

Piatetski-Shapiro sequences

by

ROGER C. BAKER (Provo, UT), WILLIAM D. BANKS (Columbia, MO),
 JÖRG BRÜDERN (Göttingen), IGOR E. SHPARLINSKI (Sydney),
 and ANDREAS J. WEINGARTNER (Cedar City, UT)

1. Introduction. Throughout the paper, the integer part of a real number t is denoted by $\lfloor t \rfloor$.

The *Piatetski-Shapiro sequences* are sequences of the form

$$(\lfloor n^c \rfloor)_{n \in \mathbb{N}} \quad (c > 1, c \notin \mathbb{N}).$$

They are named in honor of Piatetski-Shapiro, who proved (cf. [22]) that for any number $c \in (1, \frac{12}{11})$ there are infinitely many primes of the form $\lfloor n^c \rfloor$. The admissible range for c in this theorem has been extended many times over the years, and the result is currently known for all $c \in (1, \frac{243}{205})$ (cf. Rivat and Wu [23]).

In the present paper we examine various arithmetic questions about the Piatetski-Shapiro sequences. For instance, denoting by $P(m)$ the largest prime factor of an integer $m \geq 2$, we exhibit a positive function $\theta(c)$ which has the property that, for any non-integer $c > 1$ and real $\varepsilon > 0$, the inequality

$$(1.1) \quad P(\lfloor n^c \rfloor) > n^{\theta(c) - \varepsilon}$$

holds for infinitely many n . Our results extend and improve the earlier work of Abud [1] and of Arkhipov and Chubarikov [3]. The latter authors claim that for any $c \in (1, 2)$ one has

$$P(\lfloor n^c \rfloor) > n^{(27-13c)/28-\varepsilon}$$

for infinitely many n ; however, since they do not establish a result similar to our Proposition 18 (see §4) to eliminate prime powers p^k with $k \geq 2$, their result cannot be substantiated for $c \geq \frac{149}{87} = 1.712\dots$. The results presented here are much sharper than those in [3] and cover a wider range.

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Throughout the paper, we make the convention that if a result is stated in which ε appears, then ε denotes an arbitrary, sufficiently small positive number.

THEOREM 1. *Let $\theta(c)$ be the piecewise linear function given by*

$$\theta(c) = \begin{cases} 2 - c & \text{if } \frac{243}{205} \leq c < \frac{24979}{20803}, \\ 3 - 2c & \text{if } \frac{24979}{20803} \leq c \leq \frac{112}{87}, \\ (92 - 49c)/68 & \text{if } \frac{112}{87} \leq c \leq \frac{160}{117}, \\ (74 - 31c)/86 & \text{if } \frac{160}{117} \leq c \leq \frac{128}{85}, \\ (23 - 10c)/25 & \text{if } \frac{128}{85} \leq c \leq \frac{31}{20}, \\ (4 - 2c)/3 & \text{if } \frac{31}{20} \leq c \leq \frac{5}{3}, \\ (3 - c)/6 & \text{if } \frac{5}{3} \leq c < 2. \end{cases}$$

Then for any $c \in [\frac{243}{205}, 2)$ the inequality (1.1) holds for infinitely many n .

THEOREM 2. *There exists a constant $\beta > 0$ such that, for any $c > 2$, $c \notin \mathbb{N}$, the inequality*

$$P(\lfloor n^c \rfloor) > n^{\beta/c^2}$$

holds for infinitely many n .

Theorem 1 is proved in §§5–6; Theorem 2 is proved in §5.

The most important tool for our proof of Theorem 1 is the following exponential sum estimate, which is obtained by adapting the work of Cao and Zhai [11] (actually, our result is much simpler in form than that in [11]). Here and below we use notation like $m \sim M$ as an abbreviation for $M < m \leq 2M$, and $(m_1, \dots, m_k) \sim (M_1, \dots, M_k)$ means that $m_1 \sim M_1, \dots, m_k \sim M_k$.

THEOREM 3. *Let*

$$S = \sum_{(m, m_1, m_2) \sim (M, M_1, M_2)} a(m)b(m_1, m_2)e(Am^\alpha m_1^\beta m_2^\gamma),$$

where $M, M_1, M_2 \geq 1$, $A \neq 0$, $|a(m)| \leq 1$, $|b(m_1, m_2)| \leq 1$, and the constants α, β, γ satisfy $\alpha(\alpha - 1)(\alpha - 2)\beta\gamma \neq 0$. Writing $N = M_1M_2$ and $F = |A|M^\alpha M_1^\beta M_2^\gamma$ we have

$$\begin{aligned} S(MN)^{-\varepsilon} &\ll M^{5/8}N^{7/8}F^{1/8} + MN^{7/8} + M^{37/49}N^{46/49}F^{3/49} \\ &\quad + M^{23/29}N^{27/29}F^{3/58} + M^{43/58}N^{27/29}F^{2/29} \\ &\quad + M^{115/152}N^{7/8}F^{25/304} + M^{41/54}N^{25/27}F^{7/108} \\ &\quad + M^{5/6}N + M^{11/10}NF^{-1/4}. \end{aligned}$$

This is proved in §3.

As another application of Theorem 3 we give in §4 a detailed proof of a result sketched by Cao and Zhai [12]; their earlier paper [10] covers the narrower range $1 < c < \frac{61}{36}$.

THEOREM 4. For fixed $c \in (1, \frac{149}{87})$ we have

$$(1.2) \quad \#\{n \leq x : \lfloor n^c \rfloor \text{ is squarefree}\} = \frac{6}{\pi^2} x + O(x^{1-\varepsilon}).$$

A third application of Theorem 3 is the following result, which is needed for our proof of Theorem 1 and may be of independent interest; the proof is given in §4.

THEOREM 5. For fixed $c \in (1, \frac{149}{87})$ the inequality

$$\sum_{p \leq x^c} \log p \sum_{\substack{n \leq x \\ p | \lfloor n^c \rfloor}} 1 > (c - \varepsilon) x \log x$$

holds for all sufficiently large x .

A question that has not been previously considered is the following: for which values of c is it true that one has

$$P(\lfloor n^c \rfloor) \leq n^\varepsilon$$

for infinitely many n ? In this paper, we show that this is the case whenever

$$(1.3) \quad 1 < c < \frac{24979}{20803} = 1.2007\dots$$

More precisely, we prove the following result in §6.

THEOREM 6. For any number c in the range (1.3) we have

$$\#\{n \leq x : P(\lfloor n^c \rfloor) \leq n^\varepsilon\} \gg x^{1-\varepsilon}.$$

Finally, we consider a problem connected with *Carmichael numbers*, which are *composite* natural numbers N with the property that $N | a^N - a$ for every $a \in \mathbb{Z}$. The existence of infinitely many Carmichael numbers was established in 1994 by Alford, Granville and Pomerance [2]. In §7 we adapt the method of [2] to prove the following result.

THEOREM 7. For every $c \in (1, \frac{147}{145})$ there are infinitely many Carmichael numbers composed entirely of primes from the set

$$\mathcal{P}^{(c)} = \{p \text{ prime} : p = \lfloor n^c \rfloor \text{ for some } n \in \mathbb{N}\}.$$

We call the members of $\mathcal{P}^{(c)}$ *Piatetski-Shapiro primes*. The proof of Theorem 7 requires a considerable amount of information about the distribution of Piatetski-Shapiro primes in arithmetic progressions. Here, we single out one such result. Writing

$$\pi(x; d, a) = \#\{p \leq x : p \equiv a \pmod{d}\},$$

$$\pi_c(x; d, a) = \#\{p \leq x : p \in \mathcal{P}^{(c)}, p \equiv a \pmod{d}\},$$

we establish the following result in §7.

THEOREM 8. *Let a and d be coprime integers, $d \geq 1$. For fixed $c \in (1, \frac{18}{17})$ we have*

$$\begin{aligned} \pi_c(x; d, a) &= \gamma x^{\gamma-1} \pi(x; d, a) + \gamma(1-\gamma) \int_2^x u^{\gamma-2} \pi(u; d, a) du \\ &\quad + O(x^{17/39+7\gamma/13+\varepsilon}). \end{aligned}$$

We remark that, for each of the various results obtained in the present paper, the admissible range of c depends on the quality of our bounds for certain exponential sums; the particular type of exponential sum that is needed varies from one application to the next.

2. Notation and preliminaries. As usual, for all $t \in \mathbb{R}$ we write

$$\mathbf{e}(t) = e^{2\pi it}, \quad \|t\| = \min_{n \in \mathbb{Z}} |t - n|, \quad \{t\} = t - [t].$$

We make considerable use of the sawtooth function

$$\psi(t) = t - [t] - \frac{1}{2} = \{t\} - \frac{1}{2}$$

along with the well-known approximation of Vaaler [25]: there exist numbers c_h ($0 < |h| \leq H$) and d_h ($|h| \leq H$) such that

$$(2.1) \quad \left| \psi(t) - \sum_{0 < |h| \leq H} c_h \mathbf{e}(th) \right| \leq \sum_{|h| \leq H} d_h \mathbf{e}(th), \quad c_h \ll \frac{1}{|h|}, \quad d_h \ll \frac{1}{H}.$$

We use the following basic exponential sum estimates several times.

LEMMA 9. *Let f be three times continuously differentiable on a subinterval \mathcal{I} of $(N, 2N]$.*

(i) *Suppose that for some $\lambda > 0$, the inequalities*

$$\lambda \ll |f''(t)| \ll \lambda \quad (t \in \mathcal{I})$$

hold, where the implied constants are independent of f and λ . Then

$$\sum_{n \in \mathcal{I}} \mathbf{e}(f(n)) \ll N\lambda^{1/2} + \lambda^{-1/2}.$$

(ii) *Suppose that for some $\lambda > 0$, the inequalities*

$$\lambda \ll |f'''(t)| \ll \lambda \quad (t \in \mathcal{I})$$

hold, where the implied constants are independent of f and λ . Then

$$\sum_{n \in \mathcal{I}} \mathbf{e}(f(n)) \ll N\lambda^{1/6} + N^{3/4} + N^{1/4}\lambda^{-1/4}.$$

Proof. See Graham and Kolesnik [15, Theorems 2.2 and 2.6]. ■

LEMMA 10. Fix $c \in (1, 2)$, and put $\gamma = 1/c$. Let z_1, z_2, \dots be complex numbers such that $z_k \ll k^\varepsilon$. Then

$$\sum_{\substack{k \leq K \\ k = \lfloor n^c \rfloor}} z_k = \gamma \sum_{k \leq K} z_k k^{\gamma-1} + \sum_{k \leq K} z_k (\psi(-(k+1)^\gamma) - \psi(-k^\gamma)) + O(1).$$

Proof. The equality $k = \lfloor n^c \rfloor$ holds precisely when $k \leq n^c < k+1$, or equivalently, when $-(k+1)^\gamma \leq -n < -k^\gamma$. Consequently,

$$\begin{aligned} \sum_{\substack{k \leq K \\ k = \lfloor n^c \rfloor}} z_k &= \sum_{k \leq K} z_k (\lfloor -k^\gamma \rfloor - \lfloor -(k+1)^\gamma \rfloor) \\ &= \sum_{k \leq K} z_k ((k+1)^\gamma - k^\gamma) + \sum_{k \leq K} z_k (\psi(-(k+1)^\gamma) - \psi(-k^\gamma)). \end{aligned}$$

The result now follows on applying the mean value theorem and taking into account that $\sum_{k \leq K} |z_k| k^{\gamma-2} \ll 1$. ■

LEMMA 11. (Erdős–Turán) Let $t_1, \dots, t_K \in \mathbb{R}$, $\beta \in (0, 1)$, and $H \geq 1$. Then

$$\#\{k \leq K : \{t_k\} \leq \beta\} - K\beta \ll \frac{K}{H} + \sum_{h \leq H} \frac{1}{h} \left| \sum_{k=1}^K \mathbf{e}(t_k h) \right|.$$

Proof. See Baker [4, Theorem 2.1]. ■

We need a simple “decomposition result” for sums of the form

$$\sum_{X < n \leq X_1} A(n) f(n),$$

where f is any complex-valued function, and $X_1 \sim X$. A *Type I sum* is a sum of the form

$$S_I = \sum_{\substack{k \sim K \\ X < kl \leq X_1}} \sum_{l \sim L} a_k f(kl)$$

in which $|a_k| \leq 1$ for all $k \sim K$. A *Type II sum* is a sum of the form

$$(2.2) \quad S_{II} = \sum_{\substack{k \sim K \\ X < kl \leq X_1}} \sum_{l \sim L} a_k b_l f(kl)$$

in which $|a_k| \leq 1$ and $|b_l| \leq 1$ for all $(k, l) \sim (K, L)$. The following result can be derived from *Vaughan’s identity* (see Vaughan [26] or Davenport [13, Chapter 15]).

LEMMA 12. Suppose that every *Type I sum* with $L \gg X^{2/3}$ satisfies the bound

$$S_I \ll B(X)$$

and that every Type II sum with $X^{1/3} \ll K \ll X^{1/2}$ satisfies the bound

$$S_{II} \ll B(X).$$

Then

$$\sum_{X < n \leq X_1} \Lambda(n) f(n) \ll B(X) X^\varepsilon.$$

A standard procedure for estimating Type II sums with functions of the form $f(n) = \mathbf{e}(g(n))$ can be derived from the proof of [15, Lemma 4.13].

LEMMA 13. *Let $1 < Q \leq L$. If f is a function of the form $f(n) = \mathbf{e}(g(n))$, then any Type II sum (2.2) satisfies*

$$|S_{II}|^2 \ll X^2 Q^{-1} + X Q^{-1} \sum_{0 < |q| < Q} \sum_{l \sim L} |S(q, l)|,$$

where

$$S(q, l) = \sum_{k \in \mathcal{I}(q, l)} \mathbf{e}(g(kl) - g(k(l+q)))$$

for a certain subinterval $\mathcal{I}(q, l)$ of $(X, X_1]$.

3. Exponential sums with monomials. Theorem 3 is proved via the method of Cao and Zhai [11]. The upper bound in our theorem has nine terms, whereas in [11, Theorem 6] the corresponding upper bound has fourteen terms. Since Cao and Zhai omit the details of their optimization, we do not know how our optimization differs from theirs.

For the proof, we require four general results from the literature, which are reproduced here for the convenience of the reader; some other results are quoted during the course of the proof.

LEMMA 14. *Let $Y = (y_k)_{k \sim K}$ and $Z = (z_l)_{l \sim L}$ be two sequences of complex numbers with $|y_k| \leq 1$, $|z_l| \leq 1$. Let $\alpha_k, \beta_l \in \mathbb{C}$, and put*

$$S_{\alpha, \beta}(Y, Z) = \sum_{k \sim K} \sum_{l \sim L} \alpha_k \beta_l \mathbf{e}(B y_k z_l).$$

Then

$$|S_{\alpha, \beta}(Y, Z)|^2 \leq 20(1+B) S_\alpha(Y, B^{-1}) S_\beta(Z, B^{-1}),$$

where

$$S_\alpha(Y, B^{-1}) = \sum_{\substack{k, k' \sim K \\ |y_k - y_{k'}| \leq B^{-1}}} |\alpha_k \alpha_{k'}| \quad \text{and} \quad S_\beta(Z, B^{-1}) = \sum_{\substack{l, l' \sim L \\ |z_l - z_{l'}| \leq B^{-1}}} |\beta_l \beta_{l'}|.$$

Proof. See Bombieri and Iwaniec [8, Lemma 2.4]. ■

LEMMA 15. *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta \neq 0$, and let $K, L \geq 1$. Put*

$$u(k, l) = \frac{k^\alpha l^\beta}{K^\alpha L^\beta} \quad (k \sim K, l \sim L).$$

Then for any $C > 0$ we have

$$\#\{(k, \tilde{k}, l, \tilde{l}) : k, \tilde{k} \sim K, l, \tilde{l} \sim L, |u(k, l) - u(\tilde{k}, \tilde{l})| \leq C\} \\ \ll K L \log(2KL) + K^2 L^2 C.$$

Proof. See Fouvry and Iwaniec [14, Lemma 1]. ■

LEMMA 16. Let $N, Q \geq 1$, and let $Z = (z_n)_{n \sim N}$ be a sequence of complex numbers. Then

$$\left| \sum_{n \sim N} z_n \right|^2 \leq \left(2 + \frac{N}{Q}\right) \sum_{|q| \leq Q} \left(1 - \frac{|q|}{Q}\right) \sum_{n: N < n \pm q \leq 2N} z_{n+q} \bar{z}_{n-q}.$$

Proof. See [14, Lemma 2]. ■

LEMMA 17. Let

$$L(Q) = \sum_{j=1}^J C_j Q^{c_j} + \sum_{k=1}^K D_k Q^{-d_k},$$

where $C_j, c_j, D_k, d_k > 0$. Then

(i) For any $Q \geq Q' > 0$ there exists $Q_1 \in [Q', Q]$ such that

$$L(Q_1) \ll \sum_{j=1}^J \sum_{k=1}^K (C_j^{d_k} D_k^{c_j})^{1/(c_j+d_k)} + \sum_{j=1}^J C_j (Q')^{c_j} + \sum_{k=1}^K D_k Q^{-d_k}.$$

(ii) For any $Q > 0$ there exists $Q_1 \in (0, Q]$ such that

$$L(Q_1) \ll \sum_{j=1}^J \sum_{k=1}^K (C_j^{d_k} D_k^{c_j})^{1/(c_j+d_k)} + \sum_{k=1}^K D_k Q^{-d_k}.$$

Proof. See [15, Lemma 2.4] for a proof of the first assertion; the second assertion can be proved similarly. ■

Proof of Theorem 3. Let T_1, \dots, T_9 respectively denote the nine terms in the bound of the theorem.

Applying [14, Theorem 3] we have the bound

$$S \mathcal{L}^{-2} \ll M^{1/2} N^{3/4} F^{1/4} + M^{7/10} N + M N^{3/4} + M^{11/10} N F^{-1/4},$$

where $\mathcal{L} = \log(2MN)$. In the case of $F \leq M^2 N^{1/2}$ it follows that

$$S \mathcal{L}^{-2} \ll T_2 + T_8 + T_9,$$

and the theorem is proved; thus, we suppose from now on that $F \geq M^2 N^{1/2}$.

By Cauchy's inequality we have

$$|S|^2 \leq N \sum_{\substack{m_1 \sim M_1 \\ m_2 \sim M_2}} \left| \sum_{m \sim M} a(m) \mathbf{e}(A m^\alpha m_1^\beta m_2^\gamma) \right|^2.$$

Let Q be a parameter (to be optimized later) such that $10 \leq Q \leq M^{1/3-\varepsilon}$. Applying Lemma 16 to the inner sum, we obtain (after splitting the range of q into dyadic subintervals)

$$(3.1) \quad |S|^2 \mathcal{L}^{-1} \ll M^2 N^2 Q^{-1} + MNQ^{-1} \Sigma,$$

where

$$\Sigma = \sum_{\substack{m_1 \sim M_1 \\ m_2 \sim M_2}} \sum_{\substack{m \sim M \\ q_1 \sim Q_1}} c(m, q_1) \mathbf{e}(t(m, q_1) Am_1^\beta m_2^\gamma)$$

for some $Q_1 \in [\frac{1}{2}, Q]$, with

$$\begin{aligned} c(m, q_1) &= a(m + q_1) \overline{a(m - q_1)}, \\ t(m, q_1) &= (m + q_1)^\alpha - (m - q_1)^\alpha. \end{aligned}$$

Note that $|c(m, q_1)| \leq 1$ for all $(m, q_1) \sim (M, Q_1)$.

Next, we put $Q_2 = Q_1^2$ and again apply Cauchy's inequality, Lemma 16 and a dyadic splitting argument to derive the bound

$$(3.2) \quad \mathcal{L}^{-1} \Sigma^2 \ll M^2 N^2 Q_1^2 Q_2^{-1} + MNQ_1 Q_2^{-1} \Sigma_1 = M^2 N^2 + MNQ_1^{-1} \Sigma_1,$$

where

$$\Sigma_1 = \sum_{\substack{m_1 \sim M_1 \\ m_2 \sim M_2}} \sum_{\substack{m \sim M \\ q_1 \sim Q_1 \\ q_2 \sim Q_2^*}} c(m, q_1, q_2) \mathbf{e}(t(m, q_1, q_2) Am_1^\beta m_2^\gamma)$$

for some $Q_2^* \in [\frac{1}{2}, Q_2]$, with

$$\begin{aligned} c(m, q_1, q_2) &= c(m + q_2, q_1) \overline{c(m - q_2, q_1)}, \\ t(m, q_1, q_2) &= t(m + q_2, q_1) - t(m - q_2, q_1). \end{aligned}$$

Note that $|c(m, q_1, q_2)| \leq 1$ for all $(m, q_1, q_2) \sim (M, Q_1, Q_2)$.

We now partition the sum Σ_1 . To do this, we put

$$Q_a = \min\{Q_1, Q_2^*\} \quad \text{and} \quad Q_b = \max\{Q_1, Q_2^*\}.$$

Let f be the function defined by

$$f(q_1, q_2) = (q_1 q_2^{\alpha-1})^{1/(\alpha-2)},$$

and let $c' > c > 0$ be suitable constants (depending only on α) such that the interval

$$\mathcal{I} = [c f(Q_a, Q_b), c' f(Q_a, Q_b)]$$

contains all numbers of the form $f(q_1, q_2)$ with $(q_1, q_2) \sim (Q_a, Q_b)$. Let η be selected from the range

$$(3.3) \quad \max\{Q_a^2 Q_b^{-2}, 3\mathcal{L} Q_a^{-1} Q_b^{-1}\} \leq \eta \leq c'/c - 1.$$

Let $a_k = (1 + \eta)^k c f(Q_a, Q_b)$ and $\mathcal{I}_k = [a_k, (1 + \eta)a_k]$ for $0 \leq k \leq K$, where

$$K = \left\lfloor \frac{\log(c'/c)}{\log(1 + \eta)} \right\rfloor.$$

Note that $K \asymp \eta^{-1}$ for all η satisfying (3.3). Since $t(m, q_1, q_2) = t(m, q_2, q_1)$ we have

$$\Sigma_1 = \sum_{0 \leq k \leq K} \sum_{\substack{(q_1, q_2) \sim (Q_a, Q_b) \\ f(q_1, q_2) \in \mathcal{I}_k}} \sum_{\substack{m \sim M \\ m_1 \sim M_1 \\ m_2 \sim M_2}} \mathbf{e}(t(m, q_1, q_2) A m_1^\beta m_2^\gamma).$$

Let D_k be the number of 6-tuples

$$(m, \tilde{m}, q_1, \tilde{q}_1, q_2, \tilde{q}_2) \sim (M, M, Q_a, Q_a, Q_b, Q_b)$$

such that $f(q_1, q_2)$ and $f(\tilde{q}_1, \tilde{q}_2)$ lie in \mathcal{I}_k and

$$|t(m, q_1, q_2) - t(\tilde{m}, \tilde{q}_1, \tilde{q}_2)| \ll \frac{1}{|A| M_1^\beta M_2^\gamma},$$

and let E be the number of 4-tuples $(m_1, \tilde{m}_1, m_2, \tilde{m}_2) \sim (M_1, M_1, M_2, M_2)$ such that

$$|m_1^\beta m_2^\gamma - \tilde{m}_1^\beta \tilde{m}_2^\gamma| \ll \frac{1}{|A| M^{\alpha-2} Q_1 Q_2^*}.$$

An application of Lemma 14 for each value of k (taking $B \asymp M^{-2} F Q_1 Q_2^*$ and using the fact that $F \geq M^2 N^{1/2}$) yields the bound

$$\Sigma_1 \ll (M^{-2} F Q_1 Q_2^* E)^{1/2} \sum_{0 \leq k \leq K} D_k^{1/2}.$$

Using Cauchy's inequality again we have

$$(3.4) \quad \Sigma_1^2 \ll M^{-2} F Q_1 Q_2^* E \eta^{-1} \sum_{0 \leq k \leq K} D_k.$$

First assume that $\alpha \neq 3$.

If $Q_b > Q_a M^{\varepsilon/4}$ we are in a position to apply [11, Theorem 2]; the conditions $Q_b \leq M^{1-\varepsilon}$ and $Q_a Q_b \leq M^{3/2-\varepsilon}$ are certainly satisfied. For a suitably chosen η satisfying (3.3) we obtain the bound

$$(3.5) \quad M^{-\varepsilon} \eta^{-1} \sum_{0 \leq k \leq K} D_k \ll B_1,$$

where

$$\begin{aligned} B_1 &= M Q_a Q_b + M^4 F^{-1} Q_a Q_b + M^{1/4} Q_a^{7/4} Q_b^{9/4} + M^{-2} Q_a^4 Q_b^4 \\ &\quad + M^{3/4} F^{-1/8} Q_a^{7/4} Q_b^2 + M^3 F^{-1/2} Q_a + Q_a^{13/6} Q_b^{5/2} \\ &\quad + M F^{-1/4} Q_a^{7/4} Q_b^{9/4} + M^{-1/2} Q_a^{5/2} Q_b^3. \end{aligned}$$

In the case that $Q_b \leq Q_a M^{\varepsilon/4}$ we apply [11, Theorem 1] with the choices $K = 0$ and $\eta = c'/c$. Since the condition $Q_b \leq M^{2/3-\varepsilon}$ is clearly satisfied, we see that

$$M^{-\varepsilon/2} \eta^{-1} D_0 \ll M^{-\varepsilon/2} D_0 \ll M Q_a Q_b + M^4 Q_a Q_b F^{-1} + M^{-2} Q_a^2 Q_b^6 + Q_a^2 Q_b^{8/3}.$$

Since

$$M^{-2}Q_a^2Q_b^6 \leq M^{-2}Q_a^4Q_b^4 \cdot M^{\varepsilon/2} \quad \text{and} \quad Q_a^2Q_b^{8/3} \leq Q_a^{13/6}Q_b^{5/2} \cdot M^{\varepsilon/2},$$

we obtain (3.5) in this case as well.

Since $Q_a \leq Q_1$ and $Q_b \leq Q_2 = Q_1^2$ we find that

$$(3.6) \quad M^{-\varepsilon}\eta^{-1} \sum_{0 \leq k \leq K} D_k \ll B_2,$$

where

$$\begin{aligned} B_2 &= M Q_1^3 + M^4 F^{-1} Q_1^3 + M^{1/4} Q_1^{25/4} + M^{-2} Q_1^{12} + M^{3/4} F^{-1/8} Q_1^{23/4} \\ &\quad + M^3 F^{-1/2} Q_1 + Q_1^{43/6} + M F^{-1/4} Q_1^{25/4} + M^{-1/2} Q_1^{17/2}. \end{aligned}$$

We now notice that for $\alpha = 3$ we have $t(m, q_1, q_2) = 24mq_1q_2$, so the bound (3.6) is immediate in this case.

To bound E we use Lemma 15 to derive that

$$(3.7) \quad E \ll N \mathcal{L} + \frac{M^2 N^2}{F Q_1 Q_2^*}.$$

Combining (3.4), (3.6) and (3.7) leads to

$$M^{-2\varepsilon} \Sigma_1^2 \ll M^{-2} F Q_1 Q_2^* (N + M^2 N^2 / (F Q_1 Q_2^*)) B_2 \leq (M^{-2} N F Q_1^3 + N^2) B_2.$$

Taking into account (3.2) we see that

$$\begin{aligned} M^{-3\varepsilon} \Sigma^4 &\ll M^4 N^4 + M^2 N^2 Q_1^{-2} \cdot M^{-2\varepsilon} \Sigma_1^2 \\ &\ll M^4 N^4 + (F N^3 Q_1 + M^2 N^4 Q_1^{-2}) B_2. \end{aligned}$$

In the last expression only one term has a negative exponent of Q_1 , namely,

$$(M^2 N^4 Q_1^{-2})(M^3 F^{-1/2} Q_1) \ll M^5 N^4 F^{-1/2};$$

in the other terms, we replace Q_1 by Q . In view of (3.1) we derive the bound

$$\begin{aligned} |S|^8 M^{-4\varepsilon} &\ll M^8 N^8 Q^{-4} + M^4 N^4 Q^{-4} \cdot M^{-3\varepsilon} \Sigma^4 \\ &\ll M^8 N^8 Q^{-4} + M^5 N^7 F + M^8 N^7 + M^{17/4} N^7 F Q^{13/4} \\ &\quad + M^2 N^7 F Q^9 + M^{19/4} N^7 F^{7/8} Q^{11/4} + M^7 N^7 F^{1/2} Q^{-2} \\ &\quad + M^4 N^7 F Q^{25/6} + M^5 N^7 F^{3/4} Q^{13/4} + M^{7/2} N^7 F Q^{11/2} \\ &\quad + M^7 N^8 Q^{-3} + M^{10} N^8 F^{-1} Q^{-3} + M^{25/4} N^8 Q^{1/4} \\ &\quad + M^4 N^8 Q^6 + M^{27/4} N^8 F^{-1/8} Q^{-1/4} + M^9 N^8 F^{-1/2} Q^{-4} \\ &\quad + M^6 N^8 Q^{7/6} + M^7 N^8 F^{-1/4} Q^{1/4} + M^{11/2} N^8 Q^{5/2} \\ &= U_1 + \cdots + U_{19} \quad (\text{say}). \end{aligned}$$

Because $F \geq M^2$ and $Q \leq M^{1/3}$, we can discard U_{15} and U_{18} in view of the term $M^{5/6} N$ in the bound of Theorem 3. Collecting terms for which the

exponent of F is 1, we use $Q \leq M^{1/3}$ to eliminate U_5 and U_{10} :

$$U_5 \leq U_2 \quad \text{and} \quad U_{10} \leq U_8.$$

Collecting terms in which F is absent, we use $Q \leq M^{1/3}$ to eliminate U_{11} , U_{13} , U_{14} , U_{17} and U_{19} :

$$\max\{U_{11}, U_{13}, U_{14}, U_{17}, U_{19}\} \leq U_1.$$

We can also discard the term U_{16} since the bound $U_{16} \ll U_1$ follows from the inequalities $F \geq M^2$ and $Q \geq \frac{1}{2}$. Finally, the term U_{12} can be eliminated as the inequality $F \geq M^2 N^{1/2}$ implies that

$$U_{12} = M^{10} N^8 F^{-1} Q^{-3} \leq (M^8 N^8 Q^{-4})^{1/2} (M^7 N^7 F^{1/2} Q^{-2})^{1/2} = (U_1 U_7)^{1/2}.$$

After eliminating these terms, we are left with the bound

$$\begin{aligned} |S|^8 M^{-4\epsilon} &\ll M^4 N^7 F Q^{25/6} + (M^{17/4} N^7 F + M^5 N^7 F^{3/4}) Q^{13/4} \\ &\quad + M^{19/4} N^7 F^{7/8} Q^{11/4} + M^5 N^7 F + M^8 N^7 \\ &\quad + M^7 N^7 F^{1/2} Q^{-2} + M^8 N^8 Q^{-4}. \end{aligned}$$

Now we apply Lemma 17 to derive that

$$\begin{aligned} |S|^8 M^{-4\epsilon} &\ll M^5 N^7 F + M^8 N^7 + M^{223/37} N^7 F^{49/74} \\ &\quad + M^{296/49} N^{368/49} F^{24/49} + M^{131/21} N^7 F^{25/42} \\ &\quad + M^{184/29} N^{216/29} F^{12/29} + M^{125/21} N^7 F^{29/42} \\ &\quad + M^{172/29} N^{216/29} F^{16/29} + M^{115/19} N^7 F^{25/38} \\ &\quad + M^{164/27} N^{200/27} F^{14/27} + M^4 N^7 F + M^5 N^7 F^{3/4} \\ &\quad + M^{17/4} N^7 F + M^{19/4} N^7 F^{7/8} + M^{19/3} N^7 F^{1/2} + M^{20/3} N^8 \\ &= V_1 + \dots + V_{16} \quad (\text{say}). \end{aligned}$$

We can discard half of these terms using the following facts:

- (i) $V_3 \leq (V_1^{52} V_2^{22} V_9^{703})^{1/777}$;
- (ii) $V_5 = V_2^{2/21} V_9^{19/21}$;
- (iii) $V_7 = V_1^{2/21} V_9^{19/21}$;
- (iv) $\max\{V_{11}, V_{12}, V_{13}, V_{14}\} \leq V_1$;
- (v) $V_{15} \leq V_1^{1/2} V_2^{1/2}$.

Therefore, we arrive at the bound

$$\begin{aligned} |S_I|^8 M^{-4\epsilon} &\ll V_1 + V_2 + V_4 + V_6 + V_8 + V_9 + V_{10} + V_{16} \\ &= T_1^8 + T_2^8 + T_3^8 + T_4^8 + T_5^8 + T_6^8 + T_7^8 + T_8^8, \end{aligned}$$

as required. ■

4. On the divisibility of $\lfloor n^c \rfloor$ by squares. The following proposition is needed for the proofs of Theorems 4 and 5.

PROPOSITION 18. Fix $c \in (1, \frac{149}{87})$. Let $1 \leq D \leq x^{c/2}$, and let $(z_d)_{d \sim D}$ be a sequence of complex numbers such that $z_d \ll \log d$. Then

$$(4.1) \quad \sum_{d \sim D} z_d \sum_{\substack{n \leq x \\ d^2 | \lfloor n^c \rfloor}} 1 = x \sum_{d \sim D} \frac{z_d}{d^2} + O(x^{1-\varepsilon}).$$

Proof. First, suppose that $D \leq x^{2-c-6\varepsilon}$. Let S_d be the inner sum on the left-hand side of (4.1). By the argument used to prove Lemma 10, we see that

$$\begin{aligned} S_d &= \sum_{l \leq x^c/d^2} (\lfloor -(d^2 l)^\gamma \rfloor - \lfloor -(d^2 l + 1)^\gamma \rfloor) + O(1) \\ &= \sum_{l \leq x^c/d^2} ((d^2 l + 1)^\gamma - (d^2 l)^\gamma) - \sum_{l \leq x^c/d^2} \psi(-(d^2 l)^\gamma) \\ &\quad + \sum_{l \leq x^c/d^2} \psi(-(d^2 l + 1)^\gamma) + O(1). \end{aligned}$$

The mean value theorem yields the estimate

$$\sum_{l \leq x^c/d^2} ((d^2 l + 1)^\gamma - (d^2 l)^\gamma) = \gamma d^{\gamma-2} \sum_{l \leq x^c/d^2} l^{\gamma-1} + O(1) = \frac{x}{d^2} + O(1)$$

(see, e.g., LeVeque [21, pp. 138–139] for the last step). Hence, to finish the proof in this case it suffices to show that the bound

$$(4.2) \quad \sum_{l \leq x^c/d^2} \psi(-d^{2\gamma}(l + \xi)^\gamma) \ll D^{-1} x^{1-2\varepsilon}$$

holds uniformly for $0 \leq \xi < 1$. Applying [10, Lemma 3] with $\kappa = \lambda = \frac{1}{2}$, the left-hand side of (4.2) is

$$\begin{aligned} \sum_{l \leq x^c/d^2} \psi(-d^{2\gamma}(l + \xi)^\gamma) &\ll d^{2\gamma/3} (x^c/d^2)^{(1+\gamma)/3} + d^{-2\gamma} (x^c/d^2)^{1-\gamma} \\ &\ll D^{-2/3} x^{(c+1)/3} + D^{-2} x^{1-\gamma} \ll D^{-1} x^{1-2\varepsilon}, \end{aligned}$$

where we have used the inequality $D \leq x^{2-c-6\varepsilon}$ in the last step.

Next, we consider the case $D \geq x^{2-c-6\varepsilon}$. It suffices to show that the sum

$$(4.3) \quad S(D, L) = \sum_{d \sim D} \sum_{l \sim L} (\lfloor -(d^2 l)^\gamma \rfloor - \lfloor -(d^2 l + 1)^\gamma \rfloor)$$

satisfies the bound

$$S(D, L) \ll x^{1-3\varepsilon}$$

uniformly for all $L \geq 1$, $D^2L \leq x^c$. Noting that the summand in (4.3) is always either 0 or 1, and it is 0 whenever

$$\{-(d^2l)^\gamma\} > (d^2l + 1)^\gamma - (d^2l)^\gamma,$$

an application of Lemma 11 yields the bound

$$\begin{aligned} S(D, L) &\leq \sum_{\substack{d \sim D \\ \{-(d^2l)^\gamma\} \leq (D^2L)^{\gamma-1}}} \sum_{l \sim L} 1 \\ &\ll DL(D^2L)^{\gamma-1} + \frac{DL}{H_1} + \sum_{h \leq H_1} \frac{1}{h} \left| \sum_{d \sim D} \sum_{l \sim L} \mathbf{e}(h(d^2l)^\gamma) \right| \end{aligned}$$

for any number $H_1 \geq 1$; we choose $H_1 = DLx^{-1+3\varepsilon}$. Since

$$DL(D^2L)^{\gamma-1} = D^{-1}(D^2L)^\gamma \ll D^{-1}x \ll x^{1-3\varepsilon},$$

we need only show that for $\frac{1}{2} \leq H < H_1$ and any sequence $(b_h)_{h \sim H}$ of complex numbers with $|b_h| \leq 1$, the following bound holds uniformly:

$$(4.4) \quad S^* = \sum_{h \sim H} b_h \sum_{d \sim D} \sum_{l \sim L} \mathbf{e}(h(d^2l)^\gamma) \ll Hx^{1-3\varepsilon}.$$

If it is the case that $D > x^{2c-3+16\varepsilon}$ we can deduce (4.4) from Robert and Sargos [24, Theorem 3], which yields

$$(4.5) \quad S^* \ll x^\varepsilon DLH \left(\left(\frac{F}{DL^2H} \right)^{1/4} + L^{-1/2} + F^{-1} \right),$$

where

$$(4.6) \quad F = H(D^2L)^\gamma \leq Hx.$$

The second and third summands in (4.5) are easily dispatched. Indeed,

$$DL^{1/2}Hx^\varepsilon \ll Hx^{c/2+\varepsilon} \ll Hx^{1-3\varepsilon},$$

and

$$(4.7) \quad DLHF^{-1}x^\varepsilon \ll (D^2L)^{1-\gamma}x^\varepsilon \ll x^{c-1+\varepsilon} \ll Hx^{1-3\varepsilon}.$$

Taking into account (4.6) and the inequality $D > x^{2c-3+16\varepsilon}$, we have for the first summand in (4.5):

$$\begin{aligned} DLH \left(\frac{F}{DL^2H} \right)^{1/4} x^\varepsilon &= (D^2L)^{1/2} D^{-1/4} H^{3/4} F^{1/4} x^\varepsilon \\ &\leq (x^c)^{1/2} (x^{2c-3+16\varepsilon})^{-1/4} H^{3/4} (Hx)^{1/4} x^\varepsilon = Hx^{1-3\varepsilon}, \end{aligned}$$

which gives (4.4) and finishes the proof in this case.

We treat the remaining case $x^{2-c-6\varepsilon} < D \leq x^{2c-3+16\varepsilon}$ using Theorem 3. Let $\eta \asymp 1$ be a real number such that for $F \leq \eta L$ the derivative of the

function $l \mapsto h(d^2l)^\gamma$ has absolute value at most $1/2$ for $h \sim H$, $d \sim D$. If $F \leq \eta L$, the Kuz'min–Landau inequality (cf. [15, Theorem 2.1]) gives

$$S^* \ll DLHF^{-1},$$

and the proof is completed using the estimate (4.7). Now suppose that $F \geq \eta L$. We apply the B -process to the sum over l in S^* . Following the argument that yields [24, (6.10)] we have

$$\begin{aligned} S^* \ll & \frac{L}{F^{1/2}} \int_{-1/2}^{1/2} \left| \sum_{h \sim H} \sum_{d \sim D} \sum_{V < \nu \leq V_1} \mathbf{e}(\nu t) \mathbf{e} \left(\frac{Y h^{\bar{\beta}} d^{\bar{\gamma}} \nu^{\bar{\alpha}}}{H^{\bar{\beta}} D^{\bar{\gamma}} V^{\bar{\alpha}}} \right) \right| \min\{L, |t|^{-1}\} dt \\ & + DLHF^{-1/2} + DH \log D, \end{aligned}$$

where

$$V \asymp V_1 \asymp F/L, \quad Y \asymp F, \quad \bar{\beta} = \frac{1}{1-\gamma}, \quad \bar{\gamma} = \frac{2\gamma}{1-\gamma}, \quad \bar{\alpha} = \frac{\gamma}{1-\gamma}.$$

It is easy to see that

$$DLHF^{-1/2} = D^{-1}(D^2L)^{1-\gamma/2}H^{1/2} \ll H^{1/2}x^{2c-5/2+6\varepsilon}$$

since $D^2L \leq x^c$ and $D > x^{2-c-6\varepsilon}$, and that

$$DH \log D \ll Hx^{2c-3+17\varepsilon}$$

since $D \leq x^{2c-3+16\varepsilon}$. Taking into account that $c < \frac{7}{4}$ we obtain the bound

$$DLHF^{-1/2} + DH \log D \ll Hx^{1-3\varepsilon},$$

which is acceptable with regard to (4.4). To bound the integrand above, we apply Theorem 3 pointwise with $(F/L, DH)$ instead of (M, N) ; as a result, it suffices to show that

$$(F/L)^{5/8}(DH)^{7/8}F^{1/8} + \dots + (F/L)^{11/10}(DH)F^{-1/4} \ll (F^{1/2}/L)Hx^{1-4\varepsilon}.$$

Replacing F by $H(D^2L)^\gamma$, we now obtain nine separate bounds of the form

$$(4.8) \quad D^r L^s H^t (D^2L)^{\gamma u} \ll x^{v-C\varepsilon},$$

where C is a positive constant (not necessarily the same at each occurrence) and the numbers r, t, s, u, v satisfy

$$t \geq 0, \quad s + t \geq 0, \quad u \geq 0, \quad r \geq 2s + t.$$

Indeed, using the inequalities $H \leq DLx^{-1+3\varepsilon}$, $D^2L \leq x^c$, and $D \leq |x^{2c-3+16\varepsilon}|$, the left-hand side of (4.8) is

$$\begin{aligned} D^r L^s H^t (D^2L)^{\gamma u} & \leq D^{r+t} L^{s+t} (D^2L)^{\gamma u} x^{-t+3t\varepsilon} \\ & = D^{r-2s-t} (D^2L)^{s+t+\gamma u} x^{-t+3t\varepsilon} \\ & \leq (x^{2c-3+16\varepsilon})^{r-2s-t} (x^c)^{s+t+\gamma u} x^{-t+3t\varepsilon} \ll x^{v-C\varepsilon} \end{aligned}$$

provided that

$$(2c - 3)(r - 2s - t) + c(s + t) < t - u + v.$$

This leads to the bound

$$c < \min\left\{\frac{7}{4}, \frac{19}{11}, \frac{149}{87}, \frac{12}{7}, \frac{85}{49}, \frac{163}{95}, \frac{71}{39}\right\} = \frac{149}{87},$$

and the proof is complete. ■

Proof of Theorem 4. Using Proposition 18 and a dyadic splitting argument, the left-hand side of (1.2) is equal to

$$\sum_{n \leq x} \sum_{d^2 | \lfloor n^c \rfloor} \mu(d) = \sum_{d \leq x^{c/2}} \mu(d) \sum_{\substack{n \leq x \\ \lfloor n^c \rfloor \equiv 0 \pmod{d^2}}} 1 = x \sum_{d \leq x^{c/2}} \frac{\mu(d)}{d^2} + O(x^{1-\varepsilon}).$$

The theorem then follows by extending the series to infinity. ■

Next, we turn to the proof of Theorem 5, which eliminates p^k with $k \geq 2$ from a Chebyshev-style approach to establishing a lower bound for $P(\lfloor n^c \rfloor)$.

Proof of Theorem 5. Clearly,

$$(4.9) \quad \sum_{n \leq x} \log \lfloor n^c \rfloor \sim cx \log x.$$

The left-hand side of (4.9) may also be written as

$$\sum_{n \leq x} \sum_{d | \lfloor n^c \rfloor} \Lambda(d) = \sum_{d \leq x^c} \Lambda(d) \sum_{\substack{n \leq x \\ d | \lfloor n^c \rfloor}} 1 = \sum_{p \leq x^c} \log p \sum_{\substack{n \leq x \\ p | \lfloor n^c \rfloor}} 1 + E$$

where

$$0 \leq E \leq \sum_{k \geq 2, p \leq x^{c/k}} \log p \sum_{\substack{n \leq x \\ p^{2 \lfloor k/2 \rfloor} | \lfloor n^c \rfloor}} 1 = \sum_{d \leq x^c} a_d \sum_{\substack{n \leq x \\ d^2 | \lfloor n^c \rfloor}} 1.$$

Here,

$$a_d = \sum_{\substack{k \geq 2, p \leq x^{c/k} \\ p^{\lfloor k/2 \rfloor} = d}} \log p \leq 2 \log d \quad (d \leq x^c).$$

By Proposition 18 we have $E \ll x$, and Theorem 5 follows immediately. ■

5. Large prime factors of $\lfloor n^c \rfloor$

Proof of Theorem 1 for $c \in (\frac{24979}{20803}, \frac{5}{3})$. Let $\delta = \varepsilon^2$. We show that

$$(5.1) \quad \sum_{p \leq x^{\theta(c) - \delta}} \log p \sum_{\substack{n \leq x \\ p | \lfloor n^c \rfloor}} 1 \leq (\theta(c) + O(\varepsilon))x \log x$$

for all large x . In conjunction with Theorem 5 this establishes that there is a positive proportion of natural numbers $n \leq x$ divisible by some prime $p \geq x^{\theta(c)-\delta}$; thus, $P(n) > n^{\theta(c)-\delta}$ for such n .

We cover $[1, x^{\theta(c)-\varepsilon}]$ with $O(\log x)$ abutting intervals of the form

$$\mathcal{I}_D = [D, (1 + \varepsilon)D]$$

with $1 \leq D \leq x^{\theta(c)-\varepsilon}$. For each D we cover $[1, x^c/D]$ with $O(\log x)$ abutting intervals of the form

$$\mathcal{J}_L = [L, (1 + \varepsilon)L]$$

with $1 \leq L \leq x^c/D$. As in the proof of Lemma 10, the double sum in (5.1) is

$$(5.2) \quad \sum_{p \leq x^{\theta(c)-\varepsilon}} \log p \sum_{l \leq x^c/p} ([-(pl)^\gamma] - [-(pl+1)^\gamma]) + O(x^{\theta(c)-\varepsilon}).$$

Arguing as we did after (4.3), the contribution to (5.2) from the pairs (p, l) that lie in $\mathcal{I}_D \times \mathcal{J}_L$ is at most

$$W_{D,L}(\log D)(DL)^{\gamma-1}(\gamma + O(\varepsilon)) + O\left(\frac{W_{D,L}}{H_1} + \sum_{h \leq H_1} \frac{1}{h} \left| \sum_{(p,l) \in \mathcal{I}_D \times \mathcal{J}_L} \mathbf{e}(h(pl)^\gamma) \right|\right),$$

where

$$H_1 = DLx^{-1+\delta} \quad \text{and} \quad W_{D,L} = \#\{(p, l) \in \mathcal{I}_D \times \mathcal{J}_L\}.$$

Now

$$\begin{aligned} \sum_{D,L} W_{D,L}(\log D)(DL)^{\gamma-1}(\gamma + O(\varepsilon)) & \\ & \leq (1 + O(\varepsilon)) \sum_{p \leq x^{\theta(c)-\delta}} \log p \sum_{l \leq x^c/p} \gamma(pl)^{\gamma-1} \\ & \leq (1 + O(\varepsilon)) x \sum_{p \leq x^{\theta(c)-\delta}} \frac{\log p}{p} \\ & \leq (\theta(c) + O(\varepsilon)) x \log x. \end{aligned}$$

Hence it suffices to show that for any pair (D, L) , any number $H \in [1, H_1]$, and any sequence $(a_h)_{h \sim H}$ of complex numbers with $|a_h| \leq 1$, the following bound holds uniformly:

$$S^* = \sum_{h \sim H} a_h \sum_{(p,l) \in \mathcal{I}_D \times \mathcal{J}_L} \mathbf{e}(h(pl)^\gamma) \ll Hx^{1-\delta}.$$

We consider three separate cases.

CASE 1: $c \in [\frac{243}{205}, \frac{112}{87})$. We use [24, Theorem 3] to obtain the bound

$$(5.3) \quad S^* \ll x^\delta DLH \left(\left(\frac{F}{DL^2H} \right)^{1/4} + L^{-1/2} + F^{-1} \right).$$

Here we write

$$F = H(DL)^\gamma \leq Hx.$$

The last two terms in (5.3) are handled easily, for

$$x^\delta DL^{1/2}H \ll x^{c/2+\delta} D^{1/2}H \ll Hx^{1-\delta}$$

since $D \ll x^{2-c-4\delta}$, whereas

$$x^\delta DLHF^{-1} = x^\delta (DL)^{1-\gamma} \ll x^{c-1+\delta} \ll Hx^{1-\delta}.$$

For the first summand, we have

$$\begin{aligned} x^\delta DLH \left(\frac{F}{DL^2H} \right)^{1/4} &= x^\delta (DL)^{1/2} D^{1/4} H^{3/4} F^{1/4} \\ &\leq x^\delta (x^c)^{1/2} (x^{3-2c-\varepsilon})^{1/4} H^{3/4} (Hx)^{1/4} \ll Hx^{1-\delta} \end{aligned}$$

since $D \leq x^{3-2c-\varepsilon}$. This completes the proof in Case 1.

Now suppose $c \geq \frac{112}{87}$. Before separating the argument further, we observe that (using the Kuz'min–Landau inequality as in the proof of Proposition 18) it suffices to consider the case that $F \geq \eta L$ for an appropriate constant $\eta \asymp 1$. Following the argument that gives [24, (6.10)] we have

$$(5.4) \quad S^* \ll \frac{L}{F^{1/2}} \int_{-1/2}^{1/2} \left| \sum_{h \sim H} \sum_{d \sim D} \sum_{V < \nu \leq V_1} \mathbf{e}(\nu t) \mathbf{e} \left(\frac{Y h^{\bar{\beta}} d^{\bar{\alpha}} \nu^{\bar{\alpha}}}{H^{\bar{\beta}} D^{\bar{\alpha}} V^{\bar{\alpha}}} \right) \right| \min\{L, |t|^{-1}\} dt \\ + DLHF^{-1/2} + DH \log D,$$

where

$$V \asymp V_1 \asymp F/L, \quad Y \asymp F, \quad \bar{\beta} = \frac{1}{1-\gamma}, \quad \bar{\alpha} = \frac{\gamma}{1-\gamma}.$$

Since $F \gg L$ it is clear that

$$\begin{aligned} DLHF^{-1/2} + DH \log D &\ll DL^{1/2} Hx^\delta \leq D^{1/2} Hx^{c/2+\delta} \\ &\leq Hx^{(\theta(c)+c)/2+\delta} \ll Hx^{1-\delta}, \end{aligned}$$

thus it only remains to bound the integral in (5.4). We group the variables h, d, ν differently in the next two cases.

CASE 2: $c \in [\frac{112}{87}, \frac{160}{117})$. To bound the integrand, we apply Theorem 3 pointwise with (M, M_1, M_2) replaced by $(D, H, F/L)$, and thus it suffices to verify that

$$D^{5/8} N^{7/8} F^{1/8} + \dots + D^{11/10} N F^{-1/4} \ll (F^{1/2} L^{-1}) Hx^{1-2\delta}.$$

Since $F = H(DL)^\gamma$ and $N = M_1 M_2 = H^2(DL)^\gamma L^{-1}$, this gives rise to nine upper bounds of the form

$$(5.5) \quad D^r L^s H^t (DL)^{\gamma u} \ll x^{v-C\delta},$$

where C is a positive constant (not necessarily the same at each occurrence) and the numbers r, s, t, u, v satisfy

$$t \geq 0, \quad s + t \geq 0, \quad u \geq 0, \quad r \geq s.$$

Using the inequalities $H \leq DLx^{-1+\delta}$ and $DL \leq x^c$, we see that the left-hand side of (5.5) is

$$\leq D^{r+t}L^{s+t}x^{-t+u+t\delta} = D^{r-s}(DL)^{s+t}x^{-t+u+t\delta} \leq D^{r-s}x^{c(s+t)-t+u+t\delta};$$

therefore, (5.5) holds provided that

$$(5.6) \quad D \leq x^{(v+t-u-c(s+t))/(r-s)-\varepsilon}.$$

Taking all nine bounds into account, we must have $D \leq x^{\theta_1(c)-\varepsilon}$, where

$$\theta_1(c) = \min \left\{ \frac{7-4c}{4}, \frac{7-3c}{7}, \frac{92-49c}{68}, \frac{54-28c}{42}, \frac{54-29c}{39}, \frac{266-139c}{192}, \frac{100-53c}{74}, \frac{6-3c}{5}, \frac{20-5c}{22} \right\}.$$

After a simple computation one verifies that

$$\theta_1(c) = \frac{92-49c}{68} = \theta(c) \quad \text{for all } c \in \left[\frac{112}{87}, \frac{160}{117} \right),$$

so this completes the proof in Case 2.

CASE 3: $c \in \left[\frac{160}{117}, \frac{5}{3} \right)$. We proceed just as in Case 2 but with the roles of D and H interchanged, i.e., we apply Theorem 3 pointwise with (M, M_1, M_2) replaced by $(H, D, F/L)$, and we have $N = M_1M_2 = DH(DL)^\gamma L^{-1}$. We obtain nine new bounds of the form (5.6) with different values of r, s, t, u, v , and this leads to the requirement that $D \leq x^{\theta_2(c)-\varepsilon}$, where

$$\theta_2(c) = \min \left\{ \frac{5-2c}{6}, \frac{8-4c}{6}, \frac{74-31c}{86}, \frac{46-20c}{50}, \frac{43-18c}{50}, \frac{230-103c}{228}, \frac{82-35c}{92}, \frac{22-7c}{20} \right\}.$$

After a calculation, one verifies that $\theta_2(c) = \theta(c)$ for all $c \in \left[\frac{160}{117}, \frac{5}{3} \right)$. This completes the proof in Case 3 and finishes the proof of Theorem 1 for values of c in the interval $\left[\frac{24979}{20803}, \frac{5}{3} \right)$. ■

Not far to the right of $c = \frac{8}{5}$, it becomes more efficient to estimate the exponential sum

$$\sum_{n \sim N} e\left(\frac{hn^c}{q}\right)$$

in order to give a good lower bound for $P([n^c])$. We use this approach for values of $c \geq \frac{5}{3}$.

PROPOSITION 19. *Uniformly over all integers a we have:*

(a) *Fix $c \in \left(\frac{3}{2}, 2\right)$. For any natural number $q \leq N^{(3-c)/6-3\varepsilon}$ we have*

$$(5.7) \quad \#\{n \sim N : [n^c] \equiv a \pmod{q}\} = \frac{N}{q} + O\left(\frac{N^{1-\varepsilon}}{q}\right).$$

(b) *There exists a constant $\beta > 0$ with the property that for any fixed $c > 2$, $c \notin \mathbb{Z}$, the estimate (5.7) holds for any natural number $q \leq N^{\beta/c^2}$.*

From Proposition 19 we derive the following corollary, which establishes Theorem 1 for any $c \in [\frac{5}{3}, 2)$ and also establishes Theorem 2.

COROLLARY 20. *Let*

$$\theta_3(c) = \begin{cases} (3-c)/6 & \text{if } \frac{5}{3} \leq c < 2, \\ \beta/c^2 & \text{if } c > 2, c \notin \mathbb{Z}. \end{cases}$$

Then

$$(5.8) \quad P(\lfloor n^c \rfloor) > n^{\theta_3(c)-\varepsilon}$$

for infinitely many n .

Proof. Let p be a prime in the interval $[\frac{1}{2}N^{\theta_3(c)-\varepsilon/2}, N^{\theta_3(c)-\varepsilon/2}]$. Applying Proposition 19 with $\varepsilon/6$ in place of ε , the number of $n \sim N$ for which $p \mid \lfloor n^c \rfloor$ is $\gg N/p \gg N^{1-\theta_3(c)+\varepsilon/2}$ for all large N , and (5.8) holds for every such n . ■

LEMMA 21. *There is a constant $b \in (0, 1)$ such that for any $c > 2, c \notin \mathbb{Z}$, the bound*

$$\sum_{n \sim N} \mathbf{e}(\alpha n^c) \ll N^{1-b/c^2}$$

holds uniformly for all α such that $N^{-c/2} \leq |\alpha| \leq N^{c/2}$, where the implied constant depends only on c .

Proof. This is a special case of Karatsuba [20, Theorem 1]; see also Brüdern and Perelli [9, Lemma 10]. One can adapt the work of Baker and Kolesnik [7] to give an explicit value for b ; an even larger value for b would follow by incorporating the recent work of Wooley [27]. ■

Proof of Proposition 19. The condition $\lfloor n^c \rfloor \equiv a \pmod q$ is equivalent to

$$(5.9) \quad \frac{a}{q} \leq \left\{ \frac{n^c}{q} \right\} < \frac{a+1}{q}.$$

According to Lemma 11, the number of $n \sim N$ for which (5.9) holds is

$$\frac{N}{q} + O\left(\frac{N^{1-\varepsilon}}{q}\right) + O\left(\sum_{1 \leq h \leq qN^\varepsilon} \left| \sum_{n \leq N} \mathbf{e}\left(\frac{hn^c}{q}\right) \right|\right).$$

Thus, to deduce (a) it suffices show that the bound

$$(5.10) \quad \sum_{n \leq N} \mathbf{e}\left(\frac{hn^c}{q}\right) \ll \frac{N^{1-2\varepsilon}}{q} \quad (1 \leq h \leq qN^\varepsilon)$$

holds for any $q \leq N^{(3-c)/6-3\varepsilon}$. We apply Lemma 9(ii) with $\lambda \asymp hN^{c-3}q^{-1}$, which gives

$$\sum_{n \leq N} \mathbf{e}\left(\frac{hn^c}{q}\right) \ll \frac{N}{q} (h^{1/6}q^{5/6}N^{(c-3)/6} + qN^{-1/4} + h^{-1/4}q^{5/4}N^{-c/4}).$$

Taking into account the following bounds, which are valid for any $c \in (\frac{3}{2}, 3)$:

$$\begin{aligned} h^{1/6} q^{5/6} N^{(c-3)/6} &\leq q N^{(c-3)/6+\varepsilon} \leq N^{-2\varepsilon}, \\ q N^{-1/4} &\leq N^{1/4-c/6-3\varepsilon} \leq N^{-2\varepsilon}, \\ h^{-1/4} q^{5/4} N^{-c/4} &\leq q^{5/4} N^{-c/4} \leq N^{5/8-11c/24-2\varepsilon} \leq N^{-2\varepsilon}, \end{aligned}$$

we finish the proof of (a).

For part (b), choose any positive $\beta < \min\{1, b\}$, where b is the constant of Lemma 21. We must prove (5.10) for any $q \leq N^{\beta/c^2}$. Clearly, if $\varepsilon > 0$ is sufficiently small we have

$$N^{-c/2} \leq N^{-\beta/c^2} \leq \frac{h}{q} \leq N^\varepsilon \leq N^{c/2},$$

and by Lemma 21 it follows that

$$\sum_{n \leq N} \mathbf{e}\left(\frac{hn^c}{q}\right) \ll N^{1-b/c^2} \ll \frac{N^{1-2\varepsilon}}{q}$$

which completes the proof of (b). ■

6. Smooth values of $[n^c]$. The proof of Theorem 6 is based on the following result which we prove by adapting Heath-Brown [17].

PROPOSITION 22. *Fix $c \in (1, \frac{24979}{20803})$. Let $(a_k)_{k \in \mathbb{N}}$ be a bounded sequence of non-negative numbers for which*

$$(6.1) \quad \sum_{k \sim K} a_k \gg \frac{K}{\log K}$$

for all large $K \leq \frac{1}{2}x$. Put

$$K = x^{c-1+6\varepsilon}, \quad L = \frac{1}{5}x^{1-6\varepsilon}, \quad R(n) = \sum_{\substack{(k,l) \sim (K,L) \\ kl = [n^c]}} a_k a_l.$$

Then

$$\sum_{n \leq x} R(n) \gg x^{1-\varepsilon}.$$

Proof. In view of Lemma 10 we have

$$\sum_{n \leq x} R(n) = T_0 + T_1 + O(1),$$

where

$$T_0 = \gamma \sum_{(k,l) \sim (K,L)} a_k a_l (kl)^{\gamma-1} \gg (KL)^{\gamma-\varepsilon} \gg x^{1-\varepsilon}$$

from (6.1), whereas

$$T_1 = \sum_{(k,l) \sim (K,L)} a_k a_l (\psi(-(kl+1)^\gamma) - \psi(-(kl)^\gamma)).$$

Hence, it suffices to show that $T_1 \ll x^{1-2\varepsilon}$.

Using (2.1) and writing $\psi^*(t) = \sum_{0 < |h| \leq H} c_h \mathbf{e}(th)$, $y_{kl} = -(kl+1)^\gamma$, $z_{kl} = -(kl)^\gamma$, we see that $T_1 \ll S_1 + S_2 + S_3$, where

$$S_1 = \left| \sum_{(k,l) \sim (K,L)} a_k a_l (\psi^*(y_{kl}) - \psi^*(z_{kl})) \right|,$$

$$S_2 = \sum_{|h| \leq H} d_h \sum_{(k,l) \sim (K,L)} \mathbf{e}(hy_{kl}),$$

and S_3 is defined as S_2 with z_{kl} instead of y_{kl} . We choose $H = x^{c-1+\varepsilon}$, so that the contribution to $S_2 + S_3$ from $h = 0$ is $O(KLH^{-1}) = O(x^{1-\varepsilon})$.

To bound the contribution to $S_2 + S_3$ for non-zero h , we use the exponent pair $(\frac{1}{2}, \frac{1}{2})$ for the sum over l and treat the sums over k, h trivially. For example,

$$\left| \frac{d}{dt} (h(kt+1)^\gamma) \right| \asymp |h|(x^c)^{\gamma-1} K = |h|x^{6\varepsilon}.$$

Since $x^{6\varepsilon} \ll |h|x^{6\varepsilon} \ll x^{c-1+7\varepsilon}$ for any $c < 2$ we have

$$\sum_{k \sim K} \left| \sum_{l \sim L} \mathbf{e}(hy_{kl}) \right| \ll KL^{1/2} (x^{c-1+7\varepsilon})^{1/2} \ll x^{3c/2-1+7\varepsilon} \ll x^{1-\varepsilon}.$$

The sum S_1 is treated using a partial summation argument given in Heath-Brown [18] with $R(n)$ replacing $\Lambda(n)$. It suffices to show that

$$\sum_{h \leq H} \varepsilon_h \sum_{B < n \leq B_1} R(n) \mathbf{e}(hn^\gamma) \ll Bx^{-\varepsilon},$$

where $B = KL$, B_1 is an arbitrary number in $(B, 4B]$, and $|\varepsilon_h| = 1$ for each h . We can rewrite this as

$$\sum_{h \leq H} \varepsilon_h \sum_{k \sim K} a_k \sum_{B/k < l \leq B_1/k} a_l \mathbf{e}(h(kl)^\gamma) \ll Bx^{-\varepsilon}.$$

By a standard technique (explained, e.g., in Harman [16, §3.2]) we need only show that the bound

$$S = \sum_{h \sim H'} \varepsilon_h \sum_{k \sim K} b_k \sum_{l \sim L} c_l \mathbf{e}(h(kl)^\gamma) \ll KLx^{-2\varepsilon}$$

holds whenever $H' \leq H$, $|b_k| \leq 1$, $|c_l| \leq 1$. We use Baker [5, Theorem 2]. It is easy to check the hypothesis $X \gg L_1 L_2$ holds with the choice $X = H'(KL)^\gamma$, $L_1 = H'$, and $L_2 = K$; hence, for any exponent pair (κ, λ) we derive that

$$S \ll ((H'K)^{1/2} L + (HK)^{\frac{2+\kappa}{2+2\kappa}} (H'K^\gamma L^\gamma)^{\frac{\kappa}{2+2\kappa}} L^{\frac{1+\kappa+\lambda}{2+2\kappa}}) \log x.$$

Examining the ‘worst’ case in the proof of [5, Theorem 2] leads us to choose the exponent pair (see [19])

$$(\kappa, \lambda) = BA^4\left(\frac{32}{205} + \varepsilon, \frac{1}{2} + \frac{32}{205} + \varepsilon\right) = \left(\frac{3843}{8480}, \frac{4304}{8480}\right) + O(\varepsilon).$$

Noting that the bound

$$(HK)^{1/2}L \log x \ll KLx^{-2\varepsilon}$$

follows from the identity $H = Kx^{-5\varepsilon}$, it remains to show that

$$HK^{\frac{2+\kappa}{2+2\kappa}}(K^\gamma L^\gamma)^{\frac{\kappa}{2+2\kappa}}L^{\frac{1+\kappa+\lambda}{2+2\kappa}} \log x \ll KLx^{-2\varepsilon}.$$

Recalling our choices of K , L and H , we are led to the bound

$$(2 + \kappa)(c - 1) < 1 - \lambda, \quad \text{or} \quad c < \frac{24979}{20803}. \quad \blacksquare$$

Proposition 22 immediately yields the following result.

COROLLARY 23. *For any fixed $c \in (1, \frac{24979}{20803})$:*

- (a) *for at least $C_0x^{1-\varepsilon}$ natural numbers $n \leq x$ one has $P(\lfloor n^c \rfloor) \leq n^\varepsilon$,*
- (b) *for at least $C_0x^{1-\varepsilon}$ natural numbers $n \leq x$ one has $P(\lfloor n^c \rfloor) \geq n^{2-c-\varepsilon}$,*

where $C_0 > 0$ depends only c and ε .

The reader can easily obtain Corollary 23 by taking $(a_k)_{k \in \mathbb{N}}$ to be the indicator function either of the integers with $P(k) \leq x^{\varepsilon/2}$, or of the prime numbers. Note that assertion (b) completes the proof of Theorem 1 for values of c in the interval $[\frac{243}{205}, \frac{24979}{20803})$.

7. Carmichael numbers composed of Piatetski-Shapiro primes.

Our first goal is to establish two preliminary lemmas that are needed for an application of Lemma 12 with the function

$$f(x) = \mathbf{e}(mx^\gamma + xh/d),$$

where $m, h, d \in \mathbb{N}$. In what follows, we suppose that $1 < N < N_1 \leq 2N$.

LEMMA 24. *Suppose $|a_k| \leq 1$ for all $k \sim K$. Fix $\gamma \in (0, 1)$ and $m, h, d \in \mathbb{N}$. Then for any $L \gg N^{2/3}$ the Type I sum*

$$S_I = \sum_{\substack{k \sim K \\ N < kl \leq N_1}} \sum_{l \sim L} a_k \mathbf{e}(mk^\gamma l^\gamma + klh/d)$$

satisfies the bound

$$S_I \ll m^{1/2}N^{1/3+\gamma/2} + m^{-1/2}N^{1-\gamma/2}.$$

Proof. Writing $F(l) = mk^\gamma l^\gamma + klh/d$ we see that

$$|F''(l)| = m\gamma(1 - \gamma)k^\gamma l^{\gamma-2} \asymp mK^\gamma L^{\gamma-2} \quad (l \sim L).$$

Using Lemma 9 leads to

$$\sum_{\substack{l \sim L \\ N < kl \leq N_1}} \mathbf{e}(mk^\gamma l^\gamma + klh/d) \ll m^{1/2} K^{\gamma/2} L^{\gamma/2} + m^{-1/2} K^{-\gamma/2} L^{1-\gamma/2}.$$

Since $|a_k| \leq 1$ for all $k \sim K$ we see that

$$\begin{aligned} S_I &\leq \sum_{k \sim K} \left| \sum_{\substack{l \sim L \\ N < kl \leq N_1}} \mathbf{e}(mk^\gamma l^\gamma + klh/d) \right| \\ &\ll m^{1/2} K^{1+\gamma/2} L^{\gamma/2} + m^{-1/2} K^{1-\gamma/2} L^{1-\gamma/2}. \end{aligned}$$

Noting that $KL \asymp N$ (else the result is trivial) and so $K \ll N^{1/3}$, we finish the proof. ■

LEMMA 25. *Suppose $|a_k| \leq 1$ and $|b_l| \leq 1$ for $(k, l) \sim (K, L)$. Fix $\gamma \in (0, 1)$ and $m, h, d \in \mathbb{N}$. Then for any K in the range $N^{1/3} \ll K \ll N^{1/2}$ the Type II sum*

$$S_{II} = \sum_{\substack{k \sim K \\ N < kl \leq N_1}} \sum_{l \sim L} a_k b_l \mathbf{e}(mk^\gamma l^\gamma + klh/d)$$

satisfies the bound

$$S_{II} \ll m^{-1/4} N^{1-\gamma/4} + m^{1/6} N^{7/9+\gamma/6} + N^{11/12}.$$

Proof. We can assume that $KL \asymp N$. By Lemma 13 we have

$$(7.1) \quad |S_{II}|^2 \ll K^2 L^2 Q^{-1} + KLQ^{-1} \sum_{l \sim L} \sum_{0 < |q| \leq Q} |S(q; l)|,$$

where

$$S(q; n) = \sum_{k \in \mathcal{I}(q; l)} \mathbf{e}(F(k)), \quad F(k) = mk^\gamma (l^\gamma - (l+q)^\gamma) - kqh/d,$$

and each $\mathcal{I}(q; n)$ is a certain subinterval in the set of numbers $k \sim K$. Since

$$|F''(k)| = m\gamma(1-\gamma)k^{\gamma-2}((l+q)^\gamma - l^\gamma) \asymp mK^{\gamma-2}L^{\gamma-1}q \quad (k \sim K),$$

it follows from Lemma 9 that

$$S(q; l) \ll K(mK^{\gamma-2}L^{\gamma-1}q)^{1/2} + (mK^{\gamma-2}L^{\gamma-1}q)^{-1/2}.$$

Inserting this bound in (7.1) and summing over l and q , we derive that

$$\begin{aligned} |S_{II}|^2 &\ll K^2 L^2 Q^{-1} + m^{1/2} K^{1+\gamma/2} L^{3/2+\gamma/2} Q^{1/2} \\ &\quad + m^{-1/2} K^{2-\gamma/2} L^{5/2-\gamma/2} Q^{-1/2} \\ &\ll N^2 Q^{-1} + m^{1/2} K^{-1/2} N^{3/2+\gamma/2} Q^{1/2} \\ &\quad + m^{-1/2} K^{-1/2} N^{5/2-\gamma/2} Q^{-1/2}, \end{aligned}$$

where we used the fact that $KL \asymp N$ in the second step. Since the above holds whenever $0 < Q \leq L$, an application of Lemma 17 gives

$$|S_{II}|^2 \ll KN + m^{-1/2}N^{2-\gamma/2} + m^{1/3}K^{-1/3}N^{5/3+\gamma/3} + K^{-1/2}N^2.$$

Finally, for K in the range $N^{1/3} \ll K \ll N^{1/2}$ we arrive at the bound

$$|S_{II}|^2 \ll m^{-1/2}N^{2-\gamma/2} + m^{1/3}N^{14/9+\gamma/3} + N^{11/6},$$

and the result follows. ■

For any coprime integers a and $d \geq 1$, we denote by $\mathcal{P}_{d,a}^{(c)}$ the set of Piatetski-Shapiro primes in the arithmetic progression $a \pmod d$; that is,

$$\mathcal{P}_{d,a}^{(c)} = \{p \equiv a \pmod d : p = \lfloor n^c \rfloor \text{ for some } n \in \mathbb{N}\}.$$

Our next goal is to estimate the counting functions

$$\pi_c(x; d, a) = \#\{p \leq x : p \in \mathcal{P}_{d,a}^{(c)}\} \quad \text{and} \quad \vartheta_c(x; d, a) = \sum_{\substack{p \leq x \\ p \in \mathcal{P}_{d,a}^{(c)}}} \log p$$

in terms of the more familiar functions

$$\pi(x; d, a) = \#\{p \leq x : p \equiv a \pmod d\} \quad \text{and} \quad \vartheta(x; d, a) = \sum_{\substack{p \leq x \\ p \equiv a \pmod d}} \log p.$$

By Lemma 10 we have

$$\pi_c(x; d, a) = \Sigma_1(x) + \Sigma_2(x) + O(1),$$

where

$$\Sigma_1(x) = \gamma \sum_{\substack{p \leq x \\ p \equiv a \pmod d}} p^{\gamma-1}, \quad \Sigma_2(x) = \sum_{\substack{p \leq x \\ p \equiv a \pmod d}} (\psi(-(p+1)^\gamma) - \psi(-p^\gamma)).$$

Using partial summation one sees that

$$\Sigma_1(x) = \gamma x^{\gamma-1} \pi(x; d, a) - \gamma(\gamma-1) \int_2^x u^{\gamma-2} \pi(u; d, a) du.$$

Next, we turn our attention to $\Sigma_2(x)$. We begin by considering sums of the form

$$(7.2) \quad S = \sum_{\substack{N < n \leq N_1 \\ n \equiv a \pmod d}} \Lambda(n) (\psi(-(n+1)^\gamma) - \psi(-n^\gamma)).$$

Arguing as in [15, pp. 47–49], for any real number $M \geq 1$ we derive the uniform bound

$$(7.3) \quad S \ll N^{\gamma-1} \max_{N_2 \sim N} \sum_{1 \leq m \leq M} \left| \sum_{\substack{N < n \leq N_2 \\ n \equiv a \pmod d}} \Lambda(n) \mathbf{e}(mn^\gamma) \right| + NM^{-1} + N^{\gamma/2} M^{1/2}.$$

To bound the inner sum, we note that

$$\sum_{\substack{N < n \leq N_2 \\ n \equiv a \pmod{d}}} \Lambda(n) \mathbf{e}(mn^\gamma) = \frac{1}{d} \sum_{h=1}^d \sum_{N < n \leq N_2} \Lambda(n) \mathbf{e}(mn^\gamma + (n-a)h/d),$$

hence it suffices to give a bound on exponential sums of the form

$$T = \sum_{N < n \leq N_2} \Lambda(n) \mathbf{e}(mn^\gamma + nh/d),$$

where $1 < N < N_2 \leq 2N$. We do this with an application of Lemma 12, taking into account the estimates of Lemmas 24 and 25; we find that

$$TN^{-\varepsilon} \ll m^{1/2} N^{1/3+\gamma/2} + m^{1/6} N^{7/9+\gamma/6} + m^{-1/4} N^{1-\gamma/4} + N^{11/12}$$

for any fixed $\varepsilon > 0$. Inserting this bound in (7.3) and summing over m , it follows that

$$\begin{aligned} SN^{-\varepsilon} &\ll N^{-2/3+3\gamma/2} M^{3/2} + N^{-2/9+7\gamma/6} M^{7/6} \\ &\quad + N^{3\gamma/4} M^{3/4} + N^{-1/12+\gamma} M + NM^{-1}. \end{aligned}$$

Since the above holds for any real $M \geq 1$, using Lemma 17 we find that

$$\begin{aligned} SN^{-\varepsilon} &\ll N^{-2/3+3\gamma/2} + N^{-2/9+7\gamma/6} + N^{3\gamma/4} + N^{-1/12+\gamma} \\ &\quad + N^{1/3+3\gamma/5} + N^{17/39+7\gamma/13} + N^{3/7+3\gamma/7} + N^{11/24+\gamma/2}. \end{aligned}$$

Since $\pi_c(x; d, a) \ll x^\gamma$, this bound is trivial unless the exponent of each term in the parentheses is strictly less than γ . Thus, from now on we assume that $\gamma \in (\frac{17}{18}, 1)$. In this case, after eliminating lower order terms, the previous bound simplifies to

$$(7.4) \quad S \ll N^{17/39+7\gamma/13+\varepsilon}$$

for any fixed $\varepsilon > 0$.

To bound $\Sigma_2(x)$, let

$$\begin{aligned} G(x) &= \sum_{\substack{p \leq x \\ p \equiv a \pmod{d}}} (\log p) (\psi(-(p+1)^\gamma) - \psi(-p^\gamma)), \\ H(x) &= \sum_{\substack{n \leq x \\ n \equiv a \pmod{d}}} \Lambda(n) (\psi(-(n+1)^\gamma) - \psi(-n^\gamma)). \end{aligned}$$

Clearly,

$$H(x) = G(x) + O(x^{1/2}),$$

and by partial summation,

$$\Sigma_2(x) = \frac{G(x)}{\log x} + \int_2^x \frac{G(u)}{u(\log u)^2} du.$$

Splitting the sum $H(x)$ into $O(\log x)$ sums S of the form (7.2) with $2N \leq x$, and using (7.4), we see that the bound $H(x) \ll x^{17/39+7\gamma/13+\varepsilon}$ holds for any fixed $\varepsilon > 0$, and from the preceding observations we derive a similar result for $\Sigma_2(x)$. Putting everything together, we have proved Theorem 8.

Replacing the function $\pi_c(x; d, a)$ with the weighted counting function

$$\vartheta_c(x; d, a) = \sum_{\substack{p \leq x \\ p \in \mathcal{P}_{d,a}^{(c)}}} \log p = \sum_{\substack{p \leq x \\ p \equiv a \pmod{d}}} ([-p^\gamma] - [-(p+1)^\gamma]) \log p$$

and using a similar argument, we obtain the following statement.

THEOREM 26. *For any $c \in (1, \frac{18}{17})$ and $\varepsilon > 0$ we have*

$$\begin{aligned} \vartheta_c(x; d, a) &= \gamma x^{\gamma-1} \vartheta(x; d, a) + \gamma(1-\gamma) \int_2^x u^{\gamma-2} \vartheta(u; d, a) du \\ &\quad + O(x^{17/39+7\gamma/13+\varepsilon}), \end{aligned}$$

where the implied constant depends only on c, ε .

For the proof of Theorem 7 we also require the following variant of the Brun–Titchmarsh bound for Piatetski-Shapiro primes, which is a consequence of Theorem 8.

THEOREM 27. *For any $c \in (1, \frac{18}{17})$ and $A \in (0, -\frac{17}{39} + \frac{6\gamma}{13})$ there is a number $C = C(c, A) > 0$ such that if $\gcd(a, d) = 1$ and $1 \leq d \leq x^A$ then*

$$\pi_c(x; d, a) \leq \frac{C x^\gamma}{\varphi(d) \log x}.$$

Proof. Let $\varepsilon > 0$ be chosen (depending only on c, A) so that

$$\max \left\{ 2A\gamma, \frac{17}{39} + \frac{7\gamma}{13} + \varepsilon \right\} \leq \gamma - A - \varepsilon.$$

Then by Theorem 8 it follows that

$$(7.5) \quad \pi_c(x; d, a) \ll x^{\gamma-1} \pi(x; d, a) + \int_{x^{2A}}^x u^{\gamma-2} \pi(u; d, a) du + x^{\gamma-A-\varepsilon},$$

where the implied constant depends only on c, A . Since

$$x^{\gamma-A-\varepsilon} \ll \frac{x^{\gamma-A}}{\log x} \leq \frac{x^\gamma}{\varphi(d) \log x} \quad (1 \leq d \leq x^A),$$

the result follows by applying the Brun–Titchmarsh theorem to the right side of (7.5). ■

We now outline our proof of Theorem 7. We are brief since our construction of Carmichael numbers composed of primes from $\mathcal{P}^{(c)}$ closely follows the construction of “ordinary” Carmichael numbers given by Alford, Granville

and Pomerance [2]. Here, we discuss only the changes that are needed to establish Theorem 7.

The idea behind our proof is to show that the set $\mathcal{P}^{(c)}$ is sufficiently well-distributed over arithmetic progressions so that, following the method of [2], the primes used to form Carmichael numbers can all be drawn from $\mathcal{P}^{(c)}$ rather than the set \mathcal{P} of all prime numbers. For this, we apply the results derived earlier in this section.

The following statement plays a crucial role in our construction, analogous to that played by [2, Theorem 2.1].

LEMMA 28. *Fix $c \in (1, \frac{18}{17})$ and $B \in (0, -\frac{17}{39} + \frac{6\gamma}{13})$. There exist numbers $\eta > 0$, x_0 and D such that for all $x \geq x_0$ there is a set $\mathcal{D}(x)$ consisting of at most D integers such that*

$$\left| \vartheta_c(x; d, a) - \frac{x^\gamma}{\varphi(d)} \right| \leq \frac{x^\gamma}{2\varphi(d)}$$

provided that

- (i) d is not divisible by any element of $\mathcal{D}(x)$;
- (ii) $1 \leq d \leq x^B$;
- (iii) $\gcd(a, d) = 1$.

Every number in $\mathcal{D}(x)$ exceeds $\log x$, and all but at most one exceed x^η .

REMARK. In the statement and proof of Lemma 28, η , x_1 , D and $\mathcal{D}(x)$ all depend on the choice of c and B , but this is suppressed from the notation for the sake of clarity.

Proof of Lemma 28. For any such B we have $2B < \frac{5}{12}$. Applying [2, Theorem 2.1] (with $2B$ instead of B) we see that there exist numbers $\eta > 0$, x_1 and D such that for all $x \geq x_1$ there is a set $\mathcal{D}(x)$ consisting of at most D integers such that

$$(7.6) \quad \left| \vartheta(y; d, a) - \frac{y}{\varphi(d)} \right| \leq \frac{y}{10\varphi(d)} \quad (x^{1-B} \leq y \leq x)$$

whenever (i)–(iii) hold. Furthermore, every number in $\mathcal{D}(x)$ exceeds $\log x$, and all but at most one exceed x^η .

Let $\varepsilon > 0$ be chosen (depending only on c, B) so that

$$\frac{17}{39} + \frac{7\gamma}{13} + \varepsilon \leq \gamma - B - \varepsilon,$$

and suppose that d and a are integers such that (i)–(iii) hold. Then by Theorem 26 it follows that

$$\vartheta_c(x; d, a) = T_1 + T_2 + T_3 + O(T_4),$$

where

$$\begin{aligned}
 T_1 &= \gamma x^{\gamma-1} \vartheta(x; d, a), \\
 T_2 &= \gamma(1 - \gamma) \int_{x^{1-B}}^x u^{\gamma-2} \vartheta(u; d, a) du, \\
 T_3 &= \gamma(1 - \gamma) \int_2^{x^{1-B}} u^{\gamma-2} \vartheta(u; d, a) du, \\
 T_4 &= x^{\gamma-B-\varepsilon}.
 \end{aligned}$$

By (7.6) we have

$$0.9\gamma \frac{x^\gamma}{\varphi(d)} \leq T_1 \leq 1.1\gamma \frac{x^\gamma}{\varphi(d)}$$

and

$$0.9(1 - \gamma) \frac{x^\gamma}{\varphi(d)} + O\left(\frac{x^{\gamma(1-B)}}{\varphi(d)}\right) \leq T_2 \leq 1.1(1 - \gamma) \frac{x^\gamma}{\varphi(d)} + O\left(\frac{x^{\gamma(1-B)}}{\varphi(d)}\right).$$

Using the Brun–Titchmarsh bound $\vartheta(x; d, a) \ll x/\varphi(d)$ for $1 \leq d \leq x^B$ we also see that

$$T_3 \ll \frac{x^{\gamma(1-B)}}{\varphi(d)}.$$

Finally, we note that

$$T_4 \leq \frac{x^{\gamma-\varepsilon}}{\varphi(d)} \quad (1 \leq d \leq x^B).$$

Combining the above estimates, we deduce that the inequalities

$$(0.9 + o(1)) \frac{x^\gamma}{\varphi(d)} \leq \vartheta_c(x; d, a) \leq (1.1 + o(1)) \frac{x^\gamma}{\varphi(d)}$$

hold as $x \rightarrow \infty$, and the result follows. ■

As an application of Lemma 28 we derive the following statement, which extends [2, Theorem 3.1] to the setting of Piatetski-Shapiro primes.

LEMMA 29. Fix $c \in (1, \frac{18}{17})$, and let A, B, B_1 be positive real numbers such that $B_1 < B < A < -\frac{17}{39} + \frac{6\gamma}{13}$. Let $C = C(c, A) > 0$ have the property described in Theorem 27. There exists a number $x_2 = x_2(c, A, B, B_1)$ such that if $x \geq x_2$ and L is a squarefree integer not divisible by any prime q exceeding $x^{(A-B)/2}$ and for which

$$(7.7) \quad \sum_{\text{prime } q|L} \frac{1}{q} \leq \frac{1-A}{16C},$$

then there is a positive integer $k \leq x^{1-B}$ with $\gcd(k, L) = 1$ such that

$$\begin{aligned} \#\{d \mid L : dk + 1 \leq x \text{ and } p = dk + 1 \text{ is a prime in } \mathcal{P}^{(c)}\} \\ \geq \frac{2^{-D-2}(x^{1-B+B_1})^{\gamma-1}}{\log x} \#\{d \mid L : x^{B_1} \leq d \leq x^B\}, \end{aligned}$$

where $D = D(c, B)$ is chosen as in Lemma 28.

Sketch of proof. We follow the proof and use the notation of [2, Theorem 3.1]. In view of Lemma 28 we can replace the lower bound [2, (3.2)] with the bound

$$\pi_c(dx^{1-B}; d, 1) \geq \frac{1}{2} \frac{(dx^{1-B})^\gamma}{\varphi(d) \log x} \quad (d \mid L', 1 \leq d \leq x^B).$$

Also, since $dq \leq (dx^{1-B})^A$ for any natural numbers $d \leq x^B$ and $q \leq x^{(A-B)/2}$, Theorem 27 enables us to replace the upper bound that occurs after [2, (3.2)] with the bound

$$\pi_c(dx^{1-B}; dq, 1) \leq \frac{4C}{q(1-A)} \frac{(dx^{1-B})^\gamma}{\varphi(d) \log x} \quad (1 \leq d \leq x^B)$$

for every prime q dividing L' . Taking into account (7.7), we see that there are at least

$$\frac{(x^{1-B})^\gamma}{4 \log x} \sum_{\substack{1 \leq d \leq x^B \\ d \mid L'}} \frac{d^\gamma}{\varphi(d)} \geq \frac{(x^{1-B})^\gamma}{4 \log x} x^{B_1(\gamma-1)} \#\{d \mid L' : x^{B_1} \leq d \leq x^B\}$$

pairs (p, d) where $p \leq dx^{1-B}$ is a prime in $\mathcal{P}^{(c)}$, $p \equiv 1 \pmod L$, $(p-1)/d$ is coprime to L , $d \mid L'$, and $x^{B_1} \leq d \leq x^B$. Hence, there is an integer $k \leq x^{1-B}$ with $\gcd(k, L) = 1$ such that k has at least

$$\frac{(x^{1-B+B_1})^{\gamma-1}}{4 \log x} \#\{d \mid L' : x^{B_1} \leq d \leq x^B\}$$

representations as $(p-1)/d$ with a pair (p, d) as above. Since we can replace [2, (3.1)] with the lower bound

$$\#\{d \mid L' : x^{B_1} \leq d \leq x^B\} \geq 2^{-D} \#\{d \mid L : x^{B_1} \leq d \leq x^B\},$$

the proof is complete. ■

Let $\pi(x)$ be the number of primes $p \leq x$, and let $\pi(x, y)$ be the number of those for which $p-1$ is free of prime factors exceeding y . As in [2], we denote by \mathcal{E} the set of numbers E in the range $0 < E < 1$ for which

$$\pi(x, x^{1-E}) \geq x^{1+o(1)} \quad (x \rightarrow \infty),$$

where the function implied by $o(1)$ depends only on E . With only some slight modifications to the proof of [2, Theorem 4.1], using Lemma 29 in place of [2, Theorem 3.1], we have:

LEMMA 30. Fix $c \in (1, \frac{57}{56})$, and let B, B_1 be positive real numbers such that $B_1 < B < -\frac{17}{39} + \frac{6\gamma}{13}$. For any $E \in \mathcal{E}$ there is a number x_4 , depending on c, B, B_1, E and ε , such that for any $x \geq x_4$ there are at least $x^{EB+(1-B+B_1)(\gamma-1)-\varepsilon}$ Carmichael numbers up to x composed solely of primes from $\mathcal{P}^{(c)}$.

REMARK. It may seem more natural to state this result for any $c \in (1, \frac{18}{17})$ in view of our earlier results; however, it can be seen that the exponent $EB + (1 - B + B_1)(\gamma - 1) - \varepsilon$ is never positive when $c \geq \frac{57}{56}$, so the result is vacuous in that case. This point is discussed further below.

Sketch of proof of Lemma 30. Following the proof and notation of [2, Theorem 4.1], the condition (7.7) is easily verified, so we can construct a set \mathcal{P} of primes in $\mathcal{P}^{(c)}$ with $p \leq x$ with $p = dk + 1$ for some divisor d of L , which satisfies the lower bound

$$\#\mathcal{P} \geq \frac{2^{-D-2}(x^{1-B+B_1})^{\gamma-1}}{\log x} \#\{d \mid L : x^{B_1} \leq d \leq x^B\}$$

by Lemma 29 (compare to [2, (4.5)]). To complete the argument, we simply observe that the lower bound for $\#\{d \mid L : 1 \leq d \leq x^B\}$ given in [2, p. 718] is also a lower bound for $\#\{d \mid L : x^{B_1} \leq d \leq x^B\}$ if x is large enough, since the product of any

$$u = \left\lfloor \frac{\log x^B}{\log y^\theta} \right\rfloor = \left\lfloor \frac{B \log x}{\theta \log y} \right\rfloor$$

primes $q \in (y^\theta/\log y, y^\theta]$ is a divisor d of L of size $x^{B+o(1)} \leq d \leq x^B$ as $x \rightarrow \infty$. ■

Taking B and B_1 arbitrarily close to $-\frac{17}{39} + \frac{6\gamma}{13}$, and noting that \mathcal{E} is an open set by [2, Proposition 5.1], Lemma 30 implies that there are infinitely many Carmichael numbers composed of primes from $\mathcal{P}^{(c)}$ provided that

$$(7.8) \quad E\left(-\frac{17}{39} + \frac{6\gamma}{13}\right) + \gamma - 1 > 0.$$

Since $E < 1$, this inequality cannot hold if $\gamma \leq \frac{56}{57}$. Moreover, we do not know that E can be taken arbitrarily close to one, i.e., that $\mathcal{E} = (0, 1)$. At present, it is known unconditionally that $0.7039 \in \mathcal{E}$ (see Baker and Harman [6]), and taking $E = 0.7039$ in (7.8) leads to the statement of Theorem 7.

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Roger C. Baker
Department of Mathematics
Brigham Young University
Provo, UT 84602, U.S.A.
E-mail: baker@math.byu.edu

Jörg Brüderer
Mathematisches Institut
Georg-August Universität Göttingen
37073 Göttingen, Germany
E-mail: bruedern@uni-math.gwdg.de

Andreas J. Weingartner
Department of Mathematics
Southern Utah University
Cedar City, UT 84720, U.S.A.
E-mail: weingartner@suu.edu

William D. Banks
Department of Mathematics
University of Missouri
Columbia, MO 65211, U.S.A.
E-mail: bankswd@missouri.edu

Igor E. Shparlinski
Department of Computing
Macquarie University
Sydney, NSW 2109, Australia
E-mail: igor.shparlinski@mq.edu.au

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