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Jonas Kubilius (1921–2011)

by

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I. A short biography ([1]). Professor Jonas Kubilius, the luminary of the Lithuanian mathematical school, passed away on October 30, 2011, just after his ninetieth birthday celebration. He was one of the founders of probabilistic number theory, the author of monographs and influential papers in number theory, a renowned teacher and organizer of the mathematical life in Lithuania after the Second World War. His leadership as a rector of Vilnius University (1958–1991) and his contributions to the activities of the surviving Lithuanian state can hardly be overestimated.

Kubilius was born on July 27, 1921, in the village of Fermos (district of Jurbarkas, Lithuania) into a farmer's family. He was the oldest of the five sons. After primary school, he attended the pro-gymnasium in the neighboring village of Eržvilkas and gymnasium in the small western Lithuanian city Raseiniai. The latter, a school of classical type, gave good education in the humanities, including basic Latin and abilities to speak fluent German and French. Describing these years, Kubilius used to mention his attempts to write poetry, to make a radio apparatus, … and rediscovering the Pythagorean triples. The day of his school-graduation, June 16, 1940, was marked by a Red Army tank standing in the center of the city; the Molotov–Ribbentrop secret protocols went into action. Nevertheless, in October Kubilius became a freshman at the Mathematics and Natural Sciences Faculty of Vilnius University (VU).

The situation at the faculty was not easy at all. Nevertheless, the house-warming spirit was vibrant on the premises of VU, which has a long history going back to 1579. The Stefan Batory University which had functioned there between the wars stopped its activities and some faculties from Kaunas University were moved. Polish mathematicians, e.g., A. Zygmund, did not stay at the reorganized institution. The newly formed staff of the faculty was not active in mathematical research but was doing their best at

([1]) A literary biography of J. Kubilius was published by his student V. Stakėnas [39].

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lecturing. Lectures were given by Z. Žemaitis (a graduate of Odessa University), P. Katilius (with a PhD from Heidelberg University in 1928), and G. Žilinskas (the main results of his doctorate were obtained during 1937–1939 at Queen Victoria University, Manchester, under the guidance of L. J. Mordell), to mention but a few. Continuing repressions of the Soviet NKVD at the university until the Nazi invasion and the threat to be taken to an Arbeitslager during the second occupation marred the studies. After the Nazis closed the University in 1943, Kubilius returned to his native village. He spent a miserable time while crude battles waged nearby. The Lithuanian patriots resisted the return of the Soviets and their consolidation of hold in 1944. Kubilius also became a member of the resistance movement, hiding his activities under the beloved duties of a teacher of mathematics at the Eržvilkas school. Finally, the desire to return to serious studies of his favorite subject overwhelmed him. A year later, he accepted the proposed laboratory assistant job and returned to the University. In 1946, after successfully passing the missed exams and defending a graduation thesis, *Measure of transcendence*, Kubilius received the university *summa cum laude* diploma.

An assistant professor position at VU would have been very desirable for a young mathematician; nevertheless, Kubilius was pressed to become director of Preliminary Courses. The lectures and a great load of administrative work in the post-war period took all his time; therefore, in 1948, he seized the opportunity to take up doctoral studies at Leningrad (now, Sankt-Petersburg) University and to join the research group of the Russian academician Yu. V. Linnik (1915–1972, see [12]) and his prominent disciples. The candidate-of-science thesis ([2]), *Investigations in the geometry of prime numbers*, was defended on June 21, 1951, at Leningrad University (opponent B. A. Venkov).

Kubilius declined the proposal to stay at a research center in Leningrad, and returned to Lithuania where Stalin's repressions were continuing. His brother Juozas had been imprisoned for his help to the resistance. His father passed away in winter, and was thereby spared witnessing the destiny of the other family members. Kubilius' mother and another brother Antanas were deported to Siberia, to the Krasnoyarsk Region, in October. Only in 1957 could the family gather together. The university was still kept under control. Having in mind his "bad" biography, for safety reasons, Kubilius preferred to accept a research fellow's position at the Physical and Technical Institute. Nevertheless, all the time he was giving lectures at VU. In 1956, the Institute was split into two parts. At the newly organized Institute of

([2]) In the Soviet Union, two levels of scientific degrees existed: the candidate's degree corresponded to a PhD, while the requirements for the Doctor of Science degree were much higher.

Physics and Mathematics, Kubilius became vice-director and head of the Mathematics Division. He started to implement the idea to concentrate the mathematical research in Lithuania on probability theory, a branch covering a wide spectrum of theoretical and practical problems. At the Faculty, he initiated a seminar which played a decisive role during the next few decades. Kubilius was always ready to propose problems and to advise anybody who wished to become a professional mathematician. The most gifted students were directed to scientific centers of the Soviet Union, the Western direction remaining closed for a long time. This period featured great personal accomplishments, namely, at this time, Kubilius elaborated an original probabilistic approach in number theory and obtained fundamental results on the value distribution of arithmetical functions. Nevertheless, a group of number-theorists headed by I. M. Vinogradov did not accept the advance in the new branch so benevolently. The Doctor of Science thesis titled *Investigations in probabilistic number theory* had to be formally attributed to probability theory. It was defended on November 21, 1957, at the V. A. Steklov Mathematical Institute (Moscow); the opponents Yu. V. Linnik, B. V. Gnedenko, and Yu. V. Prokhorov evaluated it highly. The scientific recognition and organizational activities considerably raised Kubilius' prestige among the Lithuanian scientists and intellectuals. The ruling authorities had to listen to his proposals.

In 1958, at the age of 37, Kubilius was appointed rector of Vilnius University. One could hardly suppose then that the appointment started the Kubilius epoch at the university, filled with constant attention to the development of modern branches of science and keeping the university among the main national centers cherishing cultural heritage despite the prevailing suppression at the time. For almost 33 years at the rector's position and later, he continued to do mathematical research. The time of long and annoying official meetings, the short stays or visits abroad (Austria, Canada, France, Finland, Germany, India, Italy, USA), and vacations were filled up by writing formulae. During his long career, he founded and remained the recognized leader of a scientific school (the genealogical tree may be found at http://www.mif.vu.lt/lmm/savadas/savadas.html). He advised 29 PhD students; many of them became active researchers. He never stopped organizing mathematical life in Lithuania. First, we have to mention national mathematical competitions for schoolchildren going on since 1952 (see the report [49]) because many of its prize winners later became Kubilius' students. In 1961, he founded the *Lithuanian Mathematical Journal* and stayed on the editorial board or was Chief Editor for many years. He organized the Lithuanian Mathematical Society (officially registered in 1962) and all annual national mathematical conferences held from 1958 up to his death. He was among the organizers of the regular number-theoretic conferences and schools in the former Soviet Union (two of which, in 1966 and 1974, were held in Lithuania) and Vilnius international conferences on probability theory and mathematical statistics (since 1973; the 10th was held in 2010). For two decades, he was also a member of the Editorial Board of *Acta Arithmetica*. Kubilius' great reputation led to many honors. He became a member of the Lithuanian Academy of Sciences in 1962. He held honorary doctoral degrees from the Latvian University (Riga), Charles University (Prague), and from the universities in Greifswald and Salzburg. He received many medals and awards, including the Lithuanian State Prizes (1958, 1981) for science achievements.

In the 1990s, Kubilius was among the politically active Lithuanian intellectuals supporting the independence movement. Having a long experience of work in the former Supreme Soviets of Lithuanian Republic and of the Union, after elections he took the deputy mandate of the Lithuanian Parliament (Lietuvos Seimas, 1992–1996). He was always ready to offer his energy to science, to the university, and to Lithuania. A list of his books, interviews, newspaper articles, social polemical publications exceeds one thousand items. Most of them are listed in the bibliographical book [20] and in the collection of his speeches [22]. His draft of *The History of Mathematics in Lithuania* remained unfinished.

II. Some personal memories. As a third year student, I attended Kubilius' courses on *Function Theory of a Real Variable* and *Probability Theory and Mathematical Statistics*. The lectures were taught in a dry and pedantic way. More impressive were the oral exams during which Professor Kubilius used to smoke heavily, thus giving us a lot of time to discover the right answer. (Kubilius stopped smoking in his seventies, after having been diagnosed with serious lung problems.) Since then, I was honored to become his student. This raised my duties to appear regularly in his house at the so-called Kubilius Tuesdays where many of his disciples gathered. The first question I was asked during my first visit was which foreign language I was able to speak. My reply about poor English was followed by the request to study a paper by P. Erdős written in German. After some time the experiment was repeated with a French article. Later I learned that such attention to linguistic skills had been inherited from Linnik with whom, as a non-Russian speaking person up to some time, Kubilius had to communicate in French or German. I was lucky because Kubilius never used to ask his teacher's frequent question: *Did you improve the given paper?*, followed by the inevitable *It's sad*! after a negative answer. On the contrary, Kubilius always encouraged us and gave advice, but refused to become a coauthor. We constantly borrowed books and reprints from his rich library which actually was the only one in Lithuania containing so many recent

Western publications. He always acknowledged the foreign colleagues' help and donations.

In the 1970s, Kubilius lectured on *Probabilistic Number Theory*, including his own latest results. They were well elaborated despite the unavoidable technical difficulties. Kubilius' seminar on Mondays was going on regularly. Sitting in the front row, he sometimes seemed to be sleeping, but even then the speaker was kept under control. In addition to mathematical errors, our atypical expressions in Lithuanian were corrected. Kubilius' memory was fascinating. Once celebrating a jubilee of our department and feeling a restaurant's warmth, we succeeded in provoking him to recite poetry. A long performance followed.

In September, 2011, the fifth conference on analytic and probabilistic methods in number theory dedicated to Kubilius' 90th jubilee was held in the Lithuanian sea resort of Palanga. The celebrant seemed a little weak, but active, and he listened to all the lectures in the morning sessions.

Kubilius' attention to his family was remarkable. Now, his son Kęstutis is a professor of mathematics and his daughter Birutė is a well-known doctor. Kubilius' life took a downturn after the death of his beloved wife Valerija whom he married in 1950. They had been a harmonious couple.

The news of his death came as a shock.

III. Mathematical works. Kubilius published two research monographs with the same title in Russian [65], [75]. The revised second one was translated into English in 1964 (reprinted in 1968, 1978 with corrections, 1992, 1997). The total list of his mathematical papers contains more than 100 items. Lithuanian mathematicians owe him a few university textbooks, investigations into the mathematical thought in the country, several collections of exercises for schoolchildren competitions, and an indispensable Lithuanian–English–Russian mathematical dictionary. He edited many proceedings of mathematical conferences, books and collections of papers on the history of science. We will not discuss this part of his heritage. A survey of Kubilius' work, presented below, shows his contribution to mathematics. It is influenced by the author's taste and expertise. Hopefully, the list of mathematical papers will fill up the gaps. For instance, we do not dare to discuss Kubilius' and his PhD students' (A. Bikelis, V. N. Lazakovič, R. Merkytė, A. Mitalauskas, F. Mišeikis, P. Vaitkus) contributions to probability theory. Nor do we comment on dozens of "formulae of the classical type", as Kubilius used to call them, for values of multiple Dirichlet functions, which have been presented in several talks (see [130]–[132], [134], [135], [141]) and remain scattered in unpublished manuscripts. On the other hand, we give some hints on the subsequent development of problems more familiar to us, though without including the supporting references. We refer the reader to the books by G. J. Babu [2], W. Philipp [32], V. G. Sprindžuk [40], P. D. T. A. Elliott [5]–[8], J.-L. Mauclaire [31], W. Schwarz and J. Spilker [38], G. Tenenbaum [42], J. Knopfmacher and W.-B. Zhang [21], as well as to surveys [3], [14], [19], [23], [27]–[29], [36], [37]. Kubilius himself also wrote a few surveys: [91], [104], [137]–[139].

1. Metric number theory. In the above mentioned diploma paper, Kubilius discussed Mahler's classification of real numbers and the following question. Let $d \in \mathbb{N}$, $P(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{Z}[x]$ be a polynomial, and $H := \max\{|a_k| : 0 \le k \le d\}$. Given positive constants c and $\varkappa$, consider the inequality

$$|P(x)| < cH^{-\varkappa d} \tag{1}$$

for almost all real x in the sense of Lebesgue measure. K. Mahler [26] established that the inequality can have only finitely many solutions $(a_0, \dots, a_d) \in \mathbb{Z}^{d+1}$ for almost all x, provided that $\varkappa \ge 4$. He further continued: "Vermutlich kann diese Schranke bis auf jede Zahl oberhalb Eins herabgedrückt werden". In 1939, J. F. Koksma extended the result up to $\varkappa \ge 3$. In his first paper published in 1949, Kubilius proved the hypothesis for the case $d = 2$.

THEOREM 1 ([44]). *Let $c > 0$ and $\varepsilon > 0$ be arbitrary. If $d = 2$ and $\varkappa \ge 1 + \varepsilon$, then inequality* (1) *has only finitely many solutions $(a_0, a_1, \dots, a_d) \in \mathbb{Z}^{d+1}$ for almost all x.*

The proof was based upon an approximation of dependent variables, namely, the task was to estimate the number of solutions $q \in \mathbb{N}$ to the system of inequalities

$$\|\omega q\| < \psi(q), \qquad \|\omega^2 q\| < \psi(q), \tag{2}$$

where $\|\cdot\|$ denotes the distance to the nearest integer and $\psi(q)$ is a positive decreasing function. Actually, it was sufficient to take $\psi(q) = q^{-(1+\varepsilon)/2}$. This problem goes back to A. Khinchin's works from the 1920s.

As observed by T. Schneider [35], Kubilius started the proof of Mahler's hypothesis. Indeed, a series of improvements obtained by W. J. LeVeque, F. Kash, and B. Volkmann followed. It was in 1964 that Schneider's prediction came true, for V. G. Sprindžuk (1936–1987), a Belorussian mathematician who was Kubilius' PhD student during 1959–1961, presented a complete proof of Mahler's hypothesis. Staying in Vilnius, Sprindžuk did a lot of preparatory work, by refining the required diophantine approximations. Some years later, the Mahler–Sprindžuk theorem was extended to polynomials of two variables and degree three by Kubilius' student R. Sliesoraitienė. For a historical account, we refer to [40] and [125].

After some time Kubilius returned to system (2) and sharpened his earlier result.

Theorem 2 ([66]). *Let $\psi(q)$ be a positive function defined for $q \geq q_0$ such that $\psi(q)\sqrt{q}$ is non-increasing. If the series*

$$\sum_{q \geq q_0} \psi(q)/\sqrt{q}$$

converges, then the system of inequalities (2) *has only a finite number of solutions $q \geq q_0$ for almost all ω.*

Nowadays, the metric theory of diophantine approximations, one of the branches of the genealogical tree rooted at Kubilius' research, is flourishing in Minsk. To list but a few names, we mention that by the end of the last century the scientific grandson V. Bernik and the grand-grandson V. Beresnevich successfully extended Khinchin's results to polynomials of an arbitrary degree; for further developments, see, e.g., [3].

Kubilius left a trace in the metric theory of continued fractions and other expansions, improving results by R. Fortet, M. Kac, and I. A. Ibragimov. Let $p_n(x)/q_n(x) := [a_1(x), \ldots, a_n(x)]$ be the nth convergent of the continued fraction representation of $x \in (0,1]$. As proved by Khinchin, $q_n(x)$ grows exponentially for almost all x. Answering P. Erdős' question, Ibragimov [18] showed that $\log q_n(x)$ obeys the Gaussian law. Note that in [13] this result is wrongly attributed to Philipp [32] who also missed the following result of Kubilius.

Theorem 3 ([82]). *Let μ denote the Lebesgue measure and $\Phi(x)$ be the distribution function of the standard normal law. There exists a positive constant σ such that*

$$\mu\bigl\{x \in (0,1] : \log q_n(x) - n\pi^2/(12\log 2) < x\sigma\sqrt{n}\bigr\} - \Phi(x) \ll \frac{\log^2 n}{\sqrt{n}}. \tag{3}$$

Here and afterwards, $a \ll b$ means $a = O(b)$ as $n \to \infty$.

This result and a similar estimate in the Fortet–Kac problem was announced in [82]. The proofs were not published. The probabilistic technique for weakly dependent random variables was refined by Kubilius' student G. Misevičius who succeeded in removing one logarithm on the right-hand side in (3). More than a decade later, other methods cultivated by T. Morita or P. Flajolet and B. Vallée [13] implied the optimal order $O(n^{-1/2})$.

2. Multidimensional algebraic number theory. Advised by Linnik, for the Candidate of Sciences thesis, Kubilius chose the problem of distribution of prime ideals in algebraic number fields. By the 1950s, the asymptotic behavior of the number of such ideals with the norm up to x, as $x \to \infty$, had been intensively studied. The extension of algebraic numbers to ideal numbers proposed by E. Hecke led to a question about the asymptotic distribution in specified regions of the appropriate space. The latter could be defined via Hecke characters. This became the main objective for the thesis. In a series of papers, Kubilius elaborated the needed analytic tools, refining and generalizing results obtained earlier by Hecke and H. Rademacher.

Below we use the terminology of [48]. Let α denote an ideal integer of an algebraic number field $\mathbb{K}$ of degree $n \geq 1$ over $\mathbb{Q}$. The Hecke characters ξ of the first kind form an infinite Abelian group with generators, say, $\xi_1, \ldots, \xi_{n-1}$. Hence $\xi = \xi_1^{m_1} \cdots \xi_{n-1}^{m_{n-1}}$, where $m_j \in \mathbb{Z}$, $1 \leq j \leq n-1$. In turn, the values $\xi_j(\alpha) = \exp\{2\pi i \omega_j(\alpha)\}$ uniquely define the vector of amplitudes $\bar{\omega}(\alpha) := (\omega_1(\alpha), \ldots, \omega_{n-1}(\alpha)) \in [0,1)^{n-1}$. Let $N(\alpha)$ stand for the norm of the ideal number α and $0 \leq \omega_j' \leq \omega_j'' < 1$, where $1 \leq j \leq n-1$. Define the parallelepiped

$$\mathcal{B} = \{(\omega_1, \ldots, \omega_{n-1}) : \omega_j' \leq \omega_j \leq \omega_j'',\ 1 \leq j \leq n-1\}$$

and the set of ideal numbers

$$K(x, \mathcal{B}) := \{\alpha : |N(\alpha)| \leq x,\ \bar{\omega}(\alpha) \in \mathcal{B}\}.$$

Let $\mathcal{P}$ denote the system of non-associated ideal prime numbers of $\mathbb{K}$. The problem is to investigate the behavior of

$$\pi(x; \mathcal{B}) := \operatorname{card}\{\mathfrak{p} \in \mathcal{P} : \mathfrak{p} \in K(x, \mathcal{B})\}$$

as $x \to \infty$. Hecke [16] established an asymptotic formula for it; however, his method based on Weyl's criteria of uniform distribution did not allow obtaining a remainder term estimate. On the other hand, Rademacher [33] succeeded in doing this for a real quadratic field.

Kubilius started with a result for the Gaussian numbers [45] and soon explored the general case. To illustrate his achievements, we quote just a partial result from an extensive paper [48].

Theorem 4 ([48]). *There exists a positive constant c such that*

$$\pi(x; \mathcal{B}) = A(K) \prod_{j=1}^{n-1} (\omega_j'' - \omega_j') \int_2^x \frac{du}{\log u} + O\big(x e^{-c\sqrt{\log x}}\big). \tag{4}$$

Here $A(K)$ is a positive constant depending on the field $\mathbb{K}$.

Actually, Kubilius dealt with the number of non-associated ideal primes belonging to a class mod $\mathfrak{m}$, where $\mathfrak{m}$ is a non-zero ideal integer. The second part of that paper was devoted to imaginary quadratic fields $\mathbb{Q}(\sqrt{d})$, where $d < 0$ is a square-free rational integer. Then the error term estimate in (4) was sharpened to $O(x \exp\{-c(\log x)^{4/7}(\log\log x)^{-3/7}\})$. This was achieved by the use of Vinogradov's method of trigonometric sums (Chapter II) as well as estimates of the zero-densities of Hecke Z-functions (Chapter III). The idea to employ zero-density theorems instead of zero-free zone estimates or even analogs of the Riemann hypothesis originated in Linnik's works. Kubilius fostered it a great deal.

In the case of the imaginary quadratic field $\mathbb{K} = \mathbb{Q}(\sqrt{d})$, the group of Hecke characters of the first kind mod $\mathfrak{m}$ is the infinite cyclic group generated by $\xi(\alpha) = e^{ig \arg \alpha}$, where g is the number of units mod $\mathfrak{m}$ in $\mathbb{K}$. If χ is a group character of the multiplicative group of the reduced residue system mod $\mathfrak{m}$, then $\varXi := \chi\xi^m$, $m \in \mathbb{Z}$, is called a Hecke character of the second kind. The Hecke Z-functions are defined via the series

$$Z(s, \varXi) = \sum_{\alpha} \varXi(\alpha) N(\alpha)^{-s}, \qquad \Re s := \sigma > 1,$$

where the summation runs over a fixed class of ideal integers except zero, and have analytic continuations into the whole complex plane. Various estimates of the functions $Z(s, \varXi)$, analogous to those known for the Dirichlet L-functions, were obtained in [48] and [50]. For instance, it was established that the zeros of $Z(s, \varXi)$ lie in the half-plane $\sigma \le 0$ or in the region

$$\left|\sigma - \frac{1}{2}\right| \le \frac{1}{2} - c\big(\log(|m| + 3)(|t| + 3) \log\log(|m| + 3)(|t| + 3)\big)^{-3/4},$$

where $c > 0$ is a constant. The next result stimulated a couple of refinements of zero-density estimates for Hecke Z-functions.

Let $N(\sigma, T, \varXi)$ be the number of zeros $\rho = \beta + i\gamma$ of $Z(s, \varXi)$ satisfying $\sigma \le \beta \le 1$, $2 < \gamma \le T$, and let $M > 1$.

Theorem 5 ([48]). *If $\sigma \ge 0.8$, then*

$$\sum_{\varXi,\, |m| \le M} N(\sigma, T, \varXi) \ll_\varepsilon \big(M^{\lambda_1(\sigma)} T^{\lambda_2(\sigma)}\big)^{3(1-\sigma)+\varepsilon}$$

for every $\varepsilon > 0$. Here

$$\lambda_1(\sigma) = \frac{2\sigma + 1}{2\sigma^2 - 3\sigma + 2}, \qquad \lambda_2(\sigma) = \frac{4\sigma}{3\sigma^2 - 6\sigma + 7}.$$

Kubilius always used to stress relations of the results just mentioned to the Landau hypothesis claiming that there are infinitely many rational prime numbers of the form $p = a^2 + 1$ with $a \in \mathbb{Z}$. Investigations of $\pi(x; \mathcal{B})$ in the case of imaginary quadratic fields for "narrowing" sets $\mathcal{B}$ as $x \to \infty$ imply the expressions $p = a^2 + b^2$, $a, b \in \mathbb{Z}$, with a small b for infinitely many rational primes p. In this regard, the paper [50] written in Lithuanian played an important role. In writing it in his mother tongue, Kubilius was motivated by his self-imposed duty to develop the mathematical terminology and to press local publishers to acquire the needed characters for the printing-machines of the time. We present one of the results proved in [50].

Let $K = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field with h classes. As previously, let ν and $\mathfrak{m}$ be coprime ideal integers, and $\varphi(\mathfrak{m})$ be the Euler function.

THEOREM 6 ([50]). *Let $0 \le \omega_1 < \omega_2 \le 2\pi$. There exists a positive constant $c_1 < 53/58$ such that*

$$\operatorname{card}\{\mathfrak{p} \in \mathcal{P} : N(\mathfrak{p}) \le x,\ \mathfrak{p} \equiv \nu \pmod{\mathfrak{m}},\ \omega_1 < \arg \mathfrak{p} \le \omega_2\} = \frac{g(\omega_2 - \omega_1)x}{2\pi\varphi(\mathfrak{m})\log x}(1 + o(1)) + O(x^{c_1+\varepsilon})$$

for arbitrarily small $\varepsilon > 0$.

The result is nontrivial for the angles $\omega_2 - \omega_1 > x^{c_1-1+\varepsilon}$. In the decompositions $p = a^2+b^2$ for infinitely many rational primes, this allows one to reach the level $|b| \le p^{\theta+\varepsilon}$ for some $\theta \le 0.3904$. These primes could be found even in an arithmetical progression. The extended Riemann hypothesis for the Hecke zeta-functions would give the bound $b = O(\log^{2/3} p)$. After the works of Kubilius' students K. Bulota (1929–1990) and M. Maknys (1944–1992), we have $\theta \le 0.19$. For a few decades, Kubilius kept abreast of the subject and used to propose themes to his doctoral students. His ideas were also implemented by J. Vaitkevičius, J. Urbelis, A. Matuliauskas, E. Gaigalas, and R.-D. Cibulskytė (1939-2009). A notable advance was made by Maknys who applied modern methods based on zero-density theorems (similar to those developed by E. Bombieri, A. I. Vinogradov, and H. Montgomery for Dirichlet L-functions) and Linnik's large sieve. The investigations carried out by Kubilius' group were followed by N. Kalniņš, A. Danilov, A. F. Lavrik, T. Mitsui, R. Schultz-Arenstorff, W. Duke, and many others.

Working with Linnik, Kubilius obtained [47] first results on expressions of products of rational primes p_1p_2 and $p_1p_2p_3$ via sums of two squares. For instance, one of his results asserts that there are infinitely many different primes p_1 and p_1 such that $p_1p_2 = a^2 + b^2$, where $a, b \in \mathbb{Z}$ and $|b| = O(\log p_1p_2)$. Nowadays, using other methods, one can prove the same claim with $b = 1$, nevertheless, the above approach leads to presentations of prime numbers belonging to an arithmetic progression via quadratic norm forms. Kubilius' results in this field remain a historical milestone.

3. Probabilistic Number Theory (PNT). In the middle 1950s, Kubilius turned to additive arithmetical functions and in a short period of time proposed a novel method based upon probability theory. Some ideas had already emerged in the investigations carried out by A. Wintner, Erdős, Kac, and others. However, it was Kubilius who became the "matchmaker" of the two disciplines and whose results led to their bonding. Linnik once commented: *Jonas Kubilius' research is a great contribution to science where two domains of mathematics—probability theory and number theory—intersect. Only some considerations and isolated facts had existed in this field before. Now he has shaped a comprehensive and far-reaching theory. The essential parallelism of number theory and probability theory has been established.*

This can be considered as having even a philosophical significance (see [4, p. 240]).

By definition, a mapping $h : \mathbb{N} \to \mathbb{R}$ is called an additive function if $h(lm) = h(l)+h(m)$ for every pair of coprime natural numbers l and m. Such functions are determined by the values $h(p^k)$, where p is a prime number and $k \geq 1$, i.e., $h(1) = 0$ and

$$h(m) = \sum_p h(p^{\alpha_p(m)}), \quad m > 1,$$

where $\alpha_p(m)$ is the multiplicity of a prime factor p in the canonical representation of m. The total number $\Omega(m)$ of prime factors and the number $\omega(m)$ of distinct prime factors of m are classical instances. Let ν_n be the probability measure on subsets of $\mathbb{N}$ such that $\nu_n(\{m\}) = 1/n$ if $m \leq n$. Here and afterwards, $n \in \mathbb{N}$, and $n \to \infty$ in all asymptotical relations.

Five papers by Kubilius [51]–[55] on the value distribution of additive functions appeared in 1955. One might guess that they had been stimulated by the Erdős–Wintner theorem [10] giving necessary and sufficient conditions under which $\nu_n(h(m) < x)$ converges to a limit distribution function at the continuity points of the latter (weakly converges) or by the Erdős–Kac theorem [11] claiming that

$$F_n(x) := \nu_n\big(\omega(m) - \log\log n < x(\log\log n)^{-1/2}\big) = \Phi(x) + o(1)$$

uniformly in $x \in \mathbb{R}$. No doubt, the theorems formulated in [9] by Erdős had also called Kubilius' attention to the subject. Anyhow, in most of the following research, he aimed at finding necessary and sufficient conditions under which there exist normalizing sequences $\alpha(n)$ and $\beta(n) > 0$ such that $\nu_n(h(m) - \alpha(n) < x\beta(n))$ weakly converges to a limit law. This problem is nowadays called the *Main Problem of PNT*. Having created a very fruitful approach, Kubilius covered a great portion of it. The method opened up new horizons for other applications.

The first obstacle in the subject is the fact that there are subsets $A \subset \mathbb{N}$ which do not possess an asymptotic density, i.e., $\lim \nu_n(A)$ does not exist. Secondly, all subsets possessing this density do not form an algebra of events and the asymptotic density is not countably additive. Finally, $\alpha_p(m)$, $p \leq n$, are dependent as random variables (r.vs) with respect to ν_n. The latter is seen from $\sum_{p\leq n} \alpha_p(m) \log p \leq \log n$ if $m \leq n$. The probability theory of the 1950s (or even nowadays) contributed very little to overcoming these difficulties. To illustrate Kubilius' method, we confine ourselves to the technically simpler case of strongly additive functions, i.e., we assume that $h(p^k) = h(p)$ for every p and $k \geq 1$. Then

$$h(m) := h^{(n)}(m) := \sum_{p\leq n} h(p)\delta_p(m),$$

where $m \le n$ and $\delta_p(m) := \mathbf{1}\{m \equiv 0 \bmod p\}$, is a sum of dependent r.vs. Here and afterwards $\mathbf{1}\{\dots\}$ denotes the indicator function. If r with $2 \le r := r(n) \le n$ is sufficiently small compared with n, then $\delta_p(m)$, $p \le r$, are weakly dependent in some sense. Kubilius was the first to find a quantitative approximation of them by independent r.vs. In the following, $\{\xi_p : p \text{ is a prime}\}$ is a family of independent Bernoulli r.vs defined on some probability space $\{\Omega, \mathcal{F}, P\}$ so that $P(\xi_p = 1) = 1 - P(\xi_p = 0) = 1/p$. As usual, denote by $\pi(r)$ the number of primes up to r. The following result, though formulated in a slightly different form, is called the *Kubilius Fundamental Lemma.*

Theorem 7 ([75]). *Let $2 \le r \le n$ be arbitrary. One can define a probability space $\{\Omega, \mathcal{F}, P\}$ carrying the Bernoulli r.vs ξ_p, $p \le r$, so that the total variation distance*

$$R := \frac{1}{2} \sum_{s_2,\dots,s_{\pi(r)} \in \mathbb{Z}_+} \bigl| \nu_n(\delta_p(m) = s_p,\, p \le r) - P(\xi_p = s_p,\, p \le r) \bigr|$$

satisfies

$$R \ll \exp\Bigl\{-c\frac{\log n}{\log r}\Bigr\}. \tag{5}$$

The constant $c > 0$ and that in $\ll$ are absolute.

What does the triple $\{\Omega, \mathcal{F}, P\}$ look like? Kubilius proposed three constructions, called nowadays *Kubilius models.* In one of them, he starts with $E(p) := \{m \in \mathbb{N} : m \equiv 0 \bmod p\}$, $p \le r$, and defines the finite algebra $\mathcal{F}$ generated by the events

$$E_k := \bigcap_{p|k} E(p) \cap \bigcap_{p|\frac{\Pi}{k}} \overline{E}(p),$$

where $k \,|\, \Pi := \prod_{p \le r} p$ and $\overline{E}(p) := \{m \in \mathbb{N} : m \not\equiv 0 \bmod p\}$. In other words, $\mathcal{F}$ comprises all events $A := \bigcup_k E_k$, where k runs through some set of divisors of Π. It remains to ascribe the probabilities

$$P(E_k) = \prod_{p|k} \frac{1}{p} \prod_{p|\frac{\Pi}{k}} \Bigl(1 - \frac{1}{p}\Bigr)$$

to each $k \,|\, \Pi$ and to extend them additively to all of A.

There is one curious detail concerning his models starting with the finite sets $\{m \le n : m \equiv 0 \bmod p\}$ instead of $E(p)$. In this case, it may happen that $P(E_k) > 0$ for E_k empty. In his monographs, Kubilius left this to the reader, despite the criticism of his more pedantic students. Using the axioms of probability theory, one could easily fill up the hole of the space with the very notations E_k of these empty sets. The total probability of all such events is of the order $O(\log^{-1} n)$ only, so there has not been any harm in applying the models.

The first estimate of R was based upon Brun's sieve (see [57] and [65]). The use of the Selberg sieve led to the sharper result presented as (5) above. The paper [108] contains the following refinement:

$$R \ll \rho^{-c\rho}, \quad \rho := \frac{\log n}{\log \max\{r, \log n\}}$$

with absolute constants. In the late 1970s, Kubilius wrote two chapters for the planned third Russian edition of his monograph. They were based upon the Fundamental Lemma with the just mentioned error. A somewhat better estimate of R was given in Elliott's exhaustive books [5] which systematized all the achievements before 1980. Their appearance stopped Kubilius' work on a new edition. It is worth mentioning here that Tenenbaum's result [41] gives the most exact estimate of R obtained so far.

Theorem 6 reduced the problems of the stochastic behavior of the truncated additive function $h^{(r)}(m)$, where $\log r = o(\log n)$, to appropriate sums of independent r.vs. To estimate the influence of the remainder $h(m) - h^{(r)}(m)$, Kubilius proposed using variance estimates. Let

$$D_n := D_n(h) := \frac{1}{n} \sum_{m \le n} (h(m) - A_n)^2,$$
$$A_n := A_n(h) := \sum_{p \le n} \frac{h(p)}{p}, \qquad B_n^2 := B_n^2(h) := \sum_{p \le n} \frac{h^2(p)}{p}.$$

Theorem 8 (Turán–Kubilius inequality, [57]). *There exists an absolute constant $C > 0$ such that $D_n \le CB_n^2$ for $n \ge 1$.*

The first estimate of D_n, but in terms of A_n, for nonnegative strongly additive functions such that the values $h(p)$ are bounded was obtained by P. Turán [43]. The following story lying behind the numerical estimates of the constant C shows Kubilius' persistence.

Set

$$\lambda_n := \sup_{h \not\equiv 0} D_n(h)/B_n^2(h).$$

In [106], Kubilius proved that $1.47 < \lambda_n < 2.08$ if $n \ge n_0$ and n_0 is a sufficiently large absolute constant. Two years later, the upper bound became 1.764. My seminar notes from November 23, 1980, contain Kubilius' proof that $\lambda_n \ge 3/2 + o(1)$. The next year, the upper bound for λ_n of the same quality was obtained. Elliott, having also contributed to the problem, observes on page 423 of [6] that Kubilius' result was presented at the Budapest meeting in 1981. Kubilius published a sharper result in 1983 (see

[119], [120]). Going along this path, he elaborated an elegant approach based on the spectral analysis of integral operators. In his next paper [123], it was proved that $\lambda_n = 3/2 + O(\log^{-1} n)$. A. Hildebrand [17] obtained the asymptotic value of λ_n by a somewhat different approach. A recent deep study of the Turán–Kubilius inequality on friable (smooth) numbers [30] perfectly showed the richness of the subject. There is also another point. The investigations of the variance D_n led to estimates of all power moments of additive functions (I. Z. Ruzsa, Elliott, I. Kátai, K.-H. Indlekofer, and many others).

For a fairly large class of strongly additive functions h, one has $D_n(h) \sim B_n^2(h)$ (a simple characterization of this remains an open problem so far). The latter quantity is used in the following widely accepted definition of the so-called *Kubilius H class.*

Definition 1. A strongly additive function h belongs to class H if $B_n(h) \to \infty$ and $B_r(h) \sim B_n(h)$ for some sequence $r = r(n) \to \infty$ such that $\log r = o(\log n)$.

In the framework of the H class, Kubilius demonstrated the power of his probabilistic method.

Theorem 9 ([55]). *Let $h \in H$. The distributions $V_n(x) := \nu_n\big(h(m) - A_n < xB_n\big)$ weakly converge to a limit law with variance 1 if and only if there exists a distribution function $K(u)$ such that*

$$\frac{1}{B_n^2} \sum_{p \le n} \frac{h^2(p)}{p} \mathbf{1}\{h(p) < uB_n\} \to K(u) \tag{6}$$

weakly. The logarithm of the characteristic function of the limit law equals

$$\log \varphi(t) = \int_{-\infty}^{\infty} (e^{itu} - 1 - itu) \frac{1}{u^2} \, dK(u), \qquad t \in \mathbb{R}.$$

Hence it follows that the Lindeberg condition, i.e., (6) with $K(u) = 0$ if $u < 0$ and $K(u) = 1$ if $u \ge 0$, implies the convergence $V_n(x) \to \Phi(x)$. This generalizes the above mentioned Erdős–Kac theorem. Conversely, if $h \in H$, the convergence to the normal law implies the Lindeberg condition. In the 1970s, instances of additive functions obeying the Gaussian limit law and not satisfying the Lindeberg condition were constructed. They showed that the necessity part in the limit theorems was much more involved. On the other hand, when proving sufficiency, the probabilistic method worked even beyond the Kubilius H class. It was applied to sequences of functions (not necessarily $h_n = h/B_n$) which were isolated by the following definition proposed by Ruzsa [34].

Definition 2. A sequence of strongly additive functions h_n is *of Kubilius type* if

$$\sum_{n^\varepsilon < p \le n} \frac{1}{p} \mathbf{1}\{|h_n(p)| \ge \delta\} = o(1)$$

for every $0 < \varepsilon < 1$ and every $\delta > 0$.

In full generality, the one-dimensional limit problem for additive functions has remained open so far. The books by Elliott [5] give a panoramic picture of its development.

Kubilius' fondness of probability theory reveals itself in his application of Brownian motion to arithmetical models. Set for brevity $t_{nq} = B_q/B_n$, where q is a prime number. In 1955, Kubilius [54] examined the asymptotic behavior of the frequencies

$$\nu_n(\psi_1, \psi_2) := \nu_n\big(\psi_1(t_{nq}) < B_n^{-1}(h^{(q)}(m) - A_q) \le \psi_2(t_{nq})\big),$$

where ψ_1, ψ_2 are sufficiently smooth functions, with $\psi_1(t) < 0 < \psi_2(t)$ for $0 \le t \le 1$. Assuming the condition $\max_{p \le n} |h(p)|/B_n =: \rho_n = o(1)$, he showed that the limit of $\nu_n(\psi_1, \psi_2)$ equals the probability that the trajectories of a standard Brownian motion on $[0, 1]$ remain within certain bounds. Later, in [108], the convergence rate was obtained in terms of ρ_n. In the contemporary terminology, we could reformulate the Kubilius result as one dealing with the distribution of a functional defined on arithmetic partial sum processes. The Donsker–Prokhorov–Skorokhod theory of weak convergence of processes in function spaces was established mainly in 1956. Kubilius' followers (Babu, Philipp, P. Billingsley, B. V. Levin, N. M. Timofeev, Kh. Kh. Usmanov, Manstavičius) had the latter at their disposal; therefore, they succeeded in arriving at necessary and sufficient conditions under which the processes defined via normalized truncated additive functions weakly converge to limiting processes with independent increments. It is worth stressing here that such asymptotic behavior occurs only for Kubilius type sequences. Beyond this class, one can model only processes with dependent increments. This supports once more the importance of the definitions presented above.

Linnik and Kubilius gained great recognition for their paper [64], where the arithmetically defined processes

$$\frac{1}{\sqrt{v}} \sum_{k \le vt} \left(\frac{m+k}{Q}\right)$$

were examined. Here $\left(\frac{a}{b}\right)$ denotes the Jacobi symbol, $Q = Q(n)$ runs through some square-free sequence of odd natural numbers and $v = v(Q) \to \infty$ sufficiently slowly. It was proved that the finite-dimensional distributions of this process with respect to the probability ν_n converge to those of the standard Brownian motion. Later N. N. Liashenko extended the claim up to the weak convergence of processes. The paper [64] comprises Chapter X of the book [12], where the following *Kubilius–Linnik problem* is stated: *Does there exist a sequence* $v = v(n) \to \infty$ *such that*

$$\frac{1}{\sqrt{v}} \sum_{k \le vt} \mu(m+k),$$

where $\mu(m)$ *is the Möbius function, converges to the Brownian motion*?

A rather complete list of references concerning arithmetic models of random processes is given in the survey [29].

Kubilius used to demonstrate the power of his method dealing with additive functions with shifted arguments (see [75]). Then analytic approaches were inapplicable because of the spoiled additivity. Kubilius' first PhD student R. Uždavinys and later Z. Kryžius extended the method to superpositions of additive functions and integer-valued polynomials. Stakėnas and J. Šiaulys made a notable advance for functions defined on rational numbers, and Z. Juškys applied it to functions defined on abstract arithmetical semigroups. A similar technique was elaborated on the so-called additive arithmetical semigroups (W.-B. Zhang).

4. Analytic methods in PNT. Kubilius' method does not take into account the contribution to the asymptotic distribution of the truncated part $h(m) - h^{(r)}(m)$ which can be eliminated by some assumptions. This is a disadvantage in convergence rate estimates. For instance, in 1956, Kubilius showed [58] that the error in the Erdős–Kac theorem obeys

$$\Delta_n(x) := F_n(x) - \Phi(x) \ll \frac{1}{\sqrt{\log\log n}} \left(e^{-x^2/2} (\log\log\log n)^2 + 1\right).$$

Later M. B. Barban and Uždavinys removed the exponent 2 from the threefold logarithm (see [75]). However, the estimate for small x is worse than $\Delta_n(x) \ll (\log\log n)^{-1/2}$, as expected by LeVeque in 1949 and proved by A. Rényi and P. Turán in 1958. For this reason, in the early 1960s, to analyze the mean-value of complex multiplicative functions, Kubilius began using analytic methods elaborated mainly by A. Selberg and H. Delange. The paper [74] starts a systematic study of $\Delta_n(x)$, which was included in the second edition of his monograph as Chapter IX. Firstly, Kubilius gave an asymptotic expansion of arbitrary length and, secondly, proposed a way to deal with so-called large deviations. By that time, probability theory due to the efforts of A. C. Berry, C.-G. Esseen, H. Cramér, and V. V. Petrov already had a suitable technique for independent r.vs. Kubilius "caught the train" and adopted the technique in number theory. We include here his result concerning large deviations.

THEOREM 10 ([73]). *Let $\varepsilon > 0$ be sufficiently small, $|x| \le \varepsilon\sqrt{\log\log n}$, and $\xi := x/\sqrt{\log\log n}$. Then*

$$(7)\qquad \frac{F_n(x)}{\Phi(x)} = \exp\biggl\{(\xi - (1+\xi)\log(1+\xi))\log\log n + \frac{1}{x^2}\biggr\} \times\biggl(1 + O\Bigl(\frac{|x|+1}{\sqrt{\log\log n}}\Bigr)\biggr)$$

if $x < 0$. The same holds for the ratio $(1 - F_n(x))/(1 - \Phi(x))$ if $x > 0$.

With the next result, Kubilius was already “ahead of the train”. Namely, he succeeded in constructing the asymptotic expansions of the error in (7), overcoming all inherent difficulties. The Lithuanian probabilists V. Statulevičius and L. Saulis benefited from being the first to learn about this advance. It helped them to elaborate analogous expansions for independent r.vs. The expansions of the error in (7), presented in the Russian monograph, were not included into its English translation [75]. The latter became more transparent.

A decade later, Kubilius extended (see [92], [96], [98]) his results on $\Delta_n(x)$ to a class of additive functions h such that the values $h(p)$ are close to 1 for an overwhelming set of prime numbers. In particular, for integer-valued functions h, the condition had the form

$$(8)\qquad \sum_{h(p)\ne 1} \frac{\log p}{p} < \infty.$$

G. Halász’ paper [15] become a new turning point in analytic methods of number theory. Kubilius elaborated it in full detail during his lectures in the 1968/69 academic year. Proposing problems in PNT for dissertation works to Manstavičius, Kryžius, A. Mačiulis, and G. Bareikis, he used to recommend this pioneering paper. Kubilius himself was thinking about an analog of the Berry–Esseen estimate well known in probability theory. Is it possible to estimate the error $V_n(x) - \Phi(x)$ via the sum of Lyapunov’s ratio

$$L_n := \frac{1}{B_n^3}\sum_{p\le n}\frac{|h(p)|^3}{p},$$

where $V_n(x)$ has been defined in Theorem 9? Kubilius confirmed this in [115] under an extra condition. A complete solution was later obtained by Mačiulis who proved that the exact order of the error is $O(L_n^{2/3})$, in contrast to the case of independent r.vs (see the comments [139]).

The local limit theorems proved by Kubilius, i.e., the assertions about the behavior of $\nu_n(h(m) = k)$ for integer-valued additive functions h, deserve a special interest. This concerns even the simple observation [94] that the limits $\lim \nu_n(h(m) = k)$, $k \in \mathbb{Z}$, exist and define a discrete distribution on $\mathbb{Z}$ if and only if the series $\sum_{h(p)\neq 0} p^{-1}$ converges. Its importance becomes clear in the abstract setting of group-valued functions, which was later analyzed by Ruzsa. Assuming (8) and other unavoidable conditions, Kubilius presented an exhaustive study [87] of local asymptotic laws. It included approximations of $\nu_n(h(m) = k)$ by Poisson probabilities with increasing parameter, asymptotic expansions and, of course, large deviations, generalizing the classical works by E. Landau and L. G. Sathe on the function $\omega(m)$. In the last case, the large deviation formulas he obtained are nontrivial in the region $|k - \log\log n| \le c_1\sqrt{\log\log n}$, where $c_1 > 0$ is a sufficiently small constant. Kubilius' student R. Skrabutėnas made a lot of attempts to generalize condition (8) by exploiting the ideas of Delange. More recently, G. Stepanauskas advanced the problem by using Halász' methods. Contemporary investigations of the local laws carried out by M. Balazard, D. Hensley, C. Pomerance, Hildebrand, Tenenbaum, and others provide deep information if k lies even beyond the above mentioned region.

During his long research activity, Kubilius constantly followed the advances in probability theory. In the review of Kubilius' monograph, LeVeque called it a "sure grasp of the two fields". Kubilius was among the first who understood the importance for number theory of the Mellin transforms proposed by V. M. Zolotarev and encouraged his student A. Laurinčikas to introduce more general transforms. They perfectly substituted the characteristic functions in problems of multiplication of r.vs. Given a real-valued multiplicative function $g(m)$, one could seek some normalizing sequences $\alpha(n) \in \mathbb{R}$ and $\beta(n) > 0$ such that the distribution function

$$G_n(x) := \nu_n(e^{-\alpha(n)}|g(m)|^{1/\beta(n)} \operatorname{sgn} g(m) < x)$$

weakly converges to a limit one, say $G(x)$, including also $G_n(\pm 0) \to G(\pm 0)$. Correcting some errors in Zolotarev's original papers, Kubilius' student A. Bakštys studied this problem. Kubilius jointly with Juškys [93] proposed a version of the Esseen inequality in terms of the newly introduced transforms and obtained a convergence rate, and later jointly with Laurinčikas [97] examined large deviations. J. Galambos, Levin, Timofeev, S. T. Tulyaganov, and others went further along this path. Bareikis and Manstavičius succeeded in defining models of random processes using truncated multiplicative functions. So far, some of these results do not have analogs for products of independent r.vs.

Kubilius' influence on mathematical science is not limited to his personal results, though they have been pioneering in number theory. He actually encouraged applying more probability theory. For Dirichlet series, this has been implemented by his students E. Stankus and Laurinčikas (see [24]), and other mathematicians. The theories of value distribution of mappings defined on semigroups (see [21]) or on decomposable structures (see [1]) were other instances. I have been fortunate enough to be a witness to the whole progress.

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