+1}{2j+1} \phi(2k-2j) \phi(4k+2j+3)$$

$$+ 4 \sum_{\nu=0}^{k} \phi(2k-2\nu) \sum_{j=0}^{k} \phi(2k-2j) \sum_{\mu=0}^{j} \binom{2\nu+1+2j-2\mu}{2j-2\mu}$$

$$\times \zeta(2k+2\nu+2j-2\mu+3) \frac{(i\pi)^{2\mu}}{(2\mu+1)!}.$$

REMARK. In the same way as above, it follows from Corollary 2 that

$$A(2m;2k,j) = -2 \sum_{j=1}^{m} \phi(2m-2j) \sum_{\nu=0}^{k} \phi(2k-2\nu)$$

$$\times \sum_{\mu=0}^{\nu-1} \binom{2j+2\nu-2\mu-1}{2\nu-2\mu-1} \zeta(2j+2\nu+l-2\mu) \frac{(i\pi)^{2\mu}}{(2\mu+1)!}$$

$$+ \zeta(2k+l)\phi(2m),$$

when $2k+l \geq 2$ and $m \geq 1$. In particular, it follows from (2.13) that we can also evaluate

$$\sum_{m,n=1}^{\infty} \frac{(-1)^{m+n}}{m^{2k} n^{2k} (m+n)^{2k}} + 2 \sum_{m,n=1}^{\infty} \frac{(-1)^{n}}{m^{2k} n^{2k} (m+n)^{2k}} \tag{4.2}$$

for $k \in \mathbb{N}$; however, neither of the two sums in (4.2) can be deduced separately.

By substituting (4.1) into (3.12) in the case $l = 2p$ and $d = 2q+1$, we obtain

THEOREM 2. *For $k, p \in \mathbb{N} \cup \{0\}$ with $k+p \geq 1$ and $q \in \mathbb{N}$,*

$$T(2k+1, 2p, 2q+1) - T(2k+1, 2q+1, 2p) + T(2p, 2q+1, 2k+1)$$

$$= -2 \sum_{j=0}^{k} (2^{1-2k+2j} - 1) \zeta(2k-2j)$$

$$\times \sum_{\nu=0}^{j} \binom{2q+1+2j-2\nu}{2j+1-2\nu} \zeta(2q+2+2j+2p-2\nu) \frac{(i\pi)^{2\nu}}{(2\nu)!}$$

$$+ 4 \sum_{N=0}^{q} \left\{ \sum_{\nu=0}^{q-N} (2^{1-2q+2N+2\nu} - 1)\zeta(2q-2N-2\nu) \right.$$

$$\times \sum_{j=0}^{k} (2^{1-2k+2j} - 1)\zeta(2k-2j)$$

$$\left. \times \sum_{\mu=0}^{j} \binom{2\nu+1+2j-2\mu}{2j-2\mu} \zeta(2\nu+2+2j+2p-2\mu) \frac{(i\pi)^{2\mu}}{(2\mu+1)!} \right\} \frac{(i\pi)^{2N}}{(2N)!}.$$

Example. We list several nontrivial relations for $T(r,s,N-r-s)$ deduced from Theorem 2 when $N = 10, 12$:

$$T(3,2,5) - T(3,5,2) + T(2,5,3) = -\frac{1}{935550}\pi^{10},$$

$$T(3,2,7) - T(3,7,2) + T(2,7,3) = -\frac{17}{127702575}\pi^{12},$$

$$2T(5,2,5) - T(5,5,2) = -\frac{19}{91216125}\pi^{12},$$

$$T(5,4,3) - T(5,3,4) + T(4,3,5) = \frac{19}{182432250}\pi^{12}.$$

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Department of Management Informatics
Tokyo Metropolitan College
Akishima, Tokyo 196-8540, Japan
E-mail: tsumura@tmca.ac.jp

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